Appl. Math. and Comp. Sci., 1995, vol.5, No.2, 295–303

KINEMATICS OF PATH TRACKING AT SINGULARITIES OF THE PLANAR 2R MANIPULATOR

JON KIEFFER*

A method for determining the differential kinematics of path tracking at singular configurations of the planar 2R manipulator is presented. The method is based on Taylor series analysis of the displacement equations to determine low-order parametric curves that satisfy those equations. Results show that singularities can take the form of isolated points, turning points, nodes, and cusps and that the form of the singularity depends on the degree of contact between the path and the workspace boundary. The resulting expansions provide complete low-order information on admissible rates of path traversal and on all families of joint trajectories that track the path at the singularity.

1. Introduction

A mechanical manipulator can be thought of as a mapping $x = f(\theta)$ from joint positions $\theta \in R^n$ onto a spatial end-effector position $x \in R^n$. To control the end effector one must solve the so-called *inverse kinematics problem*. Loosely speaking this problem is to determine joint motions that provide the desired end-effector motion. One part of this problem is to solve $x = f(\theta)$ for roots θ, given x. Another part of this problem is to determine differentials in θ as a function of differentials in x. Except at *kinematic singularities*, this second problem is relatively straightforward to solve. For example, given x, $\dot{x}$, $\ddot{x}$, and a solution $\theta = f^{-1}(x)$ to the first problem, joint velocities and accelerations can be determined as follows.

$$\dot{\theta} = f_\theta^{-1}\dot{x} \tag{1}$$

$$\ddot{\theta} = f_\theta^{-1}[\ddot{x} - \dot{f}_\theta\dot{\theta}] \tag{2}$$

Kinematic singularities are defined as configurations θ where the Jacobian $f_\theta(\theta)$ is singular. At these configurations the rate solutions above do not work. Sometimes this occurs when no solution exists. In other cases a solution or multiple solutions exist.

Kinematic singularities are of interest for a number of reasons. Primarily for the nuisance they present to robot control, but also due to their fundamental relation to workspace boundaries, closed-form inverse kinematic solutions, and the possibility of using them to gain mechanical advantage. A fairly comprehensive review of related literature is given in (Kieffer, 1994).

* Department of Engineering, Faculty of Engineering and Information Technology, The Australian National University, ACT 2601, Canberra, Australia

One strand of this research has been to develop mathematical methods for determining local differential models of the manipulator kinematics in the neighbourhood of singularities. The motivations for this research are to understand singularities better, to classify them, and to offer alternatives to equations such as (1) and (2) which are not valid at singularities. Tchoń (1991) and Tchoń and Urban (1992) have applied the methods of *singularity theory* to determine *normal forms* of manipulator singularities. Kieffer (1992, 1994) has taken a different approach based on Taylor series analysis of the manipulator's path-constrained kinematics. This approach can be summarized as follows.

1. Consider that an end-effector *path* $x(\lambda)$, rather than trajectory $x(t)$ is given.
2. Assume that θ and λ are functions of a new scalar parameter s.
3. Analyse a Taylor series expansion of the constraint equations $x(\lambda(s)) = f(\theta(s))$ at the singularity to determine the first terms in the Taylor series expansions of $\theta(s)$, and $\lambda(s)$.
4. Use the resulting expansions to determine admissible end-effector rates and the corresponding joint rates

In (Kieffer, 1992; 1994) this approach was applied to general six-degree-of-freedom serial manipulators. Results showed that the locus of kinematic solutions can take the form of three types of curve singularities: isolated points, turning points, and simple nodes. In addition, general algorithms were developed to determine smooth local models (Taylor expansions) for each case. But these results were not exhaustive. They extended only to those singularities that can be unambiguously defined by the first three terms in a Taylor series expansion of the constraint equations.

In this paper we apply this method to the much simpler equations associated with the planar 2R manipulator to derive an exhaustive classification of path following singularities for any smooth path passing through a point on the outer workspace boundary. In addition, we determine smooth local models for each case that provide low-order joint rate relations as well. The results show that the topology of local models can be determined from the degree of contact between the end-point path and the workspace boundary. Section 2 presents the problem formulation. Sections 3 and 4 develop the analytic solutions for local curve expansions. Section 5 shows how these expansions can be used to determine rate solutions at the singularity. Conclusions are drawn in Section 6.

2. Problem Formulation

Figure 1 depicts the robotic path tracking problem that we wish to address. We assume that a smooth endpoint path $x(\lambda)$ is given that includes a point P on the outer workspace boundary that forces the manipulator into an *outstretched* singular configuration. The problem is to determine the locus of joint positions (θ_1, θ_2) that keep the end point on the path in the neighbourhood of P, as well as the differential rates of change of θ_1 and θ_2. The end point path $x(\lambda)$ may intersect the workspace boundary at P or have any degree of tangency with it.

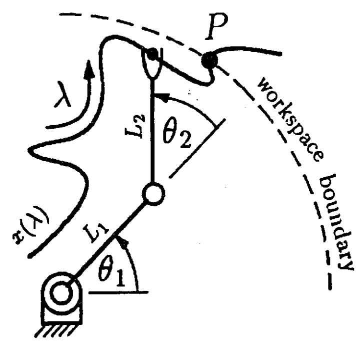


Fig. 1. 2R manipulator with path $x(\lambda)$ forcing it into a singular configuration.

To simplify explanations we choose not to consider *folded* singularities associated with the *inner* workspace boundary even though the approach can be applied to them with only minor differences. Likewise we will not consider the special folded singularity that occurs when the robot has equal link lengths $L_1 = L_2$ and its end point coincides with the workspace void which has shrunk to a point. If the method is applied to this case it will reveal a *self motion circuit* similar to the self motion circuit of the PUMA wrist singularity described in (Kieffer, 1994). Interested readers are also referred to (Tchoń, 1992) for a thorough analysis of the folded singularity of the planar 2R manipulator with equal link lengths.

It is advantageous to represent the end-point path in polar coordinates. Without loss of generality let $x(\lambda) = [r(\lambda), \phi(\lambda)]$, with $\lambda = 0$ corresponding to point P. Let us represent these functions by their Taylor series expansions about $\lambda = 0$.

$$r(\lambda) = (L_1 + L_2) + \frac{r^{(k)}}{k!}\lambda^k + O(\lambda^{k+1}) \qquad (k \geq 1) \tag{3}$$

$$\phi(\lambda) = \phi^{(0)} + \phi^{(1)}\lambda + O(\lambda^2) \tag{4}$$

Here $r^{(k)}$ denotes $r(\lambda)$'s first non-zero derivative, $\left.\frac{d^k r}{d\lambda^k}\right|_0 \neq 0$. We also require the parameterization to be regular, i.e. $(r')^2 + (r\phi')^2 > 0$.

This form of representation is chosen because it makes the path's contact with the workspace boundary explicit: e.g. $k = 1$ implies 1-point contact (intersection), $k = 2$ implies 2-point contact (simple tangency), etc. We will show that k determines the form of the singularity.

The following constraint equations relate the end point coordinates (r, ϕ) to the joint coordinates (θ_1, θ_2). Equation (5) can be derived using the law of cosines. Equation (6) follows from trigonometry.

$$r^2 = L_1^2 + L_2^2 + 2L_1L_2\cos\theta_2 \tag{5}$$

$$r\sin(\phi - \theta_1) = L_2\sin\theta_2 \tag{6}$$

Recalling that r and ϕ are parametric functions of the path parameter λ, our problem is to solve eqns. (5) and (6) for local relations between λ, θ_1, and θ_2 in the neighbourhood of the singular solution, $(\lambda, \theta_1, \theta_2) = (0, \phi^{(0)}, 0)$. Because eqn. (5) does not involve θ_1, we can solve this problem in two steps: first analyse eqn. (5) to determine local relations between λ and θ_2, then analyse eqn. (6) to locally determine θ_1. This decoupling of the problem into two parts is another advantage of representing the end point position in polar coordinates.

3. Determination of $\lambda(s)$ and $\theta_2(s)$

Substitution of the path constraint $r = r(\lambda)$ into eqn. (5) provides the following kinematic constraint.

$$r^2(\lambda) - L_1^2 - L_2^2 - 2L_1L_2\cos\theta_2 = 0 \tag{7}$$

Taylor series expansion of (7) about $(\lambda, \theta_2) = (0, 0)$ provides:

$$2(L_1 + L_2)\frac{r^{(k)}}{k!}\lambda^k + L_1L_2\theta_2^2 + O(\lambda^{k+1}) + O(\theta_2^3) = 0 \tag{8}$$

To determine the locus of solutions (λ, θ_2) to (8) in the neighbourhood of $(0, 0)$ we now assume that λ and θ_2 are functions of a new parameter s. Without loss of generality we represent these functions by the following Taylor series expansions.

$$\lambda(s) = \alpha s^p + O(s^{p+1}) \tag{9}$$

$$\theta_2(s) = \beta s^q + O(s^{q+1}) \tag{10}$$

Here α and β are unknown real scalars, whereas p and q are unknown positive integer constants. Parameters α, β, p, and q characterize $\lambda(s)$, and $\theta_2(s)$ locally. Substitution of (9) and (10) into (8) provides the following equation.

$$2(L_1 + L_2)\frac{r^{(k)}}{k!}\alpha^k s^{pk} + L_1L_2\beta^2 s^{2q} + O(s^{pk+1}) + O(s^{2q+1}) = 0 \tag{11}$$

A solution of this equation determines the first terms in the Taylor expansions of $(\lambda(s), \theta_2(s))$ that satisfies (8) near the singularity. By *solution* we mean a set of values (α, β, p, q) that ensure eqn. (11) is satisfied *for all small s*. Such solutions must satisfy the following two equations.

$$pk = 2q \tag{12}$$

$$\beta^2 = M\alpha^k, \qquad M = -\frac{2(L_1 + L_2)r^{(k)}}{L_1L_2k!} \tag{13}$$

We are not interested in determining all solutions (α, β, p, q): only those which lead to a unique curve in (λ, θ_2)-space. Four cases arise (isolated point, node, turning point, cusp) depending on the values of k and M as shown in Tab. 1 and illustrated graphically in Tab. 2. When k is even and $M < 0$, eqn. (13) admits no real solutions

Tab. 1. (α, β, p, q) solutions and singularity type.

	$M < 0$	$M > 0$
k even	*isolated point*	$p = 1$ $q = k/2$ $\alpha = 1$ $\beta = \pm\sqrt{M}$ *node*
k odd	$p = 2$ $q = k$ $\alpha =$sign(M) $\beta = \sqrt{\lvert M\rvert}$ *turning point* $(k = 1)$ or *cusp* $(k > 1)$	

for β, thus the singular solution $(\lambda, \theta_2) = (0, 0)$ must be isolated. This case is not illustrated in Tab. 2.

When k is even and $M > 0$, we can choose $p = \alpha = 1$ and then solve eqns. (12) and (13) for $q = k/2$ and $\beta = \pm\sqrt{M}$, respectively. The two solutions for β provide different curves in (λ, θ_2)-space that intersect (with or without tangency) at $(\lambda, \theta_2) = (0, 0)$ to form a node. The degree of tangency between the node branches is equal to $k/2 - 1$ as illustrated in rows 2, 4, and 6 of Tab. 2. Equations (12) and (13) also yield solutions for other choices of p and α, but these choices only lead to other parameterizations of the same curves.

When k is odd we can choose $p = 2$ and $\alpha = \text{sign}(M)$ then solve eqns. (12) and (13) for $q = k$ and $\beta = \pm\sqrt{|M|}$, respectively. In this case, the two solutions for β provide the same curve in (λ, θ_2)-space. Similarly, all other choices of p and α lead to different parameterizations of this curve. As shown in row 1 of Tab. 2, the curve has a turning point at $(\lambda, \theta_2) = (0, 0)$ if $k = 1$, otherwise it has a cusp as shown in rows 3 and 5.

The resulting expressions for $\lambda(s)$ and $\theta_2(s)$ are summarized in columns 2 and 4 of Tab. 3.

4. Determination of $\theta_1(s)$

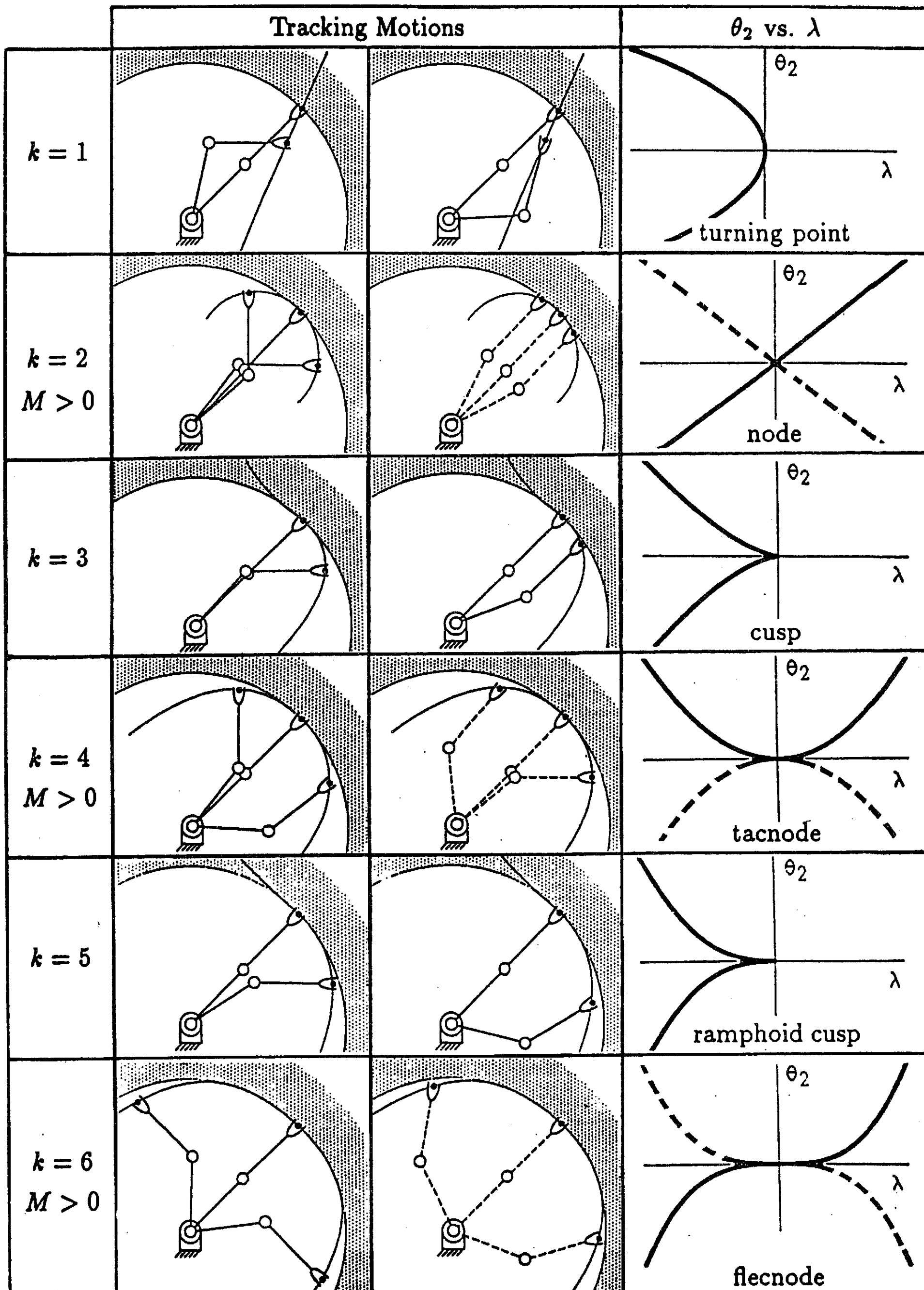
In the neighbourhood of the singularity, eqn. (6) can be solved for θ_1 as follows.

$$\theta_1 = \phi - \arcsin\left(\frac{L_2 \sin\theta_2}{r}\right) \tag{14}$$

A local expansion of $\theta_1(s)$ can be determined based on a Taylor series expansion of (14) about the singularity. Considering that $r = r(\lambda(s))$, $\phi = \phi(\lambda(s))$, and $\theta_2 = \theta_2(s)$ we obtain the following expansion of (14) about $s = 0$.

$$\begin{aligned} \theta_1(s) &= \phi + \left[\phi'\lambda' - \frac{L_2}{r}\theta_2'\right] s \\ &+ \left[\frac{1}{2}\phi'\lambda'' + \frac{1}{2}\phi''(\lambda')^2 - \frac{1}{2}\frac{L_2}{r}\theta_2'' + \frac{L_2}{r^2}\theta_2' r'\lambda'\right] s^2 + O(s^3) \end{aligned} \tag{15}$$

Tab. 2. Path tracking singularities of the planar 2R manipulator.



Tab. 3. Summary of Taylor expansions at the singularity.

k	$\lambda(s)$	$\theta_1(s)$	$\theta_2(s)$
1	$\text{sign}(M)s^2 + O(s^3)$	$\phi^{(0)} - L\sqrt{\lvert M\rvert}s + O(s^2)$	$\sqrt{\lvert M\rvert}s + O(s^2)$
2	$s + O(s^2)$	$\phi^{(0)} + [\phi^{(1)} \mp L\sqrt{M}]s + O(s^2)$	$\pm\sqrt{M}s + O(s^2)$
3,5..	$\text{sign}(M)s^2 + O(s^3)$	$\phi^{(0)} + \phi^{(1)}\text{sign}(M)s^2 + O(s^3)$	$\sqrt{\lvert M\rvert}s^k + O(s^{k+1})$
4,6..	$s + O(s^2)$	$\phi^{(0)} + \phi^{(1)}s + O(s^2)$	$\pm\sqrt{M}s^{\frac{k}{2}} + O(s^{\frac{k}{2}+1})$

Note: if $M < 0$ and k is even, then no expansion exists (solution is isolated),
$M = -\dfrac{2(L_1 + L_2)r^{(k)}}{L_1 L_2 k!}, \quad L = \dfrac{L_2}{L_1 + L_2}$

Tab. 4. First non-zero derivatives with respect to time ($\dot{s} =$ free scalar).

k	$x(t)$	$\theta_1(t)$	$\theta_2(t)$
1	$\ddot{x} = 2x'\text{sign}(M)\dot{s}^2$	$\dot{\theta}_1 = -L\sqrt{\lvert M\rvert}\dot{s}$	$\dot{\theta}_2 = \sqrt{\lvert M\rvert}\dot{s}$
2	$\dot{x} = x'\dot{s}$	$\dot{\theta}_1 = [\phi^{(1)} \mp L\sqrt{M}]\dot{s}$	$\dot{\theta}_2 = \pm\sqrt{M}\dot{s}$
3,5..	$\ddot{x} = 2x'\text{sign}(M)\dot{s}^2$	$\ddot{\theta}_1 = 2\phi^{(1)}\text{sign}(M)\dot{s}^2$	$d^k\theta_2/dt^k = k!\sqrt{\lvert M\rvert}\dot{s}^k$
4,6..	$\dot{x} = x'\dot{s}$	$\dot{\theta}_1 = \phi^{(1)}\dot{s}$	$d^{k/2}\theta_2/dt^{k/2} = \pm\frac{k}{2}!\sqrt{M}\dot{s}^{\frac{k}{2}}$

Note: same as for Tab. 3.

Evaluating the derivatives of $\lambda(s)$ and $\theta_2(s)$ at $s = 0$ using the expressions shown in columns 2 and 4 of Tab. 3 we obtain the expressions for $\theta_1(s)$ shown in column 3 of Tab. 3.

5. Admissable End-Effector Rates and Joint Rate Solutions

The low-order expansions given in Tab. 3 can be used to determine the first non-zero derivatives of $x(t)$ and $\theta(t)$ evaluated at the singularity. Evaluating the derivatives of $x(t) = x(\lambda(s(t)))$ and $\theta(t) = \theta(s(t))$ at $s = 0$ based on the expressions given in Tab. 3 we obtain the results summarized in Tab. 4. Table 4 provides expressions for first non-zero derivatives of $x(t)$ and $\theta(t)$ as functions of a free parameter $\dot{s}$.

A given rate of end-effector motion is admissable if and only if the appropriate equation in column 2 of Tab. 4 can be solved for a real value of $\dot{s}$ and any lower-order derivative of $x(t)$ is zero. If a solution for $\dot{s}$ exists, then it can be plugged into the expressions shown in columns 3 and 4 to determine the first non-zero derivatives of $\theta_1(t)$ and $\theta_2(t)$ at the singularity.

6. Conclusions

A method for determining differential kinematics of path tracking has been applied to the outstretched singular configuration of the planar 2R manipulator. Results show that isolated points, turning points, nodes, and cusps can arise in the locus of kinematic solutions depending on the end point's path's degree of contact k with the workspace boundary (Tab. 2). Explicit formulae have been derived (Tab. 3) for the first non-zero terms of the Taylor series expansions of joint displacement curves in the neighbourhood of the singularity, as well as for their first non-zero derivatives with respect to time (Tab. 4).

Physical insight into the different types of singularities can be enhanced by studying the drawings of the manipulator shown in Tab. 2. Paths that have odd degrees of contact k with the workspace boundary do not lie entirely inside the workspace. Therefore the locus of solutions must turn with respect to path parameter λ to form either a turning point ($k = 1$), or a cusp ($k = 3, 5, ...$) when the singularity is encountered.

When the end point's path has an even degree of contact k with the workspace boundary, the path in the neighbourhood of P will lie either entirely outside the workspace ($M < 0$) or entirely inside the workspace ($M > 1$). In the former case (not shown in Tab. 2) the singular solution is isolated. In the latter case the end-effector can reach all positions on the path, except point P, in two different joint configurations (elbow-up and elbow-down). At point P the two branches coincide, forming a node in the locus of joint solutions that map onto the path. This node gives rise to two distinct and smooth joint curves that give rise to two distinct families of joint rate solutions. It is interesting to note that these two curves contact each other with degree $k/2$. A switch from one curve to the other at the singularity implies a discontinuity in derivatives of order $k/2$.

Column 2 of Tab. 4 clearly defines the family of all end point rates that can be physically realized at the singularity. Odd contact k implies that the end point must stop at the singularity, but arbitrary accelerations away from the workspace boundary are allowed. Even k and $M < 0$ imply that the end point can pass the singularity with arbitrary velocity.

This case study of the planar 2R manipulator suggests that path tracking singularities for more general manipulators will also be closely related to the geometry of contact between the end effector path and the workspace boundary. For singularities that occur inside the workspace it may be the path's contact with an interior surface (the image of neighbouring singular configurations) that is important.

References

Kieffer J. (1992): *Manipulator inverse kinematics for untimed end-effector trajectories with ordinary singularities.* — Int. J. Robotics Research, v.11, No.3, pp.225–237.

Kieffer J. (1994): *Differential analysis of bifurcations and isolated singularities for robots and mechanisms.* — IEEE Trans. Robotics and Automation, v.10, No.1, pp.1–10.

Tchoń K. (1991): *Differential topology of the inverse kinematic problem for redundant robot manipulators.* — Int. J. Robotics Research, v.10, No.5, pp.492–504.

Tchoń K. (1992): *Kinematic singularities of robot manipulators: a case study.* — unpublished report.

Tchoń K. and Urban P. (1992): *Classification of kinematic singularities in planar robot manipulators.* — Systems & Control Letters, v.19, No.4, pp.293–302.

Received: December 30, 1994