Appl. Math. and Comp. Sci., 1995, vol.5, No.2, 277–293

REDUNDANCY RESOLUTION OF MANIPULATOR BY GLOBAL OPTIMIZATION

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Global optimization in the redundancy resolution, involving direct dynamic equations of a manipulator, is presented. It is carried out by using the necessary and sufficient conditions for a minimum of integral-type criteria with a free upper limit of integration. Boundary conditions resulting from the manipulator task to be performed are taken into consideration. General transversality conditions corresponding to the boundary ones are derived. As a result, a closed system of boundary dependencies, fully specifying differential equations which result from the necessary conditions for a minimum, to find an extremal joint trajectory, is obtained. In order to verify the above extremal trajectory for optimality, (local) sufficient conditions are employed. A computer example involving a planar manipulator of three revolute kinematic pairs is presented.

1. Introduction

The abilities of redundant manipulators have caused an increasing interest to employ them in performing complicated tasks in complex workspaces which may include a lot of obstacles. The redundant degrees of freedom make it possible to realize some chosen objectives, e.g. collision or singularity avoidance tasks. Most research reports in this field deal with instantaneous (i.e. at any given time moment) redundancy resolutions, obtained by instantaneous minimization of some objective functions which result in a pseudo-inverse matrix of the manipulator Jacobi matrix. Whitney (1969) minimized the kinetic energy of the manipulator. Liegeois (1977) used some vector from the null space of the Jacobi matrix to avoid joint limits. Yoshikawa (1985) and Klein (1989) considered the minimization of the manipulability measure in singularity avoidance problem. The null space vector was used by Maciejewski and Klein (1985) in the determination of the collision-free manipulator trajectory. The redundancy resolution through a torque and the acceleration minimization were carried out by Hollerbach and Suh (1987a; 1987b), and Kazerounian and Nedungadi (1987), respectively. Khatib (1983), as well as Vukobratovic and Kircanski (1984), applied the generalized inverse method (resolved at the acceleration level) to dynamic equations of the manipulator.

Although the instantaneous redundancy resolution is attractive because of a possibility of real-time computations, it does not generally guarantee global optimality of performing manipulator tasks. It seems difficult to determine e.g. a collision-free

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trajectory in the workspace including a lot of obstacles if instantaneous redundancy resolution is used. For complicated tasks in complex spaces, global optimization methods should ensure the solutions of the problems stated above. A few papers have recently been published on the global optimization. Uchijama *et al.* (1985) used parameterization of the joint trajectory by polynomials and optimized the manipulability measure. Nakamura and Hanafusa (1987) proposed a solution based on the Pontryagin maximum principle for the optimization of a general performance index. Hollerbach and Suh (1987a; 1987b) presented a solution to the global torque optimization by using the calculus of variations. The global optimization of joint rates and kinetic energy was carried out by Kazerounian and Wang in (1988). Using the classical Euler- Lagrange dynamics, Martin *et al.* (1989) presented path planning techniques involving integral cost criteria.

This study presents global optimization of the redundancy resolution using any integral criteria involving robot dynamic equations for the situation when the end-effector path, represented as a curve (in the robot workspace) parameterized e.g. by its length (a kinematic manipulator task), is given. In this case, redundancy resolution with a free final time of performing the above task is obtained. This problem was considered neither in instantaneous redundancy resolution nor in the works concerning the global redundancy resolution cited above, although it is important in practical applications. The minimization procedures obtained in (Bobrow *et al.*, 1985; Shin and McKay, 1985) are applicable only for non-redundant robots or for redundant robots if all the joint displacements have been given as specified functions of the manipulator path parameterization. This is not the case considered in the present paper. In contrast, the approach presented in this paper utilizes the necessary conditions for a minimum of functionals for constrained problems of the calculus of variations to obtain extremal joint trajectories. Boundary conditions of various types (resulting from the tasks to be performed) imposed on the ends of trajectories are taken into consideration. In order to specify fully the differential equations obtained from the necessary conditions for extrema of functionals, general transversality conditions (in minimal amount) corresponding to the boundary ones have been derived. Hence, the determination of an extremal trajectory, which is expected to be the optimal one, is reduced to solving a two-point boundary-value problem.

The trajectory thus obtained is then verified for (local) optimality based on the sufficient conditions given in the paper. The paper is organized as follows. Section 2 formulates the problem. In Section 3 Euler-Poisson equations are used in order to solve it, comments are made on both the calculus of variations and Pontryagin's maximum principle and their use in the problem under consideration. Section 4 deals with the boundary and transversality conditions. The sufficient conditions for optimality are given in Section 5. A numerical example involving a manipulator of three revolute kinematic pairs, which perform different kinematic tasks, is presented in Section 6.

2. Formulation of the Problem

Let us consider a spatially redundant manipulator whose kinematic model may be expressed in the following general form

$$x = f(q) \tag{1}$$

where $f : \mathbb{R}^n \rightarrow \mathbb{R}^m$, $f(g)$ is a vector mapping consisting of m scalar non-linear functions of the vector q, q is the n-dimensional vector of generalized coordinates (joint variables) which represents a point in the joint space of the robot, x is the m-dimensional vector being an element of the robot workspace, n is the number of manipulator kinematic pairs, and m is the dimension of the workspace. Due to the manipulator redundancy the condition $m < n$ holds. We also consider a kinematic task to be realized by the robot in the following parametric form (often encountered in practice)

$$f(g) - \varphi(s) = 0 \tag{2}$$

where $\varphi : \mathbb{R}^1 \rightarrow \mathbb{R}^m$, $\varphi(s)$ is a given vector mapping consisting of m algebraic functions of the parameter s, which presents a path to be traced by the end-effector, $s \in [0, s_{\max}]$, and $s_{\max}$ is the maximum value of the parameter s representing e.g. the length of the path.

The problem is to find a manipulator trajectory (in the joint space) being a vector function of time t, i.e. $q = q(t)$ and $s = s(t)$, which satisfies eqn. (2).

When a motion of the manipulator in the workspace is forced, other constraints are usually induced, which result e.g. from limitations imposed on the controls, or obstacle avoidance (inequality constraints). They will be taken into account later.

3. Application of the Euler-Poisson Equation

Due to the redundancy of the manipulator, a global performance criterion is introduced to determine the joint trajectory corresponding to the kinematic task defined by eqn. (2). It may be generally expressed in the following way

$$I = \int_0^T c(q, u)\,\mathrm{d}t \tag{3}$$

where $c : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^1$, $c(q, u)$ is a given cost function with continuous derivatives of a proper order with respect to its variables q and u, u is the n-dimensional vector of controls (torques/forces), $u = M(q)\ddot{q} + F(q, \dot{q})$ is the dynamic model of the robot, $M(q)$ is the $n \times n$ dimensional, non-singular, positive definite, inertia matrix, $E(q, \dot{q})$ is the n-dimensional vector of Coriolis and centrifugal forces, $t \in [0, T]$, T is a free final moment of performing the task.

The problem of optimal control with free end time results from introducing the performance index (3) and taking into account eqn. (2). The Pontryagin maximum principle (Pontryagin *et al.* 1962) may be used to find its solution. However, there are a few reasons behind inefficiency of this principle in solving the above task. First, a direct application of this principle is troublesome enough for the case considered. The optimal control may not exist although the optimal joint trajectory may. This is due to the fact that the optimal control may not exist in a strong topology (although it may exist in a weak topology) whereas the optimal joint trajectory may (Filippov, 1959). If this is the case, that trajectory (if possible, obtained in another way) should be supplied as a reference for the on-line control. In fact, the Pontryagin maximum principle can handle effectively the case of constant lower and upper limits on controls. This is a rather restrictive condition in practice. Additionally, the determination of the optimal control becomes difficult if singular arcs occur (Sontag and Sussmann, 1986). The verification of sufficient conditions for optimality is troublesome in practice. Considering a particular form of the robot dynamic model, the problem of the optimal control may be transformed into a problem of the calculus of variations. Thus, the determination of the optimal control is replaced by the determination of the optimal joint trajectory. On account of the robot dynamic model, the performance index (3) may generally be written as follows

$$I + \int_0^T k(q, \dot{q}, \ddot{q}) \, \mathrm{d}t \tag{4}$$

where $k = c(q, M(q)\ddot{q} + F(q, \dot{q}))$. In order to handle the cases where there are inequality constraints resulting, for example, from the limitations on controls (as in the case of Pontryagin's maximum principle), the theory of one-sided variations should be employed (Claf, 1970). However, it is rather difficult to apply it in practice. Additionally, an optimal trajectory thus obtained may not be smooth (this complicates the on-line control). Instead, an approximate implementation of inequality constraints may be allowed for the above purpose. The approach is to use e.g. the interior penalty function method (Fiacco and McCormick, 1968). This implies that the inequality constraints are satisfied but the performance index (4) increases somewhat since the penalty functions may force the solution away from the inequality constraint boundaries. The penalty function approach requires that the constraint set have a non-empty interior. This is a rather reasonable assumption from the practical point of view.

Summarizing, dependences (2) and (4) form a constrained optimization problem of the calculus of variations with unknown functions of time q and s. The necessary condition for a minimum of functional (4) is given by the Euler-Poisson equations of the form below (Gelfand and Fomin, 1979)

$$\begin{aligned} &\frac{\partial K}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{q}} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial K}{\partial \ddot{q}} = 0 \\ &\frac{\partial K}{\partial s} = 0 \end{aligned} \tag{5}$$

where $K = k + \langle f(q) - \varphi(s), \lambda \rangle$, λ is the m-dimensional vector function of the Lagrange multipliers, $\langle \cdot, \cdot \rangle$ denotes the scalar product of vectors, which may be written in a more suitable form, as follows

$$\begin{aligned} &W \ddddot{q} = a - J^T \lambda \\ &\langle \varphi_s, \lambda \rangle = 0 \end{aligned} \tag{6}$$

where $J = \dfrac{\partial f}{\partial q}$ is the $m \times n$ dimensional Jacobi matrix, $\varphi_s = \dfrac{\mathrm{d}\varphi}{\mathrm{d}s}$,

$$W = \frac{\partial^2 K}{\partial \ddot{q}^2}$$

$$a = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \dot{q}} - \frac{\partial k}{\partial q} - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^2 K}{\partial q \partial \ddot{q}}\dot{q} + \frac{\partial^2}{\partial \dot{q}\partial \ddot{q}}\ddot{q}\right) - \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial^2 K}{\partial \ddot{q}^2}\right)\dddot{q}$$

When employing task constraint (2) the Lagrange multiplier λ should be explicitly expressed as a function of time t.

Assuming that the matrix W is non-singular along the extremal trajectory, it follows that

$$\ddddot{q} = W^{-1}(a - J^T\lambda) \tag{7}$$

Differentation of eqn. (2) four times with respect to time results in

$$J\,\ddddot{q} - \varphi_s\,\ddddot{s} + b = 0 \tag{8}$$

where

$$b = \dot{J}\,\dddot{q} - \varphi_{sss}\dot{s}\,\dddot{s} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{J}\,\dot{q} + \frac{\mathrm{d}}{\mathrm{d}t}\left(\dot{J}\,\dot{q} - \varphi_{ss}(\dot{s})^2\right) - \varphi_{sss}\dot{s}\,\ddot{s}\right)$$

Next, the right-hand side of eqn. (7) is used in eqn. (8). The assumption of the full rank of the Jacobi matrix J results in the Lagrange function λ dependent on time t

$$\lambda(t) = \left(JW^{-1}J^T\right)^{-1}\left(JW^{-1}a - \varphi_s\,\ddddot{s} + b\right) \tag{9}$$

Substituting (9) into (7) and (6) gives

$$\begin{bmatrix} I_n & J_W^{\#}\varphi_s \\ 0 & \left\langle \varphi_s, \left(JW^{-1}J^T\right)^{-1}\varphi_s \right\rangle \end{bmatrix} \begin{pmatrix} \ddddot{q} \\ \ddddot{s} \end{pmatrix} = \begin{bmatrix} W^{-1}a + J_W^{\#}JW^{-1}a + J_W^{\#}b \\ \left\langle \varphi_s, \left(JW^{-1}J^T\right)^{-1}\left(JW^{-1}a + b\right)\right\rangle \end{bmatrix} \tag{10}$$

where I_n is the $n \times n$ dimensional identity matrix, 0 is the $1 \times n$ dimensional null matrix, $J_W^{\#} = W^{-1}J^T(JW^{-1}J^T)^{-1}$. Equations (9) and (10) are mathematically equivalent to eqns. (2) and (4) under stated assumptions. The extremal trajectory, which is expected to be the optimal one, is then specified by solving the differential eqn. (10). This equation is a fourth order differential one with the unknown final time of integration. In order to specify it fully, in general $4(n+1)+1$ scalar consistent dependences relating boundary conditions should be given. Functional I considered in a class of extremal trajectories satisfying the Euler-Poisson eqn. (10) is reduced, in general, to a function $I(q_0, s_0, \dot{q}_0, \dot{s}_0, q_T, s_T, \dot{q}_T, \dot{s}_T, T)$ of parameters q_0, s_0, $\dot{q}_0$, $\dot{s}_0$, q_T, s_T, $\dot{q}_T$, $\dot{s}_T$ and T, respectively, where $q_0 = q(0)$, $s_0 = s(0)$, $\dot{q}_0 = \dot{q}(0)$, $\dot{s}_0 = \dot{s}(0)$, $q_t = q(T)$, $s_T = s(T)$, $\dot{q}_T = \dot{q}(T)$, and $\dot{s}_T = \dot{s}(T)$. Although this function is not (usually) explicitly given, its differential is. It is expressed by the following equation (Gelfand and Fomin, 1979)

$$
\begin{aligned}
\mathrm{d}I = & \left(K - \left\langle \dot{q}, \frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}} \right\rangle - \left\langle \ddot{q}, \frac{\partial K}{\partial \ddot{q}} \right\rangle \right)_{t=T} \mathrm{d}T \\
& + \left(\left\langle \frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}, \mathrm{d}q_T \right\rangle + \left\langle \frac{\partial K}{\partial \ddot{q}}, \mathrm{d}\dot{q}_T \right\rangle \right)_{t=T} \\
& - \left(\left\langle \frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}, \mathrm{d}q_0 \right\rangle + \left\langle \frac{\partial K}{\partial \ddot{q}}, \mathrm{d}\dot{q}_0 \right\rangle \right)_{t=0}
\end{aligned}
\tag{11}
$$

where $\mathrm{d}q_0$, $\mathrm{d}q_T$, $\mathrm{d}\dot{q}_0$, $\mathrm{d}\dot{q}_T$, $\mathrm{d}T$ are arbitrarily given variations at the ends of the trajectories $q(t)$ and $\dot{q}(t)$, respectively. In the general case, the above variations are subject to constraints enforced by the boundary conditions, which result from the character of the task to be performed. This problem is considered in the next section.

4. Boundary and Transversality Conditions

A lot of robotic applications deal with split and/or mixed boundary conditions (at the moments $t = 0$ and $t = T$) on the variables q and s. The present study is limited to considering these (most frequently used in practice) conditions although the approach presented here may also be used to more general forms of boundary conditions together with derivatives at the beginning and the end of the joint trajectory. The dynamic case of the performance index (4) makes it possible to enforce the boundary conditions on the joint positions and the end-effector velocities. If the above velocities are not specified, then, in general, they are subject to the natural constraints (which result from differentiation of (2) with respect to time) below

$$
\begin{aligned}
\left(J\dot{q} - \varphi_s \dot{s}\right)_{t=0} &= 0 \\
\left(J\dot{q} - \varphi_s \dot{s}\right)_{t=T} &= 0
\end{aligned}
\tag{12}
$$

Not decreasing the character of the considerations, the following (general enough) boundary conditions (consistent with the kinematic task (2) and the constraint equations (12)) are assumed in the sequel

$$
\begin{pmatrix}
f(q_0) - \varphi(0) \\
s_0 \\
\dot{q}_0 \\
\dot{s}_0 \\
f(q_T) - \varphi(s_{\max}) \\
s_T - s_{\max} \\
\dot{q}_T \\
\dot{s}_T
\end{pmatrix} = 0
\tag{13}
$$

The purpose is to obtain other $2(n-m)+1$ scalar boundary dependences, which together with the boundary conditions (13) will fully specify eqn. (10). The use of the differential of the functional I provides these extra dependences. As it was mentioned earlier, functional I considered in a class of the extremal joint trajectories satisfying the Euler-Poisson eqn. (10) is reduced, on account of the fixed quantities s_0, $\dot{s}_0$, $\dot{q}_0$, s_T, $\dot{s}_T$ and $\dot{q}_T$ in the boundary relations (13), to a function $I(q_0, q_T, T)$ of the variables q_0, q_T and T.

According to the law of searching for a conditional extremum of the function $I(q_0, q_T, T)$ with constraints (13), constant vectors λ_0 and λ_T of dimension m exist such that the following equality holds

$$\mathrm{d}\Big(I(q_0, q_T, T) + \big\langle f(q_0) - \varphi(0), \lambda_0\big\rangle + \big\langle f(q_T) - \varphi(s_{\max}), \lambda_T\big\rangle\Big) = 0 \quad (14)$$

On account of (11), a result of the free final time T in eqn. (14) is the equation below

$$\left(K - \left\langle \ddot{q}, \frac{\partial K}{\partial \ddot{q}} \right\rangle\right)_{t=T} = 0 \quad (15)$$

and, correspondingly, arbitrary variations $\mathrm{d}q_0$, $\mathrm{d}q_T$ in (14) and $\mathrm{d}\dot{q}_0 = \mathrm{d}\dot{q}_T = 0$ from (13) result in the following transversality conditions

$$\begin{aligned} &\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}} + J^T\lambda_0\right)_{t=0} = 0 \\ &\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}} + J^T\lambda_T\right)_{t=T} = 0 \end{aligned} \quad (16)$$

Next, the vectors λ_0 and λ_T should be eliminated from (16). Multiplying the components of (16) by $J_{t=0}$ and $J_{t=T}$, respectively, results in explicit formulae for the vectors λ_0 and λ_T

$$\begin{aligned} \lambda_0 &= -\left((JJ^T)^{-1}J\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)\right)_{t=0} \\ \lambda_T &= -\left((JJ^T)^{-1}J\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)\right)_{t=T} \end{aligned} \quad (17)$$

Substituting the above quantities into (16), the transversality conditions at the initial and the final moment of executing manipulator task (2) form the following system of $2n$ scalar equations

$$\begin{aligned} &\left((I_n - J^{\#}J)\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)\right)_{t=0} = 0 \\ &\left((I_n - J^{\#}J)\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)\right)_{t=T} = 0 \end{aligned} \quad (18)$$

where $J^{\#} = J^T(JJ^T)$ is the pseudo-inverse matrix of J. Considering the first equation of (18), Nakamura and Hanafusa (1987) have also obtained in their transversality conditions the standard orthogonal projection onto the null space of $J_t = 0$ multiplied by the adjoint vector $\psi(0)$, whereas this projection is multiplied by $\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)_{t=0}$ in (18).

Summarizing, eqns. (13), (15) and (18) provide $4(n+1)+1+2m$ scalar dependent boundary and transversality conditions to find from eqn. (10) an extremal joint trajectory. Due to the dependence of these conditions it is difficult to use directly fast numerical procedures to solve the system of non-linear eqns. (13), (15) and (18).

On the other hand, in order to specify uniquely the differential eqn. (10), $4(n+1)+1$ boundary dependences are required. When $J_{t=0}$ and $J_{t=T}$ in (18) are of full rank m, the matrices $(I_n - J^{\#}J)_{t=0}$ and $(I_n - J^{\#}J)_{t=T}$ are of rank $n-m$. Hence, $n-m$ linearly independent rows for each of these matrices exist, which result in $2(n-m)$ scalar independent transversality conditions. As a consequence, the use of eqns. (18) requires the matrices constituted by choosing $n-m$ rows of $(I_n - J^{\#}J)_{t=0}$ and $n-m$ rows of $(I_n - J^{\#}J)_{t=T}$ to be of rank $n-m$. If this is not the case, another set of rows should be chosen (the assumption of the full rank of the matrices $J_{t=0}$ and $J_{t=T}$ makes it possible to choose such rows). The purpose of further considerations is to eliminate the pseudo-inverse matrices from $2n$ scalar dependent eqns. (18) and to rewrite the transversality conditions (18) in a simpler form of $2(n-m)$ independent relations. The assumption $\mathrm{rank}J_{t=0} = m = \mathrm{rank}J_{t=T}$ makes it possible to select from each of matrices $J_{t=0}$, $J_{t=T}$ m linearly independent columns, which are, without loss of generality, the first columns of $J_{t=0}$ and $J_{t=T}$, respectively, and to constitute non-singular matrices $(J^R)_{t=0}$, $(J^R)_{t=T}$ of dimension $m \times m$ (otherwise another set of independent columns should be chosen). The other columns of $J_{t=0}$, $J_{t=T}$ constitute matrices $(J^F)_{t=0}$, $(J^F)_{t=T}$ of dimension $m \times (n-m)$, respectively. Following the derivation method presented in (Galicki, 1992), general transversality conditions are obtained below

$$
\begin{aligned}
&\left(\left[\left((J^R)^{-1}J^F\right)^T - I_{n-m}\right]\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)\right)_{t=T} = 0 \\
&\left(\left[\left((J^R)^{-1}J^F\right)^T - I_{n-m}\right]\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)\right)_{t=0} = 0
\end{aligned}
\tag{19}
$$

where I_{n-m} is the $(n-m)\times(n-m)$ dimensional identity matrix. The above expressions present the transversality conditions obtained at the initial and the final moment of executing the kinematic task (2), which the extremal joint trajectory has to satisfy. Similary, as for the case of the transversality conditions given by (18), where the $(n-m)\times n$ dimensional matrices constituted from $(I_n - J^{\#}J)_{t=0}$ and $(I_n - J^{\#}J)_{t=T}$ should be of rank $n-m$, the use of eqns. (19) requires the matrices $(J^R)_{t=0}$ and $(J^R)_{t=T}$ to be of rank m. Nevertheless, dependences (19) do not require the pseudo-inverse matrix calculations and seem to be simpler for numerical computations than relations (18). Taking into account eqns. (13), (15), and (19), a system of $2(n+1)$ independent boundary and transversality conditions at the initial moment is obtained

$$\begin{pmatrix} f(q_0) - \varphi(0) \\ s_0 \\ \left(\left[\left((J^R)^{-1}J^F\right)^T - I_{n-m}\right]\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)\right)_{t=0} \\ \dot{q}_0 \\ \dot{s}_0 \end{pmatrix} = 0 \tag{20}$$

whereas, at the final moment, there are $2(n+1)+1$ such dependences

$$\begin{pmatrix} f(q_T) - \varphi(s_{\max}) \\ s_T - s_{\max} \\ \left(\left[\left((J^R)^{-1}J^F\right)^T - I_{n-m}\right]\left(\frac{\partial K}{\partial \dot{q}} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}\right)\right)_{t=T} \\ \dot{q}_T \\ \dot{s}_T \\ \left(K - \left\langle \ddot{q}, \frac{\partial K}{\partial \ddot{q}}\right\rangle\right)_{t=T} \end{pmatrix} = 0 \tag{21}$$

Summarizing, an extremal joint trajectory results from solving the two-point boundary-value problem, specified by the differential eqn. (10) and $4(n+1)+1$ boundary and transversality conditions (20), (21).

Due to the integration of the differential eqn. (10) from the initial moment to the final one, the quantities q_T, s_T and their time derivatives occuring in (20), (21) depend on $4(n+1)+1$ coordinates of the vector $\alpha = (q_0, s_0, \dot{q}_0, \dot{s}_0, \ddot{q}_0, \ddot{s}_0, \dddot{q}_0, \dddot{s}_0, T)$, i.e. $q_T = q(\alpha)$, $s_T = s(\alpha)$, $\dot{q}_T = \dot{q}(\alpha)$, $\dot{s}_T = \dot{s}(\alpha)$, $\ddot{q}_T = \ddot{q}(\alpha)$, $\ddot{s}_T = \ddot{s}(\alpha)$, $\dddot{q}_T = \dddot{q}(\alpha)$ and $\dddot{s}_T = \dddot{s}(\alpha)$. Hence, the left-hand sides of eqns. (20), (21) are, in fact, functions of α and may be written in a general form as $e(\alpha)$, where $(e(\alpha))_i$ denotes the left-hand side of the i-th scalar equation of systems (20), (21), $i = 1, ..., 4(n+1)+1$. Finally, the determination of an extremal joint trajectory is reduced to finding a root (or roots) of $4(n+1)+1$ non-linear equations below

$$e(\alpha) = 0 \tag{22}$$

with the same number of unknowns being the coordinates of vector α (note that for the case under consideration some of the components of α are given *a priori*, and as a consequence the number of unknowns to be found and their corresponding scalar relations of system (22) is less than $4(n+1)+1$). A numerical procedure is proposed in (Galicki, 1992) to solve the above system in order to find its roots which determine uniquely the extremal joint trajectories. They should be then verified for optimality.

5. Locally Sufficient Conditions

To check the extremal trajectory obtained above for a minimum, sufficient conditions are needed. They may be provided by means of second variation of functional (4) and by the task constraint (2). The method of its derivation (omitted herein) is based on a similar technique as for the case of the first variation (11). The first variation is differentiatied for the second time. The second variation is then obtained by taking into account the quadratic terms of the differentiatied variation with respect to the variations of the above extremal trajectory and its ends. As a consequence, the quadratic functional of (4), obtained at this trajectory, is given as follows

$$
\begin{aligned}
\mathrm{d}^2 I = & \left(\frac{\partial K}{\partial t} - \left\langle \frac{\partial K}{\partial q}, \dot{q} \right\rangle - \frac{\partial K}{\partial s}\dot{s} - \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}, \ddot{q}, \right\rangle \right)_{t=T} \mathrm{d}T^2 \\
& +2\left(\left\langle \frac{\partial K}{\partial q}, \mathrm{d}q_T \right\rangle + \frac{\partial K}{\partial s}\mathrm{d}s_T, + \left\langle \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial K}{\partial \ddot{q}}, \mathrm{d}\dot{q}_T \right\rangle \right)_{t=T} \mathrm{d}T \\
& + \int_0^T \omega(\delta q, \delta s, \delta \dot{q}, \delta \ddot{q})\,\mathrm{d}t
\end{aligned}
\tag{23}
$$

where

$$
\begin{aligned}
\omega = & \left(\left\langle \frac{\partial^2 K}{\partial q^2}\delta q, \delta q \right\rangle + \left\langle \frac{\partial^2 K}{\partial \dot{q}^2}\delta \dot{q}, \delta \dot{q} \right\rangle + \left\langle \frac{\partial^2 K}{\partial \ddot{q}^2}\delta \ddot{q}, \delta \ddot{q} \right\rangle + \frac{\partial^2 K}{\partial s^2}\delta s^2 \right. \\
& \left. + \left\langle 2\frac{\partial^2 K}{\partial q \partial \dot{q}}\delta \dot{q}, \delta q \right\rangle + \left\langle 2\frac{\partial^2 K}{\partial \ddot{q} \partial \dot{q}}\delta \dot{q}, \delta \ddot{q} \right\rangle + \left\langle 2\frac{\partial^2 K}{\partial \ddot{q} \partial q}\delta q, \delta \ddot{q} \right\rangle \right)
\end{aligned}
$$

δq, δs, $\delta \dot{q}$ and $\delta \ddot{q}$ are variations of the extremal trajectories $q(t)$, $s(t)$ and their derivatives, in their neighbourhoods, respectively. It seems reasonable in practice to consider sufficient conditions in a class of extremal joint trajectories with fixed end moments, which were obtained in the previous sections of this paper from the necessary conditions for a minimum. This assumption does not decrease the character of considerations and makes further calculations easier. If this is the case, the quadratic functional (23) can be simplified, as follows

$$
\mathrm{d}^2 I = \int_0^T \omega(\delta q, \delta s, \delta \dot{q}, \delta \ddot{q})\,\mathrm{d}t \tag{24}
$$

The sufficient condition for a (local) minimum is that the quadratic functional (24) is strongly positive definite for the extremal trajectory under consideration (Gelfand and Fomin, 1979). To check the above trajectory for a minimum, an accessory problem related to the prior one should be solved (Claf, 1970). For the case considered here, it is defined as follows: minimize the quadratic functional (24) subject to the following constraints

$$
\left[J(t) - \varphi_s(t) \right] \begin{pmatrix} \delta q \\ \delta s \end{pmatrix} = 0 \tag{25}
$$

where the quantities $J(t) = \left(\frac{\partial f}{\partial q}\right)_{q=q(t)}$, $\varphi_s(t) = \left(\frac{\mathrm{d}\varphi}{\mathrm{d}s}\right)_{s=s(t)}$ are computed along the extremal trajectory $(q(t)\ s(t))$, $t \in [0, T]$, and

$$\begin{pmatrix} J(0)\delta q_0 \\ \delta s_0 \\ \delta \dot{q}_0 \\ \delta \dot{s}_0 \\ J(T)\delta q_T \\ \delta s_T \\ \delta \dot{q}_T \\ \delta \dot{s}_T \end{pmatrix} = 0 \tag{26}$$

where $\delta q_0 = \delta q(0)$, $\delta s_0 = \delta s(0)$, $\delta \dot{q}_0 = \delta \dot{q}(0)$, $\delta \dot{s}_0 = \delta \dot{s}(0)$, $\delta q_T = \delta q(T)$, $\delta s_T = \delta s(T)$, $\delta \dot{q}_T = \delta \dot{q}(T)$ and $\delta \dot{s}_T = \delta \dot{s}(T)$. Explicit formulae for finding the unknown functions $(\delta q, \delta s)$ are based on the Euler-Poisson equations of the form below

$$\begin{aligned} &\frac{\delta\Omega}{\partial(\delta q)} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\Omega}{\partial(\delta \dot{q})} + \frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\delta\Omega}{\partial(\delta \ddot{q})} = 0 \\ &\frac{\delta\Omega}{\partial(\delta s)} = 0 \end{aligned} \tag{27}$$

where $\Omega = \omega + (\mu, J(t)\delta q - \varphi_s(t)\delta s)$, μ is the m-dimensional vector of the Lagrange multipliers, with the boundary constraints (27) and the transversality conditions, which can be derived (similarly as in Section 4) using the first differential of functional (24) equal to zero, given below

$$\begin{aligned} d(\mathrm{d}^2 I) = &\left(\left\langle \frac{\delta\Omega}{\partial(\delta \dot{q})} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\Omega}{\partial(\delta \ddot{q})}, \mathrm{d}(\delta q_T), \right\rangle + \left\langle \frac{\delta\Omega}{\partial(\delta \ddot{q})}, \mathrm{d}(\delta \dot{q}_T) \right\rangle\right)_{t=T} \\ &- \left(\left\langle \frac{\delta\Omega}{\partial(\delta \dot{q})} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\delta\Omega}{\partial(\delta \ddot{q})}, \mathrm{d}(\delta q_0), \right\rangle + \left\langle \frac{\delta\Omega}{\partial(\delta \ddot{q})}, \mathrm{d}(\delta \dot{q}_0) \right\rangle\right)_{t=0} = 0 \end{aligned} \tag{28}$$

where $\mathrm{d}(\delta q_0)$, $\mathrm{d}(\delta q_T)$ are arbitrarily given variations at the ends of the trajectory $(\delta q, \delta s)$. The extremal trajectories $(\delta q(t), \delta s(t))$, where $t \in [0, T]$, being the solutions of the above problem are called accessory extremal trajectories. They may be found (numerically) using the methods given by Galicki (1992).

The quadratic functional (24) is strongly and positive definite at the extremal trajectory $(q(t)\ s(t))$, where $t \in [0, T]$, when the following conditions are satisfied (i.e. local, sufficient conditions):

- quadratic matrix $\dfrac{\partial^2 K}{\partial \ddot{q}^2}$ is positive definite for each $t \in [0, T]$,

- interval $[0, T]$ does not include any moment of time, which is conjugate to $t = 0$. A moment $t' \in (0, T]$ is said to be conjugate to the moment $t = 0$ iff an accessory extremal trajectory exists, such that it equals zero at $t = 0$, $t = t'$ and is not identical to zero between the moments 0 and t'.

Thus, an extremal trajectory $(q(t)\ s(t))$, where $t \in [0, T]$, minimizes (locally) functional (4) when the null trajectory $(\delta q(t)\ \delta s(t)) = (0\,0)$ is the only solution for the accessory problem. In order to verify (numerically) the trajectory $(q(t)\ s(t))$ for a minimum, a number of initial guesses for finding accessory trajectory $(\delta q(t), \delta s(t))$ should be made. If each solution thus obtained is the null accessory trajectory, then it may be concluded that the extremal trajectory $(q(t)\ s(t))$ is the optimal one. If this is not the case, this trajectory is not the optimal one.

6. Numerical Example

A planar manipulator of three revolute kinematic pairs is considered. The link length are as follows: $l_1 = 3$, $l_2 = 2.5$ and $l_3 = 2$. The kinematic task is to trace a circle, described analytically in the form

$$\varphi(s) = \begin{pmatrix} 2.25\cos(s) + 2.5 \\ 2.25\sin(s) \end{pmatrix}$$

where $s \in [0, s_{\max} = 6.28]$, by the end-effector (Fig. 1) so as to minimize the functional

$$I = \int_0^T \Big(1 + c\langle \dot{q}, \dot{q}\rangle\Big)\, \mathrm{d}t \tag{29}$$

where c is a positive coefficient, which equals $c = 0.03$.

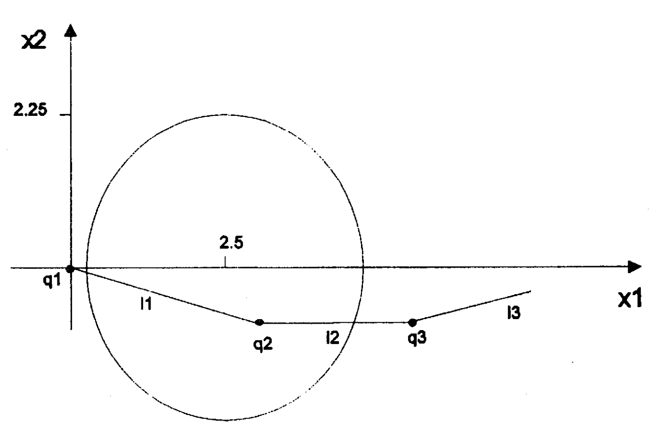

Fig. 1. A periodic task to be realized by the manipulator.

A kinematic model of the above manipulator is expressed, as follows

$$\mathbb{R}^2 \ni x = f(q) = \begin{pmatrix} \sum_{i=1}^{3} l_i c_i \\ \sum_{i=1}^{3} l_i s_i \end{pmatrix}$$

where $c_i = \cos(q^1 + q^2 + \cdots + q^i)$, $s_i = \sin(q^1 + q^2 + \ldots + q^i)$, $i = 1, \cdots, 3$. The boundary and the transversality periodic condition to find optimal joint trajectory $(q(t)\ s(t))$ assumes for the case under consideration the following form (note that functional (29) does not depend on $\ddot{q}$)

$$e(\alpha) = \begin{pmatrix} f(q_0) - \varphi(0) \\ s_0 \\ \left\langle \left(\left((J^R)^{-1} J^F \right)^T - 1 \right)_{t=0}, \dot{q}_0 \right\rangle \\ q(\alpha) = q_t - q_0 \\ s(\alpha) = s_t - s_{\max} \\ 1 - 0.5c\left\langle \dot{q}(\alpha) = \dot{q}_T, \dot{q}(\alpha) = \dot{(q)}_T \right\rangle \end{pmatrix} = 0 \tag{30}$$

where $\alpha = (q_0, s_0, \dot{q}_0, \dot{s}_0, T)$, $(J^R)_{t=0}$ is the 2×2 dimensional matrix constituted from the first two columns of the Jacobi matrix $\left(\frac{\partial f}{\partial q}\right)_{t=0}$, which are assumed to be linearly independent; $(J^F)_{t=0}$ is the 2×1 dimensional matrix obtained by excluding $(J^R)_{t=0}$ from $\left(\frac{\partial f}{\partial q}\right)_{t=0}$. The initial guess α_0 for solving system (30) equals $\alpha_0 = (-0.1, 0.1, 0.1, 0.0, 0.1, -0.1, -0.1, 0.2, 0.1)$. Employing numerical method to find a solution to system (30), the following root is found:

$$\alpha = (-1.09, 1.48, 0.61, 0.0, 5.63, 0.60, -0.89, 12.37, 0.84)$$

The corresponding extremal trajectory as a function of time is presented in Fig. 2.

Next, this extremal joint trajectory is verified for a minimum using the sufficient conditions given in Section 5. In this case, the accessory Euler-Poisson equations (here reduced to the Euler-Lagrange equations) assume the following form

$$2\left(\frac{\partial^2 K}{\partial q^2}\right)_{q=q(t)} \delta q + J^T(t)\mu - 4c\delta\ddot{q} = 0$$

$$2\left(\frac{\partial^2 K}{\partial s^2}\right)_{s=s(t)} \delta s - \langle \mu, \varphi_s(t) \rangle = 0$$

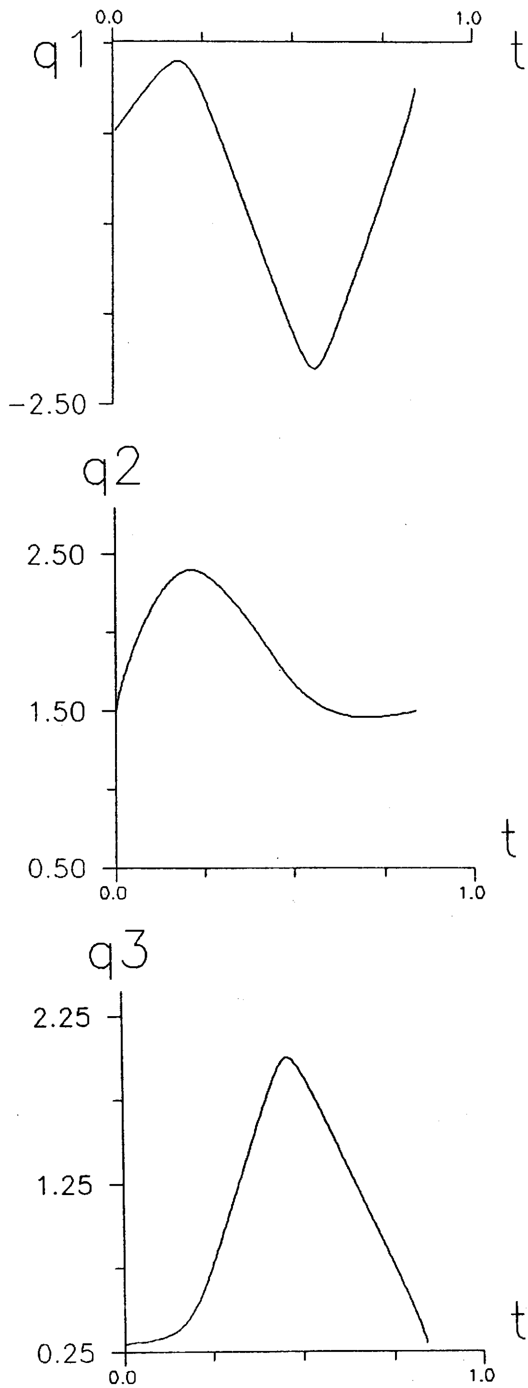

Fig. 2. Extremal trajectories v. time for α.

with the accessory boundary and transversality conditions

$$e(\alpha) = \begin{pmatrix} J_{t=0}\delta q_0 \\ \delta s_0 \\ \left\langle \left(\left((J^R)^{-1} J^F \right)^T - 1 \right)_{t=0}, \delta \dot{q}_0 \right\rangle \\ \delta q_T = \delta q(\alpha) - \delta q_0 \\ \delta s_T = \delta s(\alpha) \end{pmatrix} \tag{31}$$

where $\alpha = (\delta q_0, \delta s_0, \delta \dot{q}_0, \delta \dot{s}_0)$, $\delta \dot{q}_0 = \delta \dot{q}(0), \delta \dot{s}_0 = \delta \dot{s}(0)$. In the computer example six values of the initial guess α_0 are taken for computations. They are presented below.

No.	$\alpha_0 = (q_0, s_0, \dot{q}_0, \dot{s}_0)$
1	(10, -1, 5, 0.0, -6, -2, 4, 5)
2	(8, -5, 3, 0.0, 7, -4, 5, -2)
3	(-5, -4, 7, 0.0, 9, -2, 7, -5)
4	(-10, 5, 5, 0.0, -10, 5, 5, -10)
5	(5, -10, -10, 0.0, -10, 5, -10, -10)
6	(-9, 2, -3, 0.0, 7, -2, -1, 1)

Each of them results in the null accessory trajectory. Hence, it is concluded that the extremal trajectory presented in Fig. 2 is the optimal one.

7. Conclusions

Global redundancy resolution, based on minimization of any integral criteria (using the calculus of variations) is given. Boundary conditions imposed and transversality ones derived fully specify the Euler-Poisson differential equations, which result from the necessary conditions for extrema of functionals, and thus reduce the amount of numerical computations needed to find the optimal solution. This solution may be found using shooting methods (given e.g. in (Muszynski and Myszkis, 1984)) which involve, in general, guessing the initial configuration, initial directions of acceleration and jerk of the extremal trajectory, final time, and refining the guess until the transversality and the boundary conditions are satisfied with accuracy.

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Received: December 21, 1994
Revised: February 27, 1995