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FREQUENCY DOMAIN APPROACH TO MINIMIZING DETECTABLE FAULTS IN FDI SYSTEMS

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Problems related to fault detection and isolation in uncertain dynamical systems are studied. The purpose of this study is to develop a scheme to design residual generators and furthermore to evaluate residuals. Different from the most of previous studies, the basic idea of this work is to minimize the size of detectable faults. This physically means that a fault will be detected even if its size may be small. With the aid of frequency domain fault detection approach, a relationship between residual generators as well as residual evaluation function and the minimum size of detectable faults is established. This enables us to formulate the design problems as optimization problems that are solvable by using frequency domain optimization techniques. The study reveals how far a fault could be detected using a suitable residual generator and evaluation scheme and answers some questions that have been studied over the years.

1. Introduction

In view of an increasing demand for higher performance as well as for more safety and reliability of dynamic systems, fault detection and isolation (FDI) has received more and more attention. One area of the active research is the development of analytical redundancy management. In the course of this development a capable strategy has emerged and is increasingly discussed. It is based on the use of modern observer theory (Frank, 1990, 1991; Gertler, 1988, 1991; Patton and Chen, 1991; Patton *et al.*, 1989).

1.1. Background of the Study

The basic idea of the observer–based approach to FDI is, in contrast to the physical redundancy approach that makes use of physical replica for the residual generation, to compare the measurable process variables with their estimate given by observers. The difference, also called residual, is nominally near zero and evidently deviates from zero when a fault has occured in the system. Evidently, this is realizable if we could acutely understand the dynamic processes arising in any physical plant so that we would be able to precisely estimate the desired process variables. Unfortunately, this idealized prerequisite does generally not apply to real technical systems.

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Neither there exist perfect models nor are characteristics of possible disturbances, available which are unvoidable in real technical systems. The consequence of the existence of such model uncertainties is that the measurements will not match the corresponding estimation even if there are no faults present in the system. For this reason, the robustness problem becomes one of the most important issues of observer–based FDI schemes.

A classical way to increase the robustness to the model uncertainties is to use evaluation thresholds to distinguish a fault (Frank, 1990; Gertler, 1988; Patton *et al.*, 1989). The crux with thresholds is that they do not only reduce the sensitivity to faults, but also vary with the input of the actual system and the magnitude and nature of the model uncertainties. Choosing the threshold too low increases the rate of false alarms; choosing it too large reduces the net effect of fault detection. To overcome this difficulty, Clark (see Patton *et al.*, 1989) has presented an adaptive threshold selecting method which has been demonstrated to be suitable for the instrument fault detection regarding a class of system uncertainties. A more generalized version of this method was lately proposed by Emami–Naeini *et al.* (1988), where they have introduced the concept of threshold selector.

Another way to solve the robustness problem is to use robust residual generators. The philosophy of this scheme originated from the idea that if one is able to generate residuals that are on the one hand insensitive or even invariant to the model uncertainty and on the other sensitive to the faults, then FDI is achieved. Notice that in this case evaluation thresholds can be set as zero or nearly zero so that a residual evaluation is separeable. Over the last two decades a great deal of research has been devoted to the realization of this idea, and in company with it, also supported by the development in modern control theory and computer technology, a number of sophisticated residual generation techniques has been developed. During the eighties, the study was concentrated on designing robust residual generators that are *invariant* to unknown disturbances. Significant researches on this topic have been made e.g. by Chow and Willsky (1984), Ge and Fang (1988), Massoumnia (1986), Patton *et al.* (1989), Viswanadham and Srichander (1987) and Wuennenberg (1990), using the modern linear control theory such as geometrical theory, algebraical theory. The results reveal that it may be very difficult to achieve a residual that is invariant to the unknown disturbances. Encouraged by this knowledge, a new study has been carried out. As a logical alternative, it was tried, instead of making residuals invariant to the disturbances, to generate residuals such that they are *approximately invariant* to the disturbances. Based on this idea, new approaches have been developed during the last years (Chow and Willsky, 1984; Ding and Frank, 1989, 1991; Lou *et al.*, 1986; Patton *et al.*, 1989; Wuennenberg, 1990).

Recently, it has been noticed that although the new approaches are effective to enhance the robustness of the residuals against the disturbances, which, from the modelling point of view, may be considered as structured model uncertainties, they may fail in dealing with unstructured model uncertainties (Ding and Frank, 1991; Frank, 1991). It was furthermore recognised, also due to the publication of the results achieved by Emami–Naeini *et al.* (1988), that a possible way to solve this problem is considering residual generation and evaluation as one problem (Ding and Frank, 1991; Frank, 1991; Patton and Chen, 1991). Encouraged by this idea, the first study has been done by Ding and Frank (1991).

By the study on this topic, one is forced to define a performance index that mathematically describes the original robustness idea and plays the role of connecting residual generation and evaluation problems. On the other hand, one may remark that although the robustness of FDI may be increased due to the introduction of the threshold, only a part of faults, namely the faults that may cause evaluated residuals larger than the threshold, are detectable. It is naturally desirable to have a residual generator that makes the size of detectable faults as small as possible. The basic idea of this paper originated from this consideration. We formulate it as follows:

> Design a residual generator that minimizes the size of detectable faults, instead of optimizing some performance index.

Obviously, our objective has a clear, direct physical meaning that is also the aim of FDI, while previous works dealt with the robustness problem in an indirect way.

The mathematical tool used for realizing the above idea is frequency domain approaches. The decision on such approaches has the following background:

- many frequency domain properties can be utilized by the design of residual generators as well as by the residual evaluation (Ding and Frank, 1989, 1990);
- the rapid development of $\mathcal{H}_\infty$–theory (Francis, 1987) provides us a powerful tool to solve robustness problems associated with FDI (Frank, 1991; Patton and Chen, 1991).

The application of frequency domain approaches to FDI was initiated by Viswanadham *et al.* (1987) and Viswanadham and Minto (1988). They have proposed a simple form for constructing the residual generator and suggested to apply $\mathcal{H}_\infty$–theory to FDI. A more intensive study and extension has been made by Ding and Frank (1989, 1990, 1991), in which the FDI problem was systematically formulated and solved by using factorization and $\mathcal{H}_\infty$–optimization techniques. A part of these results constructs the basis of this paper.

1.2. Notation

Standard notation is used whenever possible. For simplicity of terminology, the term *transfer function* will be used to refer to transfer function matrices as well. $G^\top(s)$ and $G^*(s)$ denote the transpose and Hermitian transpose of a transfer function $G(s)$, respectively.

$$\|r(j\omega)\|_2 := (r^*(j\omega)r(j\omega))^{1/2}$$

denotes the 2–norm of a vector and $||r(j\omega)||_\epsilon$ denotes

$$||r(j\omega)||_\epsilon = \left((2\pi)^{-1}\int_{\omega_2}^{\omega_1} r^*(j\omega)r(j\omega)\mathrm{d}\omega\right)^{1/2}, \quad \epsilon = \omega_2 - \omega_1.$$

$\sigma(G(j\omega))$ as well as $\bar{\sigma}(G(j\omega))$ and $\underline{\sigma}(G(j\omega))$ denote the singular as well as maximum and minimum singular value of transfer function $G(j\omega)$, $\sup(\cdot)$ the supremum, and $\inf(\cdot)$ the infimum. The subset of Hardy space $\mathcal{H}_\infty$ (Francis, 1987) consisting of real–rational functions, i.e., all transfer functions that are realizable using stable linear systems, are denoted by $\mathcal{RH}_\infty$.

2. Problem Formulation

2.1. Mathematical Model

We consider nominal linear systems described as follows:

$$y_o(s) = G_u(s)u(s), \tag{1}$$

where $u(t) \in \mathcal{R}^p$ is the input vector and $y_o(t) \in \mathcal{R}^m$ is the nominal observation vector, $G_u(s)$ is an $m \times p$ dimentional real–rational transfer function matrix.

In the context of FDI, a fault is understood as any kind of malfuntions in an actual dynamic system that leads to an unacceptable anomaly in the overall system performance. The effect of the faults on the system dynamics can be in general modelled as follows:

$$y(s) = G_u(s)u(s) + G_f(s)f(s), \tag{2}$$

where $f \in \mathcal{R}^q$ represents fault vector and $G_f(s)$ denotes a known distribution transfer matrix which, without loss of generality, is assumed to be stable. It is worth mentioning that transfer function $G_f(s)$ also serves as a weighting factor which addresses possible information about the fault f.

Since in practice no nominal models can describe a physical plant perfectly, model uncertainties should be taken into account. Model uncertainties refer to the mismatch between the nominal model and the actual system. With respect to system model (2) a uncertain system may be expressed by

$$y(s) = y_o(s) + \Delta y(s) + G_f(s)f(s) \tag{3}$$

with $\Delta y(s)$ representing the effects of the model uncertainties on the measurement. We call the model uncertainties structured if it can be written as

$$\Delta y(s) = G_w(s)w(s)$$

with known distribution matrix $G_w(s)$ and a unknown function vector $w(s)$, otherwise it is called unstructured. In the remainder of this paper we consider systems of the form

$$y(s) = (G_u(s) + \Delta G_u(s))u(s) + (G_f(s) + \Delta G_f(s))f(s) + G_w(s)w(s) \tag{4}$$

where $G_w(s)$ is known and assumed to be stable, $\Delta G_u(s), \Delta G_f(s), w(s)$ are unknown, but their sizes are restricted:

$$\bar{\sigma}(\Delta G_u(j\omega)) \leq \delta_u(\omega), \quad \bar{\sigma}(\Delta G_f(j\omega)) \leq \delta_f(\omega), \quad \|w(j\omega)\|_2 \leq \delta_w(\omega). \tag{5}$$

2.2. Outline of Tasks

A complete FDI process essentially consists of two stages (Frank, 1990). The first stage is residual generation and the second stage residual evaluation. A residual, $r(t)$, is a vector of functions that are accentuated by the fault vector $f(t)$. To generate a residual, a dynamic system, also called residual generator, has to be constructed. It makes use of the *a priori* knowledge contained in the system model and processes on–line measurements $y(t)$ and $u(t)$ to perform some kind of validation of the nominal relationships of the system. If a fault occurs, the redundancy relations should no longer be satisfied. In general, a residual has the following definition:

$$r(s) = 0, \quad \text{for} \quad f(s) = 0, \ \Delta y(s) = 0, \tag{6}$$

$$r(s) \neq 0, \quad \text{for} \quad f(s) \neq 0. \tag{7}$$

To decide whether a fault occurs in case $r(s) \neq 0$, the generated residual has to be evaluated to form some appropriate decision function, denoted by $J(r)$. A fault f is detectable and isolable if the following specifications are fulfilled:

$$J(r) < J_{th} \qquad \text{for } f(s) = 0 \tag{8}$$

$$J(r) > J_{th} \qquad \text{for } f(s) \neq 0 \qquad \text{(Fault detection)} \tag{9}$$

$$J(r_i) < J_{th_i} \qquad \text{for } f_i(t) = 0 \qquad (i = 1, \ldots, q) \tag{10}$$

$$J(r_i) > J_{th_i} \qquad \text{for } f_i(t) \neq 0 \qquad \text{(Fault isolation)} \tag{11}$$

where J_{th} as well as J_{th_i} define thresholds. Evidently, the ideal case would be

$$J_{th} = 0, \quad J_{th_i} = 0, \quad (i = 1, \ldots, q),$$

called perfect fault detection and isolation (PFDI), so that every fault, independent of its size and form, could be detected or isolated. This is, however, normally not realizable. In order to avoide false alarms, the thresholds should differ from zero. As a result faults under certain size cannot be detected or isolated. Nevertheless, we can try to make the size of detectable faults as small as possible. The tasks of this paper are to

- develop an approach of designing residual generators,
- introduce decision function $J(r)$ and
- define thresholds J_{th} (J_{thi})

so that the *size of detectable faults could be minimized.*

3. Preliminaries

This section reviews some preliminaries concerned with FDI, in particular, the frequency domain approach to residual generation and evaluation.

3.1. Parametrization of Residual Generators and their Dynamics

Over the last decades, FDI using observation techniques has been widely studied and a number of significant approaches have been proposed such as detection filter scheme (Beard, 1971; Jones, 1973), unknown input observer approach, (Viswanadham and Srichander, 1987; Patton *et al.*, 1989) and parity check (Gertler, 1988; Lou *et al.*, 1986). Ding and Frank (1990) have proposed a frequency domain approach, with the help of factorization theory, to construct residual generators. It is the basis of this paper.

As mentioned earlier, the useful signals for the residual generation are the measurable input vector and the output vector. Thus, the most general form of a linear residual generator in the frequency domain is

$$\boldsymbol{r}(s) = \boldsymbol{F}(s)\boldsymbol{u}(s) + \boldsymbol{H}(s)\boldsymbol{y}(s), \tag{12}$$

where $\boldsymbol{F}(s)$, $\boldsymbol{H}(s) \in \mathcal{RH}_\infty$. Let

$$\boldsymbol{G}_u(s) = \hat{\boldsymbol{M}}_u^{-1}(s)\hat{\boldsymbol{N}}_u(s) \tag{13}$$

be a left coprime factorization of $\boldsymbol{G}_u(s)$. Ding and Frank (1990) have proved the following lemma.

Lemma 1. *Given system (4), then (linear) residual generators can be parametrized as follows:*

$$\boldsymbol{r}(s) = \boldsymbol{R}(s)(\hat{\boldsymbol{M}}_u(s)\boldsymbol{y}(s) - \hat{\boldsymbol{N}}_u(s)\boldsymbol{u}(s)), \quad \boldsymbol{R}(s)(\neq 0) \in \mathcal{RH}_\infty. \tag{14}$$

In equation (14) $\boldsymbol{R}(s)$ is a parametrization matrix, the transfer functions $\hat{\boldsymbol{M}}_u(s)$ and $\hat{\boldsymbol{N}}_u(s)$ can easily be determined using the existing algorithms, see e.g. (Francis, 1987). Thus, the problem of designing residual generators is reduced to finding a suitable parametrization matrix $\boldsymbol{R}(s)$ to meet different specifications of FDI.

We now observe the dynamics of the residual vector corresponding to the fault $\boldsymbol{f}$ as well as to the model uncertainties. To do this we substitute equation (4) into (14), which yields

$$\boldsymbol{r}(s) = \boldsymbol{R}(s)\hat{\boldsymbol{M}}_u(s)\big((\boldsymbol{G}_f(s) + \Delta \boldsymbol{G}_f(s))\boldsymbol{f}(s) + \Delta \boldsymbol{G}_u(s)\boldsymbol{u}(s) + \boldsymbol{G}_w(s)\boldsymbol{w}(s)\big). \tag{15}$$

Equation (15) characterizes the achievable residuals with respect to the fault vector and the model uncertainties. Based on this relationship we are able to suitably choose the parametrization matrix $\boldsymbol{R}(s)$.

It is worth mentioning that the parametrization of residual generators provides a systematic and straightforward way to construct residual generators, while the parametrization of residual dynamics suggests how to choose parametrization matrix $R(s)$. This makes the design especially suitable for the implementation on a computer.

3.2. About Fault Detection and Isolation

Following (15) it is clear that FDI problem may be looked upon as being virtually a decoupling problem, i.e.,

- for fault detection the fault effects on the residual r must be distinguishable from the effects of the model uncertainties and
- for fault isolation the effects of a fault on the residual r must be distinguishable from the effects of the model uncertainties as well as the other faults.

We re–write system model (4) as

$$y(s) = G_u(s)u(s) + \bar{G}_d(s)\bar{d}(s) + \bar{G}_f(s)f(s) \tag{16}$$

or

$$y(s) = G_u(s)u(s) + \bar{G}_d(s)\bar{d}(s) + \bar{G}_{fi}(s)\bar{f}_i(s) + \bar{g}_{fi}(s)f_i(s) \tag{17}$$

where

$$\bar{G}_d(s) = [I \quad G_w(s)],\ \bar{d}(s) = \begin{bmatrix} \Delta G_u(s)u(s) \\ w(s) \end{bmatrix},$$

$$\bar{G}_f(s) = G_f(s) + \Delta G_f(s) = [\bar{g}_{f1}(s) \cdots \bar{g}_{fi}(s) \cdots \bar{g}_{fq}(s)],$$

$$\bar{G}_{fi}(s) = [\bar{g}_{f1}(s) \cdots \bar{g}_{f(i-1)}(s) \quad \bar{g}_{f(i+1)}(s) \cdots \bar{g}_{fq}(s)],$$

$$\bar{g}_{fi}(s) = g_{fi}(s) + \Delta g_{fi}(s), \quad i = 1, \cdots, q,$$

$$\bar{f}_i(s) = [f_1(s) \cdots f_{i-1}(s) \quad f_{i+1}(s) \cdots f_q(s)]^{\mathsf{T}}.$$

Substituting (16) or (17) into (15) gives

$$r(s) = R(s)\hat{M}_u(s)(\bar{G}_d(s)\bar{d}(s) + \bar{G}_f(s)f(s)) \tag{18}$$

or

$$r_i(s) = R_i(s)\hat{M}_u(s)(\bar{G}_d(s)\bar{d}(s) + \bar{G}_{fi}(s)\bar{f}_i(s) + \bar{g}_{fi}(s)f_i(s)). \tag{19}$$

This indicates that

- the fault detection problem is that of choosing a suitable parametrization matrix $R(s)$ so that the generated residual $r(s)$ is sensitive to the fault f and robust against $\bar{d}$ and
- the fault isolation problem is that of choosing q parametrization matrices $R_i(s)$, $i = 1, \cdots, q$, so that the generated residuals are sensitive to the fault f_i and robust against $\bar{d}$ and $\bar{f}_i(s)$.

In view of the above statement we focus our attention, without loss of generality and also for the sake of simplicity, on the systems of the form

$$y(s) = G_u(s)u(s) + G_d(s)d(s) + \bar{G}_f(s)f(s) \tag{20}$$

with known distribution matrices $G_d(s), G_f(s)$ and unknown input vector $d(s)$ whose size is restricted by

$$\|d(j\omega)\|_2 \leq \delta_d(\omega) \neq 0 \quad \text{for all} \quad \omega \tag{21}$$

where $\delta_d(\omega)$ is an even rational function in ω. Notice that $d(s)$ is a function of $u(s)$. This implies that $\delta_d(\omega)$ is (off–line) known only if $u(j\omega) \in \mathcal{RL}_\infty$ and is given before the process observed is in operation. Unfortunately, this is not always the case, since in practice the input signal is often on–line calculated to adapt real operating conditions. Taking this into account we consider in the remainder of this paper two cases

- $\delta_d(\omega)$ is known;
- $\delta_d(\omega)$ consists of two parts

$$\delta_d(\omega) = \hat{\delta}_d(\omega)\hat{\delta}_u(\omega) \tag{22}$$

where $\hat{\delta}_d(\omega)$ is known and $\hat{\delta}_u(\omega)$ is only on–line achievable.

Substituting (20) into (14) gives a residual of the form

$$r(s) = R(s)\hat{M}_u(s)(G_d(s)d(s) + \bar{G}_f(s)f(s)). \tag{23}$$

Thus, our problem is to find out a parametrization matrix $R(s)$ such that the generated residual is sensitive to the fault f and robust against d. Here we would like to emphasize that the column number of transfer function $G_d(s)$ may be much larger than its row number. That results in, in general,

$$\text{rank } G_d(s) = \dim(y) = m. \tag{24}$$

We would also like to point out that although the system is re–modelled so that the model uncertainties are summarized by a unknown input vector, a scheme for the residual generation and evaluation that is different from the existing methods will be developed in the next section.

3.3. A Remark on PFDI

If a perfect FDI is realizable, thresholds can then be set equal to zero or nearly zero and the FDI procedure becomes very simple. Unfortunately, in most practical cases this is impossible. To show this clearly, we give the following lemma.

Lemma 2 (Ding and Frank, 1991). *Let system (20) be given. Then, a PFDI is achievable iff*

$$\text{rank } [\bar{G}_f(s) \quad G_d(s)] = \text{rank } \bar{G}_f(s) + \text{rank } G_d(s), \tag{25}$$

$$\text{rank } \bar{G}_f(s) = q. \tag{26}$$

Lemma 2 can physically be interpreted as follows: a PFDI is achievable iff $\boldsymbol{f}$ and $\boldsymbol{d}$ have totally decoupled effects on the measurement $\boldsymbol{y}$. Clearly, if the system considered contains unstructured uncertainty, this is almost not achievable, since in this case rank $\boldsymbol{G}_d(s)$ becomes very large so that condition (25) cannot be satisfied. This demonstrates how difficult it may be to achieve a PFDI.

3.4. Evaluation Function and Threshold

Fault detection decisions are based on evaluating the characteristics of the residual vector $\boldsymbol{r}(s)$. Similarly to the so-called root mean square (rms) value, defined as:

$$J(\tau) = ((1/\tau) \int_0^{\tau} \boldsymbol{r}^{\mathrm{T}}(t)\boldsymbol{r}(t)\mathrm{d}t)^{1/2}, \tag{27}$$

Ding and Frank (1991) have proposed a frequency domain evaluation function defined as follows:

$$J(\phi) = ((2\pi\epsilon)^{-1} \int_{\omega_2}^{\omega_1} \boldsymbol{r}^*(j\omega)\boldsymbol{r}(j\omega)\mathrm{d}\omega)^{1/2} = \epsilon^{-1/2}\|\boldsymbol{r}(j\omega)\|_\epsilon, \quad \epsilon = \omega_2 - \omega_1, \tag{28}$$

where ϕ denotes the frequency range (ω_1, ω_2). In this index $r(j\omega)$ will be on-line calculated by using Fourier transformation and the frequency window ϕ will be determined by the designer (this will be dicussed in the next section). This evaluation function is adopted in this paper.

Thresholds can be established under residual measure (28). It is well known that a major requirement on the fault detection is to reduce or prevent false alarms. Thus, in the absence of any faults, $J(\phi)$ should be less than a threshold value, J_{th}, i.e.,

$$J_{th} = \sup_{\Delta y, f=0} J(\phi). \tag{29}$$

Setting $\boldsymbol{f}(s)$ in (23) equal to zero gives

$$J_{th} = \sup_d \Big((2\pi\epsilon)^{-1} \times$$

$$\times \int_{\omega_1}^{\omega_2} \boldsymbol{d}^*(j\omega)\boldsymbol{G}_d^*(j\omega)\hat{\boldsymbol{M}}_u^*(j\omega)\boldsymbol{R}^*(j\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\boldsymbol{d}(j\omega)d\omega\Big)^{1/2}$$

$$= \sup_d \|\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\boldsymbol{d}(j\omega)\|_\epsilon \epsilon^{-1/2}. \tag{30}$$

4. Main Results

This section presents the main results of the paper. Remember that our objective is to minimize the size of detectable faults. Since a FDI procedure consists of residual generation and evaluation, the size of detectable faults evidently depends on the construction of residual generator, the selection of evaluation function as well as thresholds. This, considering the results presented in the last section, concretely means that the size of detectable faults is in fact a function of the parametrization matrix $R(s)$ and the frequency window $\phi = (\omega_1, \omega_2)$. If we are able to find out the relationship between the minimum size of the detectable faults and the parametrization matrix as well as the frequency window and to present it in an appropriate mathematical form, our problem then reduces to an optimization problem for which we may get an optimal or at least a suboptimal solution. This is the basic idea of the following study.

4.1. Minimum Detectable Fault

It is physically clear that a fault is detectable if it makes the evaluation function larger than threshold, i.e.,

$$J_f(\phi) > J_{th} \tag{31}$$

where $J_f(\phi)$ denotes the evaluation function corresponding to some f. Notice, however, that a fault may, due to the model uncertainties, have different effects on the residual and furthermore on evaluation function. Taking this into account, we define a fault as detectable if

$$\inf_{\Delta y} J_f(\phi) > J_{th}. \tag{32}$$

Denote the set of all detectable faults by Ω_f, i.e.,

$$\Omega_f := \left\{ f : \inf_{\Delta y} J_f(\phi) > J_{th} \right\}, \tag{33}$$

then our first problem to be solved is to find out $f_{\min}$ satisfying

$$\|f_{\min}\|_\epsilon = \inf\left\{ \|f\|_\epsilon : \ f \in \Omega_f \right\} \tag{34}$$

with respect to the parametrization matrix $R(s)$ and the frequency window ϕ. This is given in the following theorem.

Theorem 1. *Given a residual generator (14) and residual evaluation function (28), then we have*

$$\|\boldsymbol{f}_{\min}\|_\epsilon = 2J_{th}\epsilon^{1/2}/k(\phi), \tag{35}$$

where the threshold is defined by (30) and

$$k(\phi) = \min_{\omega\in\phi}\Big(\underline{\sigma}^2\big(R(j\omega)\hat{M}_u(j\omega)G_f(j\omega)\big) - \bar{\sigma}^2\big(R(j\omega)\hat{M}_u(j\omega)\big)\delta_f^2(\omega)\Big)^{1/2}. \tag{36}$$

Remarks.

- Theorem 1 presents a result that can be taken as a frequnecy domain extension of the time domain results given by Emami–Naeini *et al.* (1988).
- Somewhat different from the work in (Emani–Naeini *et al.*, 1988), here the effects of the model uncertainties on $G_f(s)$ have been taken into account; there is also no special restriction on the faults.

Proof. Following (23) and (28) $J_f(\phi)$ may be re–written as

$$J_f(\phi) = \|R(s)\hat{M}_u(s)G_d(s)d(s) + R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon \epsilon^{-1/2}. \tag{37}$$

Observe inequality

$$\begin{aligned} &\|R(s)\hat{M}_u(s)G_d(s)d(s) + R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon \\ &\geq \|R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon - \|R(s)\hat{M}_u(s)G_d(s)d(s)\|_\epsilon \end{aligned} \tag{38}$$

for some $f \in \Omega_f$. According to the defintion of a detectable fault we have

$$\|R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon - \|R(s)\hat{M}_u(s)G_d(s)d(s)\|_\epsilon > 0 \tag{39}$$

so the equality in (38) holds for some $d(s), \bar{G}_f(s)$. This yields

$$\begin{aligned} \inf_{\Delta y} J_f(\phi) &= \inf_{d(s),\Delta G_f(s)} \big(\|R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon - \|R(s)\hat{M}_u(s)G_d(s)d(s)\|_\epsilon\big)\epsilon^{-1/2} \\ &= \big(\inf_{\Delta G_f(s)} \|R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon - \sup_{d(s)} \|R(s)\hat{M}_u(s)G_d(s)d(s)\|_\epsilon\big)\epsilon^{-1/2} \\ &= \inf_{\Delta G_f(s)} \|R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon \epsilon^{-1/2} - J_{th}. \end{aligned} \tag{40}$$

It can then be concluded that for a detectable fault the relation

$$\inf_{\Delta G_f(s)} \|R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon > 2J_{th}\epsilon^{1/2} \tag{41}$$

holds. Now, we observe the left side of (41), which can be re–written as

$$\begin{aligned} \|R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon &= \Big((2\pi)^{-1} \int_{\omega_1}^{\omega_2} f^*(j\omega)\big(G_f^*(j\omega) \\ &+\Delta G_f^*(j\omega)\big)\hat{M}_u^*(j\omega)R^*(j\omega)R(j\omega)\hat{M}_u(j\omega) \times \\ &\times \big(G_f(j\omega) + \Delta G_f(j\omega)\big)f(j\omega)\mathrm{d}\omega\Big)^{1/2}. \end{aligned} \tag{42}$$

Note that

$$\left(G_f^*(j\omega) + \Delta G_f^*(j\omega)\right)\hat{M}_u^*(j\omega)R^*(j\omega)R(j\omega)\hat{M}_u(j\omega)\left(G_f(j\omega) + \Delta G_f(j\omega)\right)$$
$$\geq G_f^*(j\omega)\hat{M}_u^*(j\omega)R^*(j\omega)R(j\omega)\hat{M}_u(j\omega)G_f(j\omega)$$
$$-\Delta G_f^*(j\omega)\hat{M}_u^*(j\omega)R^*(j\omega)R(j\omega)\hat{M}_u(j\omega)\Delta G_f(j\omega). \tag{43}$$

Hence, using the well known relation

$$\underline{\sigma}(A)\|b\|_2 \leq \|Ab\|_2 \leq \bar{\sigma}(A)\|b\|_2, \tag{44}$$

we finally have

$$\inf_{\Delta G_f(s), f(s)} \|R(s)\hat{M}_u(s)\bar{G}_f(s)f(s)\|_\epsilon$$

$$= \inf_{\Delta G_f, f(s)} \Big((2\pi)^{-1} \int_{\omega_1}^{\omega_2} f^*(j\omega)\big(G_f^*(j\omega)\hat{M}_u^*(j\omega)R^*(j\omega)R(j\omega)\hat{M}_u(j\omega)G_f(j\omega)$$
$$-\Delta G_f^*(j\omega)\hat{M}_u^*(j\omega)R^*(j\omega)R(j\omega)\hat{M}_u(j\omega)\Delta G_f(j\omega)\big) f(j\omega)d\omega\Big)^{1/2} \tag{45}$$
$$= \min_{\omega\in\phi}\Big(\underline{\sigma}^2\big(R(j\omega)\hat{M}_u(j\omega)G_f(j\omega)\big) - \bar{\sigma}^2\big(R(j\omega)\hat{M}_u(j\omega)\big)\delta_f^2(j\omega)\Big)^{1/2}\|f_{\min}\|_\epsilon. \tag{46}$$

Summarizing (37), (41) and (46) leads to (35). This completes the proof.
Notice that $d(j\omega)$ can be written as

$$d(j\omega) = \bar{d}(j\omega)\delta_d(\omega) \tag{47}$$

with

$$\|\bar{d}(j\omega)\|_2 \leq 1.$$

This leads to

$$J_{th} = \sup_d\Big((2\pi\epsilon)^{-1} \times$$
$$\times \int_{\omega_1}^{\omega_2} d^*(j\omega)G_d^*(j\omega)\hat{M}_u^*(j\omega)R^*(j\omega)R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)d(j\omega)\mathrm{d}\omega\Big)^{1/2}$$
$$= \sup_{\bar{d}}\Big((2\pi\epsilon)^{-1} \times$$
$$\times \int_{\omega_1}^{\omega_2} \bar{d}^*(j\omega)G_d^*(j\omega)\hat{M}_u^*(j\omega)R^*(j\omega)\delta_d^2(\omega)R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)\bar{d}(j\omega)\mathrm{d}\omega\Big)^{1/2}$$
$$= \max_{\omega\in\phi}\bar{\sigma}\big(\delta_d(\omega)R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)\big)\epsilon^{-1/2}. \tag{48}$$

With this equation the following corollary is obvious.

Corollary 2. *Given a residual generator (14) and residual evaluation function (28), then we have*

$$\|\boldsymbol{f}_{\min}\|_\epsilon = 2\max_{\omega\in\phi}\bar{\sigma}\big(\delta_d(\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\big)/k(\epsilon). \tag{49}$$

Remark. We would like to emphasize that $\delta_d(\omega)$ depends on $\|u(j\omega)\|_2$ which is often achievable only under on–line operating conditions. This requires an *on–line calculation* for J_{th} according to (48) which may cause some troubles due to the somewhat involved computation for the maximum singular value. For this reason we suggest that instead of equation (48) we use the following formula for the threshold calculation:

$$J_{th} = \max_{\omega\in\phi}\bar{\sigma}\big(\hat{\delta}_d(\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\big)\Big((2\pi\epsilon)^{-1}\int_{\omega_1}^{\omega_2}\hat{\delta}_u^2(\omega)\mathrm{d}\omega\Big)^{1/2} \tag{50}$$

where $\delta_d(\omega)$ is assumed to be of the form (see Section 3.2)

$$\delta_d(\omega) = \hat{\delta}_d(\omega)\hat{\delta}_u(\omega).$$

Since $\hat{\delta}_d(\omega)$ is known, the on–line calculation needed is only the integration of $\hat{\delta}_u^2(\omega)$.

4.2. Minimization of Minimum Detectable Fault

Theorem 1 reveals the dependence of minimun detectable fault on the parametrization matrix $\boldsymbol{R}(s)$ and frequency window (ω_1, ω_2). With the aid of this relationship we now study how to minimize the minimum detectable fault by suitably choosing $\boldsymbol{R}(s)$ and frequency window.

It follows from Theorem 1 and Corollary 2 that the minimization of minimum detectable fault is equivalent to solving the following optimization problem

$$\inf_{R(s),\phi}\|f_{\min}\|_\epsilon$$

$$= 2\inf_{R(s),\phi}\frac{\max_{\omega\in\phi}\bar{\sigma}\big(\delta_d(\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\big)}{\min_{\omega\in\phi}\Big(\underline{\sigma}^2\big(\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_f(j\omega)\big)-\bar{\sigma}^2\big(\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\big)\delta_f^2(\omega)\Big)^{\frac{1}{2}}}. \tag{51}$$

Evidently, this minimum problem can be re–formulated as the following maximum problem

$$\sup_{R(s),\phi}\frac{\min_{\omega\in\phi}\Big(\underline{\sigma}^2\big(\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_f(j\omega)\big)-\bar{\sigma}^2\big(\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\big)\delta_f^2(\omega)\Big)^{\frac{1}{2}}}{\max_{\omega\in\phi}\bar{\sigma}\big(\delta_d(\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\big)}. \tag{52}$$

In the following, for the sake of simplicity, we will study the optimization problem (52).

It is worth pointing out that $f_{\min}$ may also be zero, which, as mentioned in the last sections, means a PFDI. In this paper, we do not consider this ideal case. For this reason, the following assumptions are made:

- **Assumption 1**: Condition (25) is not satisfied.
- **Assumption 2**: $\bar{G}_f(s)$ has as its zeros all the zeros on the $j\omega$–axis of $G_d(s)$ together with their structure in the sense that for

  $$\alpha G_d(z) = 0,$$

  condition

  $$\alpha \bar{G}_f(z) = 0$$

  also holds.

Ding and Frank (1991) have demonstrated physical meaning of these two assumptions, which is now briefly mentioned below.

It is clear that if condition (25) holds, then there exists $R(s)$ so that

$$\underline{\sigma}(R(j\omega)\hat{M}_u(j\omega)\bar{G}_f(j\omega)) \neq 0 \quad \text{for some } \omega \tag{53}$$

$$\bar{\sigma}(R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)) = 0 \quad \text{for all } \omega. \tag{54}$$

This obviously leads to

$$f_{\min} = 0.$$

Assume that $G_d(s)$ has a zero and $\bar{G}_f(s)$ has no zero at $j\omega_o$. Then, there exists $R(s)$ so that

$$\underline{\sigma}(R(j\omega)\hat{M}_u(j\omega)\bar{G}_f(j\omega)) \neq 0 \quad \text{for some } \omega \in (\omega_1, \omega_2) \tag{55}$$

$$\bar{\sigma}(R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)) \simeq 0 \quad \text{for all } \omega \in (\omega_1, \omega_2), \tag{56}$$

where the frequency interval (ω_1, ω_2) is defined as a small neighbourhood of ω_o. This shows that if Assumption 2 does not hold, then a near PFDI is achievable.

In the following, we will study the solution of the optimization problem (52). We process this in two steps. We first discuss how to achieve a minimization of minimum detectable fault for a given frequency range $\phi = (\omega_1, \omega_2)$ under the assumption that $\delta_d(\omega)$ is known. The results will then be extended to the cases where the frquency range ϕ is arbitarily selectable and $\delta_d(\omega)$ is only partly known.

4.2.1. Case 1: Frequency Range is Given

If the frequency range is given, problem (52) reduces to

$$\sup_{R(s)} \frac{\min_{\omega\in\phi}\left(\underline{\sigma}^2\big(R(j\omega)\hat{M}_u(j\omega)G_f(j\omega)\big) - \bar{\sigma}^2\big(R(j\omega)\hat{M}_u(j\omega)\big)\delta_f^2(\omega)\right)^{\frac{1}{2}}}{\max_{\omega\in\phi}\bar{\sigma}\big(\delta_d(\omega)R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)\big)}. \tag{57}$$

In (Ding and Frank, 1991) a systematic procedure for dealing with such kind of optimization problem has been proposed, and is adopted here.

It is straightforward to demonstrate that the optimization problem (57) is equivalent to:

$$\sup_{R(s)} \min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(R(j\omega) \hat{M}_u(j\omega) G_f(j\omega) \big) - \bar{\sigma}^2 \big(R(j\omega) \hat{M}_u(j\omega) \big) \delta_f^2(\omega) \Big)^{1/2} \tag{58}$$

subject to

$$\max_{\omega \in \phi} \bar{\sigma} \big(\delta_d(\omega) R(j\omega) \hat{M}_u(j\omega) \bar{G}_d(j\omega) \big) \leq 1. \tag{59}$$

Denote $\hat{M}_u(s) G_d(s)$ by $\hat{G}_d(s)$. Performing the so–called extended co–inner–outer factorization (ECIOF) for $\hat{G}_d(s)$, which is introduced by Ding and Frank (1991), gives

$$\hat{G}_d(s) = G_{do}(s) G_{dz}(s) G_{di}(s), \tag{60}$$

with $G_{do}(s)$ as co–outer, $G_{di}(s)$ as co–inner and $G_{dz}(s)$ having as its zeros all the zeros on the imaginary axis of $\hat{G}_d(s)$. Choosing

$$R(j\omega) = Q(j\omega) G_{do}^{-}(j\omega) \tag{61}$$

with $G_{do}^{-}(s)$ as the inverse of $G_{do}(s)$ yields

$$\bar{\sigma} \big(R(j\omega) \hat{G}_d(j\omega) \big) = \bar{\sigma} \big(Q(j\omega) G_{dz}(j\omega) \big). \tag{62}$$

Taking into account Assumption 2, $\hat{M}_u(s) G_f(s)$ can be factorized as (Ding and Frank, 1991)

$$\hat{M}_u(s) G_f(s) = G_{do}(s) G_{dz}(s) G_{f1}(s) \quad \text{for some} \quad G_{f1}(s) \in \mathcal{RH}_\infty. \tag{63}$$

We write, for the sake of simplicity, $\hat{M}_u(s) \Delta G_f(s)$ as follows

$$\hat{M}_u(s) \Delta G_f(s) = G_{do}(s) G_{dz}(s) \Delta(s) \tag{64}$$

for some unknown transfer function $\Delta(s)$. In view of the restriction on the size of $\Delta G_f(s)$, it follows

$$\bar{\sigma} \big(\Delta(j\omega) \big) \leq \bar{\sigma} \big(\hat{M}_u(j\omega) \big) \delta_f(\omega) / \bar{\sigma} \big(G_{do}(j\omega) G_{dz}(j\omega) \big) := \delta(\omega). \tag{65}$$

The optimal problem (58) thus reduces to

$$\sup_{Q(s)} \min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(Q(j\omega) G_{dz}(j\omega) G_{f1}(j\omega) \big) - \bar{\sigma}^2 \big(Q(j\omega) G_{dz}(j\omega) \big) \delta^2(\omega) \Big)^{1/2} \tag{66}$$

subject to

$$\max_{\omega \in \phi} \bar{\sigma} \big(\delta_d(\omega) Q(j\omega) G_{dz}(j\omega) \big) \leq 1. \tag{67}$$

With the aid of the following lemma the above optimization problem can be solved.

Lemma 3. (Ding and Frank, 1991) *Let* $G(s)(k \times k) \in \mathcal{RH}_\infty$ *with rank* $G(s) = k$ *be given, whose zeros lie on the* $j\omega$*-axis. Then there exists* $Q(s) \in \mathcal{RH}_\infty$ *so that*

$$G^*(j\omega)Q^*(j\omega)Q(j\omega)G(j\omega) \leq I, \quad \text{for all} \quad \omega \tag{68}$$

$$G^*(j\omega_0)Q^*(j\omega_0)Q(j\omega_0)G(j\omega_0) = I \quad \text{for some} \quad \omega_0 \in [0, \infty]. \tag{69}$$

We now state the solution to our optimization problem.

Theorem 3. *The optimum of (66) is given by*

$$\sup_{Q(s)} \min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(Q(j\omega)G_{dz}(j\omega)G_{f1}(j\omega)\big) - \bar{\sigma}^2 \big(Q(j\omega)G_{dz}(j\omega)\big)\delta^2(\omega)\Big)^{\frac{1}{2}}$$

$$= \min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(\delta_d^{-1}(\omega)G_{f1}(j\omega)\big)$$

$$- \big(\delta_d^{-1}(\omega)\bar{\sigma}\big(\hat{M}_u(j\omega)\big)\delta_f(\omega)/\bar{\sigma}\big(G_{do}(j\omega)G_{dz}(j\omega)\big)\big)^2\Big)^{\frac{1}{2}} \tag{70}$$

Proof. First, we show that for all $Q(s)G_{dz}(s)$ satifying (67) the following inequality holds:

$$\min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(Q(j\omega)G_{dz}(j\omega)G_{f1}(j\omega)\big) - \bar{\sigma}^2 \big(Q(j\omega)G_{dz}(j\omega)\big)\delta^2(\omega)\Big)^{1/2}$$

$$\leq \min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(\delta_d^{-1}(\omega)G_{f1}(j\omega)\big) - \delta_d^{-2}(\omega)\delta^2(\omega)\Big)^{1/2}. \tag{71}$$

If this is not the case, then there exists some $Q(s)$ such that

$$\min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(Q(j\omega)G_{dz}(j\omega)G_{f1}(j\omega)\big) - \bar{\sigma}^2 \big(Q(j\omega)G_{dz}(j\omega)\big)\delta^2(\omega)\Big)^{1/2}$$

$$= \min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(Q(j\omega)G_{dz}(j\omega)\delta_d(\omega)\delta_d^{-1}(\omega)G_{f1}(j\omega)\big)$$

$$-\bar{\sigma}^2 \big(Q(j\omega)G_{dz}(j\omega)\delta_d(\omega)\delta_d^{-1}(\omega)\big)\delta^2(\omega)\Big)^{1/2}$$

$$> \min_{\omega \in \phi} \Big(\underline{\sigma}^2 \big(\delta_d^{-1}(\omega)G_{f1}(j\omega)\big) - \delta_d^{-2}(\omega)\delta^2(\omega)\Big)^{1/2}. \tag{72}$$

This leads, by using the relations

$$\sigma(kA) = k^{1/2}\sigma(A), \ \underline{\sigma}(AB) \leq \underline{\sigma}(A)\underline{\sigma}(B) \tag{73}$$

with a constant k, to

$$\min_{\omega\in\phi}\Big(\underline{\sigma}^2\big(Q(j\omega)G_{dz}(j\omega)\delta_d(\omega)\big)\big(\underline{\sigma}^2\big(\delta_d^{-1}(\omega)G_{f1}(j\omega)\big)-\delta_d^{-2}(\omega)\delta^2(\omega)\big)\Big)^{1/2}$$

$$> \min_{\omega\in\phi}\Big(\underline{\sigma}^2\big(\delta_d^{-1}(\omega)G_{f1}(j\omega)\big)-\delta_d^{-2}(\omega)\delta^2(\omega)\Big)^{1/2}. \tag{74}$$

Since

$$\underline{\sigma}^2\big(Q(j\omega)G_{dz}(j\omega)\delta_d(\omega)\big)\leq 1, \tag{75}$$

(74) does not hold. This certifies, by contrast, that (71) is true.
Assume now at frequency $\omega_o \in \phi$

$$\min_{\omega\in\phi}\Big(\underline{\sigma}^2\big(\delta_d^{-1}(\omega)G_{f1}(j\omega)\big)-\delta_d^{-2}(\omega)\delta^2(\omega)\Big)^{1/2}$$

$$= \Big(\underline{\sigma}^2\big(\delta_d^{-1}(\omega_o)G_{f1}(j\omega_o)\big)-\delta_d^{-2}(\omega_o)\delta^2(\omega_o)\Big)^{1/2}. \tag{76}$$

According to (71) we have

$$\sup_{Q(s)}\min_{\omega\in\phi}\Big(\underline{\sigma}^2\big(Q(j\omega)G_{dz}(j\omega)G_{f1}(j\omega)\big)-\bar{\sigma}^2\big(Q(j\omega)G_{dz}(j\omega)\big)\delta^2(\omega)\Big)^{1/2}$$

$$\leq \Big(\underline{\sigma}^2\big(\delta_d^{-1}(\omega_o)G_{f1}(j\omega_o)\big)-\delta_d^{-2}(\omega_o)\delta^2(\omega_o)\Big)^{1/2}. \tag{77}$$

Using Spectral Factorization Theory (Aström, 1970) we are able to factorize $\delta_d(\omega)$ as

$$\delta_d^2(\omega) = \bar{\delta}_d^*(j\omega)\bar{\delta}_d(j\omega) \tag{78}$$

with $\bar{\delta}_d(s)$, $\bar{\delta}_d^{-1}(s) \in \mathcal{RH}_\infty$. Let

$$Q(j\omega) = \bar{\delta}_d^{-1}(j\omega)\bar{Q}(j\omega) \tag{79}$$

and choose $\bar{Q}(j\omega)$ satisfying

$$\bar{\sigma}(\bar{Q}(j\omega)G_{dz}(j\omega)) \leq 1 \quad \text{for all} \quad \omega\in\phi \tag{80}$$

$$\underline{\sigma}(\bar{Q}(j\omega_o)G_{dz}(j\omega_o)) = \bar{\sigma}(\bar{Q}(j\omega_o)G_{dz}(j\omega_o)) = 1, \tag{81}$$

which, according to Lemma 3, does exist. Furthermore, we notice the relation

$$\sigma\big(\bar{\delta}(j\omega)G(j\omega)\big) = \sigma\big(\delta(\omega)G(j\omega)\big) \tag{82}$$

with $\bar{\delta}^*(j\omega)\bar{\delta}(j\omega) = \delta^2(\omega)$, then it can be, on the other hand, concluded that there exists $Q(s)$ such that

$$\sup_{Q(s)} \min_{\omega\in\phi}\left(\underline{\sigma}^2\left(Q(j\omega)G_{dz}(j\omega)G_{f1}(j\omega)\right) - \bar{\sigma}^2\left(Q(j\omega)G_{dz}(j\omega)\right)^2\delta^2(\omega)\right)^{1/2}$$

$$= \min_{\omega\in\phi}\left(\underline{\sigma}^2\left(\delta_d^{-1}(j\omega)G_{f1}(\omega)\right) - \left(\delta_d^{-1}(\omega)\bar{\sigma}(\hat{M}_u(j\omega))\delta_f(\omega)/\bar{\sigma}(G_{do}(j\omega)G_{dz}(j\omega))\right)^2\right)^{1/2}$$

for which the constraint

$$\sup_{\omega\in\phi} \bar{\sigma}\left(Q(j\omega)G_{dz}(j\omega)\delta_d(\omega)\right) \leq 1 \tag{83}$$

is also satisfied. This proves the theorem.

With the aid of Theorem 3 it is staightforward to find out the solution to the original problem (51) with ϕ being given. It is summarized in the following corollary.

Corollary 4. *Let system (20), residual generator of the form (14) and evaluation function (28) be given. Then*

$$\inf_{R(s)} \|f_{\min}\|_\epsilon = 2\max_{\omega\in\phi}\left(\underline{\sigma}^2\left(\delta_d^{-1}(\omega)G_{f1}(j\omega)\right) - \delta_d^{-2}(\omega)\delta_f^2(\omega)\right)^{-1/2}, \tag{84}$$

and the optimal parametrization matrix $R(s)$ is determined by

$$R(s) = Q(s)G_{do}^-(s) = \bar{\delta}_d^{-1}(s)\bar{Q}(s)G_{do}^-(s) \tag{85}$$

with $\bar{Q}(s)$ satisfying

$$\bar{\sigma}\left(\bar{Q}(j\omega)G_{dz}(j\omega)\right) \leq 1 \quad \text{for all} \quad \omega\in\phi$$

$$G_{dz}^*(j\omega_o)\bar{Q}^*(j\omega_o)\bar{Q}(j\omega_o)G_{dz}(j\omega_o) = I,$$

where $G_{f1}(s), G_{do}, G_{dz}$ and ω_o have the meaning stated by (60), (63), (76), respectively.

4.2.2. Case 2: Frequency Range is Selectable

In this sub–section we study the solution of optimization problem (51) or equivalently (52). The results obtained in the last sub–section is the basis of the research. We assume that the lenth of the frequency range $\epsilon = \omega_2 - \omega_1$ is constant and may be suitably small.

Observe equation (84). It can be seen that for a given frequency range the the minimal value of $\|f_{\min}\|_\epsilon$ given by the right side of (84) is independent of $R(s)$. This ensures

$$\inf_{R(s),\phi\in[0,\infty]} \|f_{\min}\|_\epsilon = 2\inf_{\phi\in[0,\infty]}\max_{\omega\in\phi}\left(\underline{\sigma}^2\left(\delta_d^{-1}(\omega)G_{f1}(j\omega)\right) - \delta_d^{-2}(\omega)\delta_f^2(\omega)\right)^{-1/2}. \tag{86}$$

This suggests the following way to compute $\inf \|f_{\min}\|_\epsilon$: select a ϕ; calculate $\min_{\omega\in\phi}\left(\underline{\sigma}^2(\delta_d^{-1}(\omega)G_{f1}(j\omega)) - \delta_d^{-2}(\omega)\delta_f^2(\omega)\right)^{-1/2}$; store the smaller one by comparing with the previour value; repeat. In fact, (86) is a nonlinear optimization problem which can be solved by a search over ω using the convetional nonlinear optimization techniques (see e.g. Luenberger, 1969).

A simple way to achieve a sub–optimum to the optimization problem (86) is to determine the frequency ω_o, at which $\min_{\omega\in[0,\infty]}\left(\underline{\sigma}^2(\delta_d^{-1}(\omega)G_{f1}(j\omega)) - \delta_d^{-2}(\omega)\delta_f^2(\omega)\right)^{-1/2}$ reaches its minimum, i.e.,

$$\min_{\omega\in[0,\infty]}\left(\underline{\sigma}^2(\delta_d^{-1}(\omega)G_{f1}(j\omega)) - \delta_d^{-2}(\omega)\delta_f^2(\omega)\right)^{-1/2}$$

$$= \left(\underline{\sigma}^2(\delta_d^{-1}(\omega_o)G_{f1}(j\omega_o)) - \delta_d^{-2}(\omega_o)\delta_f^2(\omega_o)\right)^{-1/2}, \tag{87}$$

and then let

$$\phi = (\omega_o - \epsilon/2,\ \omega_o + \epsilon/2). \tag{88}$$

It is obvious that in this case

$$\inf_{\phi\in[0,\infty]}\max_{\omega\in\phi}\left(\underline{\sigma}^2(\delta_d^{-1}(\omega)G_{f1}(j\omega)) - \delta_d^{-2}(\omega)\delta_f^2(\omega)\right)^{-1/2}$$

$$\rightarrow \left(\underline{\sigma}^2(\delta_d^{-1}(\omega_o)G_{f1}(j\omega_o)) - \delta_d^{-2}(\omega_o)\delta_f^2(\omega_o)\right)^{-1/2} \tag{89}$$

corresponds to $\epsilon \rightarrow 0$.

4.2.3. Case 3: Input Signal is Only On–Line Achievable

As mentioned above, there exists such a situation that the input signal $u(s)$ becomes known only under process operating conditions. In this case, $\delta_d(\omega)$, written as $\hat{\delta}_d(\omega)\hat{\delta}_u(\omega)$, is partly known before the observed process comes into operation. Taking into account on–line realizability, it was suggested in Section 4.1 that instead of (48) expression (50) may be used for the threshold determination. According to Theorem 1, minimizing detectable faults therefore reduces to solving

$$\inf_{R(s),\phi}\frac{\max_{\omega\in\phi}\bar{\sigma}\left(\hat{\delta}_d(\omega)R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)\right)\left((2\pi\epsilon)^{-1}\int_{\omega_1}^{\omega_2}\hat{\delta}_u^2(\omega)d\omega\right)^{\frac{1}{2}}}{\min_{\omega\in\phi}\left(\underline{\sigma}^2(R(j\omega)\hat{M}_u(j\omega)G_f(j\omega)) - \bar{\sigma}^2(R(j\omega)\hat{M}_u(j\omega))\delta_f^2(\omega)\right)^{\frac{1}{2}}}. \tag{90}$$

Notice that the term $\left((2\pi\epsilon)^{-1}\int_{\omega_1}^{\omega_2}\hat{\delta}_u^2(\omega)d\omega\right)^{1/2}$ is independent of $\boldsymbol{R}(s)$, so the optimization problem (90) is equivalent to

$$\inf_{\boldsymbol{R}(s),\phi}\frac{\max_{\omega\in\phi}\bar{\sigma}\left(\hat{\delta}_d(\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\right)}{\min_{\omega\in\phi}\left(\underline{\sigma}^2\left(\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_f(j\omega)\right)-\bar{\sigma}^2\left(\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\right)\delta_f^2(\omega)\right)^{\frac{1}{2}}} \tag{91}$$

which has a similar form to (51) and can be solved *off-line* using the results obtained above.

By now, we have arrived at our aim of minimizing the detectable faults. The results obtained will be interpreted in the next section and summarized in the form of algorithms to realize FDI.

5. Discussions

This section consists of discussions of several issues concerned with the use of the results achieved in the last section and their interpretation.

5.1. Threshold

In Section 3.4 a general expression for a threhold is given, which is re-formulated in Section 4.1. Bearing in mind that these expressions were derived based on system model (20) and hold for all parametrization matrix $\boldsymbol{R}(s)$, the question may arise: how to determine a threshold if the original system model (4) is considered and the parametrization matrix $\boldsymbol{R}(s)$ is optimal in the sense of (51). We now answer this question. Without loss of generality we only study the case of fault detection.

We begin by observing (48) as well as (50) that are re-written here:

$$J_{th}=\max_{\omega\in\phi}\bar{\sigma}\left(\delta_d(\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\right)\epsilon^{-1/2}\quad\text{or}$$

$$J_{th}=\max_{\omega\in\phi}\bar{\sigma}\left(\hat{\delta}_d(\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\right)\left((2\pi\epsilon)^{-1}\int_{\omega_1}^{\omega_2}\hat{\delta}_u^2(\omega)\mathrm{d}\omega\right)^{1/2}.$$

We first assume that $\boldsymbol{u}(j\omega)\in\mathcal{RL}_\infty$ and is given before the process comes into operation. By proving Theorem 3 it has been shown that

$$\max_{\omega\in\phi}\bar{\sigma}\left(\delta_d(\omega)\boldsymbol{R}(j\omega)\hat{\boldsymbol{M}}_u(j\omega)\boldsymbol{G}_d(j\omega)\right)=1 \tag{92}$$

if $\boldsymbol{R}(s)$ is optimally chosen in the sense of (51). This means, somewhat surprising, that the threshold is a constant equal to $\epsilon^{-1/2}$. As a matter of fact, one may expect a J_{th} changing with the input signal. However, if we observe $\delta_d(\omega)$ in detail, the reason becomes evident. As shown in Section 3.2,

$$d(s) = \begin{bmatrix} \Delta G_u(s)u(s) \\ w(s) \end{bmatrix},$$

so we have

$$\|d(j\omega)\|_2^2 \le \delta_w^2(\omega) + \delta_f^2(\omega)\|u(j\omega)\|_2^2 := \delta_d^2(\omega). \tag{93}$$

This shows that the information on the input signal is included in $\delta_d(\omega)$ which is further processed by the residual generator design. This is no more the case if $u(j\omega)$ is unknown before the process is in operation. To establish a threshold according to (50) we have first to divide $\delta_d(\omega)$ into two parts. Since

$$\delta_w^2(\omega) + \delta_f^2(\omega)\|u(j\omega)\|_2^2 = \delta_w^2(\omega)(1 + \delta_f^2(\omega)\|u(j\omega)\|_2^2/\delta_w^2(\omega)) \tag{94}$$

$$= \delta_f^2(\omega)(\delta_w^2(\omega)/\delta_f^2(\omega) + \|u(j\omega)\|_2^2), \tag{95}$$

$\hat{\delta}_d(\omega)$ may be difined as

$$\hat{\delta}_d(\omega) = \delta_f(\omega) \quad \text{or} \quad \hat{\delta}_d(\omega) = \delta_w(\omega) \tag{96}$$

and $\hat{\delta}_u(\omega)$ as

$$\hat{\delta}_u(\omega) = (1 + \delta_f^2(\omega)\|u(j\omega)\|_2^2/\delta_w^2(\omega))^{1/2}$$

or

$$\hat{\delta}_u(\omega) = (\delta_w^2(\omega)/\delta_f^2(\omega) + \|u(j\omega)\|_2^2)^{1/2}. \tag{97}$$

Note that this factorization is not unique. Remembering the discussion in Section 4.2.3, we finally have

$$J_{th} = \left((2\pi\epsilon)^{-1} \int_{\omega_1}^{\omega_2} \hat{\delta}_u^2(\omega)\mathrm{d}\omega\right)^{1/2} \tag{98}$$

which can be on–line calculated so far $u(j\omega)$ is known.

Due to its adaptabilty to input signals the threshold derived above may be called adaptive threshold (Frank, 1991). Emami–Naeini *et al.* (1988) have first systematically studied this problem in the time domain, followed by the work (Ding and Frank, 1991) in the frequency domain. Comparing the results one may notice that the threshold introduced here is presented in a very simple form and can also be easily calculated. Especially, in contrast to the time domain case, there exists a suitable algorithm to determine the frequency window as given above.

5.2. Two Schemes to FDI

Remember that we have studied the problem of minimizing detectable faults in two steps corresponding to two cases. The first one is the frequency window being given, and the other one the frequency window being selectable. Both of these two cases may be met in practice and therefore are of interest for engineers. In the following we summarize the main results of the last section into two schemes and present them in the form of algorithms.

Case 1: The Frequency Window is Given

This case is especially of interest if one is sure that the fault f is dominant over some frequency range (ω_0, ω_k). In this case, one may divide (ω_0, ω_k) into k sub–ranges $(\omega_i, \omega_{i+1}), i = 0, \cdots, k-1$,. Over each of the sub–ranges a residual generator will be designed by solving the optimization problem (57). This will be realized by using the following algorithm.

Algorithm to Residual Generator Design

- formulate process model into the form (20);
- calculate left coprime factorization of $G_u(s)$ for $\hat{M}_u(s), \hat{N}_u(s)$ using the standard algorithms (Francis, 1987);
- do an ECIOF for $\hat{G}_d(s) = \hat{M}_u(s)G_d(s)$, for which Ding and Frank (1991) have proposed an effective algorithm;
- do factorization for $\hat{G}_f(s) = \hat{M}_u(s)G_f(s)$ to achieve $G_{f1}(s)$ (see also Ding and Frank, 1991);
- determine frequency ω_o satisfying

$$\min_{\omega \in \phi}\Big(\underline{\sigma}^2\big(\delta_d^{-1}(\omega)G_{f1}(j\omega)\big) - \delta_d^{-2}(\omega)\delta_f^2(\omega)\Big) = \underline{\sigma}^2\big(\delta_d^{-1}(\omega_o)G_{f1}(j\omega_o)\big)$$

$$-\delta_d^{-2}(\omega_o)\delta_f^2(\omega_o);$$

- the optimal parametrization matrix $R(s)$ reads:

$$R(s) = Q(s)G_{do}^{-}(s) = \bar{\delta}_d^{-1}(s)\bar{Q}(s)G_{do}^{-}(s)$$

  with $\bar{Q}(s)$ satisfying

$$\bar{\sigma}\big(\bar{Q}(j\omega)G_{dz}(j\omega)\big) \le 1 \quad \text{for all} \quad \omega \in \phi$$

$$G_{dz}^*(j\omega_o)\bar{Q}^*(j\omega_o)\bar{Q}(j\omega_o)G_{dz}(j\omega_o) = I,$$

  where $G_{f1}(s), G_{do}, G_{dz}$ and ω_o have the meanings stated by (60), (63), (76), respectively;

- the residual generator is finally given by

$$r(s) = R(s)\big(\hat{M}_u(s)y(s) - \hat{N}_u(s)u(s)\big).$$

Using the obtained residual generators one may now realize FDI for the process considered. For each frequency interval a residual generator will be used and the corresponding *on–line operations* are summarized as follows:

On–Line Operations for Residual Generation and Evaluation

- calculate $r(t)$ according to

$$r(s) = R(s)\big(\hat{M}_u(s)y(s) - \hat{N}_u(s)u(s)\big);$$

- transform $r(t)$ into the frequency domain for $r(j\omega)$ using Fourier transformation;
- calculate $J(\phi)$ according to (28);
- calculate the threshold J_{th} if necessary;
- compare $J(\phi)$ with J_{th}, and if $J(\phi) > J_{th}$ then make alarm, otherwise repeat the procedure.

Case 2: The Frequency Window is Selectable

If one has no information on the frequency ranges over which the fault may present, one is forced to choose a or a series of frequency windows for the residual evaluation. In this case, an additional step has to be carried out in the above algorithm, namely, a nonlinear optimization problem should be solved for the frequency window according to (86). The on–line operations remain unchanged.

5.3. Interpretation and Comparison

We first study the physical meaning of expression (49) for the minimum size of detectable fault. It can be seen that $\|f_{\min}\|_\epsilon$ is determined by $\bar{\sigma}\big(\delta_d(\omega)R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)\big)$ as well as $k(\epsilon)$. Re–write $k(\epsilon)$ as

$$k(\epsilon) = \underline{\sigma}\big(R(j\omega)\hat{M}_u(j\omega)\bar{G}_f(j\omega)\big) \tag{99}$$

and assume that ϕ is selectable over $[0,\ \infty]$. This makes it clear that $\|f_{\min}\|_\epsilon$ becomes smaller by increasing $\min_{\omega\in\phi}\underline{\sigma}\big(R(j\omega)\hat{M}_u(j\omega)\bar{G}_f(j\omega)\big)$ or decreasing $\max_{\omega\in\phi}\bar{\sigma}\big(\delta_d(\omega)R(j\omega)\hat{M}_u(j\omega)G_d(j\omega)\big)$. It is known (Francis, 1987; Maciejowski, 1989) that $\max_\omega \bar{\sigma}(\cdot)$ and $\min_\omega \underline{\sigma}(\cdot)$ have the following meanings

- $\max_\omega \bar{\sigma}(\cdot)$ is indeed the $\mathcal{H}_\infty$–norm of a transfer function that measures the greatest increase in energy that can occur between the input and the output for a given system
- in contrast, $\min_\omega \underline{\sigma}(\cdot)$ describes the possible smallest increase in energy that can occur between the input and the output for a given system.

It is worth mentioning that for the above two cases no assumption on the input signal is made beside that its size is restricted. This corresponds to our case where no information about the fault as well as the model uncertainties is assumed to be given.

effects of the input signal, which, as mentioned above, is just what we need for FDI. Notice that it would be possible that

$$\sup_{f} \|G(s)f(s)\| \gg \inf_{f} \|G(s)f(s)\|, \quad \text{for} \quad \|f(s)\| \leq 1. \tag{100}$$

This means that using the optimal residual generator in the sense of (51) the minimum size of detectable fault may be strongly reduced in comparison with the use of residual generators proposed in previous works. This certifies, on the other hand, that considering residual generation and evaluation problems together, as opposed to previous works, may offer more potential for the enhancement of robustness in the overall schemes of observer–based FDI.

Here, we would like to emphasize the fact that *the minimum size of detectable faults, under the use of the optimal residual generator in the sense of (51), is independent of the residual generator used (see (84))*. In fact, it is calculatable as long as the system model and the restrictions on the model uncertainties are given. This provides us with knowledge about how far we are able to detecte a fault before we begin solving FDI problem for a given system.

Finally, it is worth pointing out that although we have treated the problem by modelling the model uncertainties as a part of the unknown input vector, we did not follow the classic way that simply compares the effects of the unknwon inputs and faults. On the contrary, we devoted the attention to minimizing the size of detectable faults. In this way, information on the model uncertainties is utilized.

6. Conclusions

In this paper several problems related to fault detection and isolation have been studied. The basic idea of the study, different from the majority of previous works, is to minimize the size of detectable faults using suitable residual generation and evaluation scheme. This may enable a fault detection even if the size of the fault may be small.

The key point of our study is the derivation of the relationship between the residual generator as well as the residual evaluation function and the minimum size of detectable faults. This reveals, on the one hand, how far a fault could be detected using a suitable residual generation and evaluation scheme and, on the other hand, reduces the problems into optimization problems that are solved using frequency domain optimization techniques. The results have finally been presented in the form of algorithms so that they are calculatable on a computer. The needed on–line operations have also been investigated.

We finally discussed the physical meanings of the results achieved here. With this we have found an answer to the question which performance index should be adopted for designing residual generators, a question that has been, due to its importance, studied and discussed over the years. With the aid of these results we have achieved more insight into the observer–based FDI. It is believed that this provides a valuable basis for further research.

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