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OPTIMAL CONTROL OF FAIR DISCRETE-EVENT SYSTEMS

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In this paper, we formulate and solve a class of optimal control problems for discrete-event systems modelled as finite-state automata. The controlled discrete-event system is considered optimal if the performance index consisting of a behaviour gain and a control cost is maximized, plus that the controller satisfies a structural constraint. A hybrid computational algorithm is devised to reduce the solution space for the optimal controller design. An example is given to illustrate the results.

1. Introduction

Recently there have been several studies on optimal supervisory control of logical Discrete-Event Systems (DESs). Passino and Antsaklis (1989) discussed the following problem: for a given system G and a set of marking states, find a minimal cost path that leads from the initial state to the set of marking states. Sengupta and Lafortune (1991) studied another optimal control problem in which both the control cost and the trajectory cost are considered, employing the traditional idea of linear quadratic (LQ) control for linear time-invariant systems. Kumar and Garg solved an optimal control problem, in which the cost function was the sum of a positive gain (desired behaviour) and a negative cost (undesired behaviour as well as the control cost), by using the technique of network flow (Kumar, 1991; Kumar and Garg, 1995). Li (1991) used the A^* algorithm to deal with the optimal control of a class of DESs. A hybrid computation algorithm for finding the optimal state feedback control for a class of optimal control problems was given in (Li, 1993).

Our point of view regarding the optimal control of DESs is that in formulating optimal control problems for DESs a trade-off between the gain (the reachable closed-loop performance) and the cost (the control effort in order to reach such a closed-loop performance) should be made in order to keep consistence with the minimally restrictive control by Ramadge and Wonham (1987). Toward this end we will follow the ideas of Kumar and Garg (Kumar, 1991; Kumar and Garg, 1995). However, our treatment of the optimal control problem will be different in the following aspects. First, instead of enumerating desired and undesired states, we explicitly impose a closed-loop behaviour specification (without loss of generality, we restrict ourselves to the predicate control problem (Lin *et al.*, 1988)). Second, instead of assuming the

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"one-time cost", we explicitly impose a structural constraint on the controller (that is, the controller should be global, see Section 2), and we require that all the closed-loop system state trajectories are fair, a concept that we will formalize in Section 2. Third, we use a hybrid algebraic/computation approach instead of the network flow technique in finding the optimal controls.

This paper is organized as follows. In Section 2, we formulate the global predicate control problem and formalize the concept of fair-in-state. In Section 3, we construct the performance index. From this index, we find the optimal control from elements in the solution space in the global predicate control problem. In Section 4, we propose a hybrid computational procedure to find the optimal control. We include an example to illustrate the design procedure in Section 5. Finally in Section 6, we give a conclusion.

2. The Global State Feedbacks

Let $G = (\Sigma, Q, \delta)$ be the controlled plant, where Σ is the finite set of events, $Q = \{q_1, q_2, \ldots, q_N\}$ is the finite set of states, $\delta : Q \times \Sigma \to Q$ is the state transition map, which is partial (Hopcroft and Ullmann, 1979). We leave the initial state unspecified. Instead, we generalize the state transition map to $\delta_j : Q \times \Sigma^* \to Q$ $(j = 1, 2, \ldots, N)$ such that $\delta(q_j, s)$ is defined for $s \in \Sigma^*$, where Σ^* is the Kleene closure. The language generated by $G_j = (\Sigma, Q, \delta, q_j)$ is denoted by $L_j(G) = \{s \mid s \in \Sigma^*, \delta_j(q_j, s)$ is defined$\}$, where G_j is a realization of G when the initial state is q_j. The event set Σ is decomposed into two disjoint subsets $\Sigma = \Sigma_c \cup \Sigma_u$. Events in Σ_c are controllable while events in Σ_u are uncontrollable. The active event set $\Sigma(q)$ represents all $\sigma \in \Sigma$ such that $\delta(\sigma, q)$ is defined for a state $q \in Q$. A subset γ of $\Sigma(q)$ is called a control pattern at q if $\Sigma_u \cap \Sigma(q) \subset \gamma \subset \Sigma(q)$. Let Γ be the set of all control patterns. Then $\Gamma = \{\gamma \mid \gamma \in 2^{\Sigma}\}$, where 2^{Σ} is the power set of Σ. The DES G equipped with Γ is called the controlled DES. Control of DES G is realized by repeatedly changing the control patterns according to the observation of the operation history of the system. The switching of control patterns is made so as to guarantee that the resultant closed loop system operates at a prespecified permissive region. Such a permissive region can be specified either as a desired closed-loop language or a predicate (Ramadge and Wonham, 1987a; 1987b). In this paper we focus our attention on the predicate-type specification. It has been shown that for this kind of specification a state feedback control is sufficient to realize the control task (Lin *et al.*, 1988; Ramadge and Wonham, 1987b; Ushio, 1989). It has also been shown that state feedback control is superior to the supervisory control (event feedback) for predicate-type specification in the sense that even the initial state is uncertain inside $T(P)$ $(T(P) = \{q \mid q \in Q, P(q) = 1\})$, and the same state feedback controller can accomodate the related control task (Ushio, 1989).

Let $f : Q \times \Sigma \to \{0, 1\}$ be a state feedback control. For a given G, $q \in Q$, an event $\sigma \in \Sigma(q)$ is said to be enabled by f if $f(q, \sigma) = 1$. Otherwise, if $f(q, \sigma) = 0$, then σ is disabled by$f$. According to the definition of control patterns, $\forall \sigma \in \Sigma_u \cap \Sigma(q)$, $f(q, \sigma) = 1$. The closed-loop system G^f, i.e. G under control of f, is another automaton $G^f = (\Sigma, Q^f, \delta^f)$ such that $Q^f \subset Q$ and

$$\delta^f(q,\sigma) = \begin{cases} \delta(q,\sigma) & \text{if } f(q,\sigma) = 1 \\ \text{undefined} & \text{otherwise} \end{cases}$$

The state feedback control f is said to solve P if $Q^f \subset T(P)$. It has been shown that there is a state feedback control f which solves P if and only if the system starts at $q \in T(P)$ and P is control-invariant with respect to f (Ramadge and Wonham, 1987b; Ushio, 1989), i.e.

$$P \leq wlp^f_\sigma(P) \qquad \forall \sigma \in \Sigma$$

where

$$wlp_\sigma(P)(q) = \begin{cases} 1 & \text{if } \delta(q,\sigma) \text{ is defined and } \delta(q,\sigma) \in T(P) \\ & \text{or } \delta(q,\sigma) \text{ is undefined} \\ 0 & \text{otherwise} \end{cases}$$

$$wlp^f_\sigma(P)(q) = wlp_\sigma(P)(q) \vee \overline{f(q,\sigma)} \qquad \forall q \in Q$$

and $\overline{f(q,\sigma)}$ is the usual negation operation in Boolean algebra. It has been shown (Ramadge and Wonham, 1987b) that P is control-invariant with respect to f if and only if P is Σ_u-invariant, i.e.

$$P \leq wlp_\sigma(P) \qquad \forall \sigma \in \Sigma_u$$

A state feedback control f_a is said to be more permissive than f_b for P if $f_b(q,\beta) \leq f_a(q,\beta)$ $\forall q \in Q$ and $\beta \in \Sigma(q)$. For a given control-invariant predicate P there is a unique maximal permissive state feedback control f^* that solves P (Ramadge and Wonham, 1987b).

As shown by Li (1991), even for a control-invariant predicate, not all states specified by this predicate are reachable from the given initial state, i.e. $R(L(G^f_j)) \subseteq T(P)$ with respect to q_j where $R(L(G^f_j)) = \{q \mid \exists s \in \Sigma^*, \delta(q_j, s) = q\}$.

In order to characterize the controllability of predicate, i.e. a predicate for which all specified states can be reached from the initial state, a set of states $Re(G_j, P)$ is recursively defined as follows (Li, 1991):

1. $q_j \in Re(G_j, P)$;
2. If $q_i \in Re(G_j, P)$, $\delta(q_i, \sigma)$ is defined and $\delta(q_i, \sigma) \in P$ for $\sigma \in \Sigma$, then $\delta(q_i, \sigma) \in Re(G_j, P)$;
3. Every state in $Re(G_j, P)$ is obtained as in (1) or (2).

With a little abuse of notation, in the sequel $Re(G_j, P)$ also denotes a predicate defined on Q. $\forall q \in Q$, $Re(G_j, P)(q) = 1$ if q is reachable from q_j via legal states specified by P.

Definition 1. (Li, 1991) Given DES G and a predicate P defined on Q, P is controllable with respect to G_j if

$$P \leq Re(G_j, P) \wedge wlp_\sigma(P) \qquad \forall \sigma \in \Sigma_u$$

The following theorem relates a state feedback control f and a controllable predicate P.

Theorem 1. (Li, 1991) *Let P be a predicate defined on Q and $q_j \in T(P)$. Then there is a state feedback f such that $R(L(G_j^f)) = T(P)$ if and only if P is controllable with respect to G_j.*

However, even with a controllable predicate P (with respect to G_j), if the system starts at $q_i \in T(P)$, $q_i \neq q_j$, then it is not necessary that all the states in $T(P)$ are reachable from q_i under the same control f as given in the above theorem. In order to solve this problem, a concept called the absolutely controllable predicate is introduced in (Li, 1995).

Definition 2. (Li, 1995) Suppose $T(P) = \{q_1, q_2, \ldots, q_M\}$. The corresponding predicate P is called absolutely controllable if

$$P \leq \wedge_{i=1}^{M} Re(G_i, P) \wedge wlp_\sigma(P) \qquad \forall \sigma \in \Sigma_u$$

Remark 1. It can be verified that a predicate P is absolutely controllable if and only if P is control-invariant and the states in $T(P)$ are mutually reachable.

In order to characterize the absolutely controllable predicate, a class of automatons is defined as follows.

Definition 3. A DES $G = (\Sigma, Q, \delta)$ is called strongly connected if $\forall q_i, q_j \in Q$, $\exists s, t \in \Sigma^*$ s.t. $\delta(q_i, s) = q_j$ and $\delta(q_j, t) = q_i$.

Definition 4. A state feedback f is called global if and only if G^f is strongly connected.

In the sequel we abbreviate the Global State Feedbacks to GSF.

Definition 5. A DES is said Fair-In-States if it is strongly connected.

Actually, for a strongly-connected DES, given any state trajectory $s_t = q^1 q^2 \ldots q^n$, we see that $\forall q_j \in Q$, as $n \to \infty$, q_j appears on the state trajectory infinitely often, provided that the underlying transition probablity from one state to any other state is not zero.

With the above notations, the following problem is formulated (Li, 1995):

Global Predicate Control Problem (GPCP): *Given any G and predicate P defined on Q, find a GSF f such that $R(L(G_j^f)) = T(P)$ $\forall q_j \in T(P)$.*

The following result gives the necessary and sufficient condition for the solvability of GPCP.

Theorem 2. (Li, 1995) *For a given DES G and a predicate P defined on Q, the GPCP is solvable if and only if P is absolutely controllable.*

In order to deal with the connectivity of the state space, we give the definition of the set of coreachable states in G with respect to P and $q_j \in T(P)$.

Definition 6. (Li, 1995) The set of coreachable states in G with respect to P and $q_j \in T(P)$, denoted by $Cre(G_j, P)$, is defined recursively as follows:

1. $q_j \in Cre(G_j, P)$;
2. If $q^x \in Cre(G_j, P)$, and $\exists \sigma \in \Sigma$ such that $q^x = \delta(q^y, \sigma)$, $q^y \in T(P)$, then $q^y \in Cre(G_j, P)$;
3. All states in $Cre(G_j, P)$ are obtained as in steps (1) and (2).

As shown in (Li, 1995), for a given predicate, the state space of the controlled DES can actually be partitioned into equivalence classes with respect to the reachable/coreachable relations. Consider a DES G and a predicate P defined on Q. Two states $q_i, q_j \in Q$ are said to be satisfying the relation $AC(P)$ if any one of the following conditions holds:

1. $q_i, q_j \notin T(P)$;
2. $q_i, q_j \in T(P)$, and $(q_i \in Re(G_j, P)) \wedge (q_i \in Cre(G_j, P))$.

Proposition 1. (Li, 1995) *$AC(P)$ is an equivalence relation.*

Remark 2. The relation $AC(P)$ associated with a given predicate P actually partitions the state space into different equivalence classes. Since no control-invariant property is concerned, any equivalence class may not correspond to a global state feedback. Fortunately, according to (Li, 1995), it is always possible to find the maximal elements corresponding to every equivalence class of states partitioned by $AC(P)$.

Based on the above discussion, for any predicate P, we can first partition the state space of G into different equivalence classes, i.e. $Q = Q_1 \cup Q_2 \ldots \cup Q_{|AC(P)|}$. For any specific equivalence class, we can find the set of absolutely controllable state subsets contained in it. Consequently, the corresponding global state feedback can be constructed.

The following result characterizes the number of possible GSF's.

Proposition 2. *We have $2^{|Q|} \geq |AC(P)| \times 2^{|Q_{\max}|} \geq \Sigma_{i=1}^{|AC(P)|} 2^{|Q_i|}$, where $|AC(P)|$ is the number of elements in the equivalence class $AC(P)$, and $Q_{\max} = \max_{i=1,\ldots,|AC(P)|} Q_i$.*

The proof consists in using the induction method. We omit it here.

Our interest here is, what is the optimal GSF if a performance index is given along with the GPCP? This is the topic of the next section.

3. The Optimal GSF

3.1. Performance Index

Our interest is to find a trade-off between the behaviour gain E_1 and the related control cost E_2. This is essentially the same as in (Kumar and Garg, 1995). First, let us construct the performance index E.

3.1.1. Choice of E_1

For our problem as mentioned above, the minimally restrictive control in the sense of Ramadge and Wonham is the state feedback f such that $|Q^f|$ is the largest while the predicate control task is realized.

In order to accommodate this observation, let

$$E_1 = \sum_{j=1, f\in F}^{|Q^f|} K_1(q_j)$$

Here Q^f is a subset of Q, $K_1 : Q \to R^+ \cup \{0\}$ is a mapping, and F is the set of all state feedbacks.

3.1.2. Choice of E_2

The criterion for determining E_2 is to consider the control cost for every disabling action issued by the state feedback f. Formally, it is defined as

$$E_2 = \sum_{j=1}^{|\Sigma|} K_2(l_j \sigma_j) = \sum_{j=1}^{|\Sigma|} l_j K_2(\sigma_j)$$

where l_j is the number of times the event σ_j is disabled, and

$$K_2(\sigma_j) = \begin{cases} -\infty < N_j < 0 & \text{if } \sigma_j \in \Sigma_c \\ -\infty & \text{otherwise} \end{cases}$$

where N_j is a negative real number.

Remark 3. In our performance index, the "gain" E_1 corresponds exactly to the activities that the system can perform. Recalling the results in (Lin, 1991), maximizing E_1, i.e. to let the closed loop system behaviour be as large as possible, is equivalent to let the system run the fastest when certain fairness condition of the system is satisfied. So our formulation here can be considered as a trade-off between the system operation speed and the control cost while the given supervisory control task is realized. Notice that the fairness condition in (Lin, 1991) is different from the FIS condition we stated above.

The optimal control problem can now be framed as follows.

Optimal State Feedback Control Problem (OSFCP): *For a DES G and a predicate P, find a state feedback f such that*

1. *f is a GSF that solves P;*
2. *$E = E_1 + E_2$ is maximized with respect to G^f.*

In the above problem, (1) is the logic specification for the closed-loop system as well as the structural constraint for the related controller. The logic specification is the control task the controller is supposed to realize. The structural constraint on f (i.e. it is a GSF) is employed mainly to clarify the "one-time cost " as assumed in (Kumar and Garg, 1995). Let us just look at one example. Suppose G is the controlled automaton with $Q = \{q_1, q_2, q_3\}$ with the property that once the controlled system starts at q_1, it will leave q_1 and never comes back, that is, q_1 will be a "transient" state while q_2, q_3 are "steady-state" states. It is unlikely that we should pay the same attention to q_1 as well as to q_2, q_3. The same idea applies to the control cost E_2. This is why we impose the GSF constraint on the controller f.

3.2. The Set of GSFs

Definition 7. Let $G = (\Sigma, Q, \delta), S = (\Sigma, X, \eta)$. S is called a subautomaton of G if

1. $X \subset Q$;
2. $\eta = \delta \mid_X$.

Here $\delta \mid_X$ is the restriction of δ on X.

Based on the definition of controllability of languages (Ramadge and Wonham, 1987a), fix $q_j \in Q$, for $L \subset L(G_j)$. Then L is called controllable with respect to $(L(G_j), \Sigma_u)$ if

$$\bar{L}\Sigma_u \cap L(G_j) \subset \bar{L}$$

where $\bar{L}$ is the prefix closure of L.

Definition 8. A subautomaton $S = (\Sigma, X, \eta)$ of G is called realizable if $\forall q_j \in X$, $L(S_j)$ is controllable with respect to $(L(G_j), \Sigma_u)$.

In fact, a realizable subautomaton S of G is obtained by only removing some controllable edges from G and at the same time the states in S are isolated from the states in $G - S$.

In the sequel, we denote the set of realizable subautomata of G by G^R.

Definition 9. For a realizable subautomaton $S \subset G^R$, $B(S) = \{x \mid x \in X, \exists \sigma \in \Sigma$ s.t. $\delta(q, \sigma)$ is defined but $\delta \mid_X$ is undefined$\}$ is called the set of boundary states.

It is trivial to have the following result.

Proposition 3. *Any GSF corresponds to a strongly-connected realizable subautomaton of G.*

Let us suppose that S is a strongly-connected realizable subautomaton of G corresponding to the GPCP, the associated GSF is f_S. Numbering the states in Q as $q_1, q_2, \ldots, q_{|Q|}$ and events in Σ as $\sigma_1, \ldots, \sigma_{|\Sigma|}$, we can build a control matrix as follows:

$$C(\Sigma, Q) = \begin{bmatrix} * & * & * & \cdots & * \\ * & * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ * & * & * & \cdots & * \end{bmatrix}_{|\Sigma_c| \times |Q|}$$

where the elements are the values of $\overline{f(q_j, \sigma_i)}$ for $i \in \overline{(1, |Q|)}$, $j \in \overline{(1, |\Sigma|)}$. If $f(q_j, \sigma_i)$ is undefined, the corresponding entry in $C(\Sigma, Q)$ is set to "0".

Recalling the definition of $B(S)$, we have

$$C_B(\Sigma, Q) = \begin{bmatrix} * & * & \cdots & * \\ * & * & \cdots & * \\ \vdots & \vdots & \vdots & \vdots \\ * & * & \cdots & * \end{bmatrix}_{|\Sigma_c| \times |B(S)|}$$

The corresponding column is denoted by C_i $(i = 1, 2, \ldots, |B(S)|)$.

Definition 10. $\Delta_S = C_1 + \cdots + C_{|B(S)|}$ is called the extended control vector.

In fact, the extended control vector is the sum of all the control actions of the GSF for realizing S.

Sometimes, we can compare GSFs without explicitly stating the performance index.

Proposition 4. *For $S_1, S_2 \in G^R$, if $S_1 \subset S_2$ and $\Delta_{S_1} \geq \Delta_{S_2}$, then $E(S_1) \leq E_(S_2)$.*

Proof. We have $S_1 \subset S_2 \to E_1(S_1) \leq E_1(S_2)$. Moreover, $\Delta_{S_1} \geq \Delta_{S_2} \to \forall \sigma_j \in \Sigma_c$, $\Delta^j_{S_1} \geq \Delta^j_{S_2}$, where $\Delta^j_{S_1}(\Delta^j_{S_2})$ is the j-th row of $C_B(\Sigma, Q_1)$ $(C_B(\Sigma, Q_2))$. $\Delta^j_{S_1} \leq \Delta^j_{S_2}$ implies that in realizing S_1, the number of disabling actions for σ_j is greater than that in realizing S_2, or equivalently, $l^1_j \geq l^2_j$. Since $\Delta_{S_1} \geq \Delta_{S_2}$, we have $l^1_i \geq l^2_i (\forall i)$. Recalling the definition of E_2, we have $E_2(S_1) \leq E_2(S_2)$. Combining the fact that $E_1(S_1) \leq E_1(S_2)$, we have the claim verified. ■

A subsequent development of the above proposition is as follows.

Corollary 1. *For $S_1, S_2 \in G^R$, if $S_1 = S_2$ and $\Delta_{S_1} \geq \Delta_{S_2}$, then $E(S_1) \leq E_(S_2)$.*

The above corollary can be explained as follows. Given a DES G, predicate P and performance index E, if two GSFs give the same closed-loop state space, but one takes less control than the other, then the former is better according to the performance index E. The state feedback that is minimally restrictive is called parsimonious in (Kumar and Garg, 1995). In fact, we can have another result parallel to the above corollary.

Corollary 2. *For the OSFCP problem, there is at least a maximal solution in the sense of Ramadge and Wonham.*

Proof. We construct the subset F^{RW} of F as follows: Select an $f \in F$, its corresponding realized subautomaton is S, its extended control vector is Δ_s. Control $f_j \in F^{RW} (j = 1, 2, \ldots, m)$ if and only if the realizable subautomaton corresponding to f_j is S_j with the property that $\Delta_{S_j} = \Delta_S$ $(j = 1, 2, \ldots, m)$. We then have $E(S_k) \leq E(S_n)(k, n \in (1, 2, \ldots, m))$ if and only if $f_k \leq f_n$. That is, for the chain of permissive state feedbacks $f_i \in F^{RW}$, there is a maximal element (by Zorn's lemma).

4. Synthesis Procedure

In this section, we present an algorithm for solving the OSFCP. This algorithm is of hybrid nature. We first reduce the solution space to a sufficiently small subspace by employing an algebraic operation on the set of all GSFs and then use the brute force computation to get the optimal control. The algorithm is as follows:

ALGORITHM "HYBRID"

1. Partition the state space Q according to the relation $AC(P)$. Choose one element, say Q_1, in $AC(P)$;
2. For all $f_i \in F_1$ $(i = 1, 2, \ldots, |F_1|)$, where F_1 is the set of GSFs with respect to AC_1, compute its corresponding extended control vector Δ_{f_i};
3. If for some $i, j, \Delta_{f_i} = \Delta_{f_j}$, then f_i and f_j are equivalent with respect to Δ. The equivalence classes in F_1 are $(F^1, F^2, \ldots, F^M)$;
4. For all $i \in (1, 2, \ldots, M)$, compute the maximal elements in F^i under the relation "$\leq$"; Denote the set of these elements by F_1^{reduced};
5. For the set of state feedbacks in F_1^{reduced}, for $(i \neq j)$ $i, j \in (1, 2, \ldots, M)$, if $f \in F^j$, $g \in F^i, g \leq f$ and $\Delta^j \geq \Delta^i$, then remove g from the solution space. Denote the resultant GSFs by $F_1^{0,\text{reduced}}$;
6. Find $\max_{f \in F_1^{0,\text{reduced}}} E$ via computation.

For this algorithm, we have the following result.

Proposition 5. *If Σ, and Q are finite, then the algorithm HYBRID gives the optimal solution in finite steps.*

Proof. Since Q is finite, $AC(P)$ contains at most $|Q|$ elements, so Step-1 can be done in finite steps.

Notice that in Step 2, $Q_1 \subset Q$ so $|Q_1| < |Q|$. Since we have at most $2^{|Q_1|}$ elements in F_1, computation in Step 2 can be done in finite steps. As long as F_1 is finite, computations in Steps 3–5 can be done in finite steps. Finally, notice that

$F_1^{0,\text{reduced}} \subset F_1^{\text{reduced}} \subset F_1$, and that F_1 is finite, so the computation in Step 6 can be done in finite steps. ■

5. Example

Consider the system depicted in Fig. 1. Let $T(P) = Q - \{5\}$. We can find that $AC(P) = \{1,2,3,4\}, \{5\}, \{6\}$. Suppose all the events are controllable. Let us concentrate on the equivalence class $\{1,2,3,4\}$ in $AC(P)$.

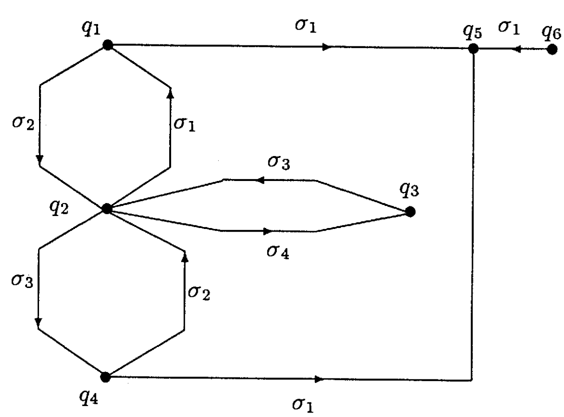

Fig. 1. The controlled plant.

We enumerate all the GSFs and their corresponding extended control vectors as follows. For the state feedback f_1, the corresponding state subset is $\{1,2,3,4\}$:

$$f_1 : C_{\{1,4\}}^{\{1,2,3,4\}}(\Sigma, Q) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta_1 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Note that $C_B^A(\Sigma, Q)$ stands for the control matrix for the state subset A with the corresponding boundary state subset B.

Similarly, we have

$$f_2 : C_{\{2,4\}}^{\{2,3,4\}}(\Sigma, Q) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \Delta_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$f_3 : C^{\{1,2,3\}}_{\{1,2\}}(\Sigma, Q) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \Delta_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$f_4 : C^{\{1,2,4\}}_{\{1,2\}}(\Sigma, Q) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad \Delta_4 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$f_5 : C^{\{1,2\}}_{\{1,2\}}(\Sigma, Q) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}, \qquad \Delta_5 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$f_6 : C^{\{2,3\}}_{\{2\}}(\Sigma, Q) = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \Delta_6 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$f_7 : C^{\{2,4\}}_{\{2,4\}}(\Sigma, Q) = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \end{bmatrix}, \qquad \Delta_7 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Using the algorithm HYBRID, we simply get the following elimination process:

1. f_2 should be eliminated (compared with f_1);
2. f_4 should be eliminated (compared with f_1);
3. f_5 should be eliminated (compared with f_3);
4. f_6 should be eliminated (compared with f_3);
5. f_7 should be eliminated (compared with f_1).

Thus, we have just two feedbacks left: f_1 and f_3. Once we have the performance index E, we can decide which GSF is better.

6. Conclusion

An optimal control problem is formulated and solved in this paper. It has the following features: the performance index is a trade-off between the control cost and behaviour gain; the control specification is explicitly given, and a structural constraint is imposed on the candidate optimal controllers so as to clarify the assumption of one-time cost; the synthesis algorithm is of interest in the sense that the required numeric calculation can be partly reduced by qualitative reasoning.

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