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BASIC SYSTEMS THEORY FOR DISCRETE LINEAR REPETITIVE PROCESSES USING 2D ROESSER MODEL INTERPRETATIONS

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Previous work has shown that the stability conditions for discrete linear repetitive processes and 2D linear systems recursive in the positive quadrant can be tested using the same tests. This does not provide a suitable basis for studying the application of 2D linear systems theory to key, currently open, systems theoretic questions for discrete linear repetitive processes, such as e.g. what (if anything) is meant by reachability/controllability and observability and how these properties are characterised. The objective of this paper is to develop 2D systems models for the repetitive processes which remove this difficulty. Here the main results are a range of 2D linear systems models, with particular emphasis on the well-known Roesser model structure, a proof of stability equivalence, and some key results regarding reachability.

1. Introduction

Repetitive, or multipass, processes constitute a class of 2D linear systems whose unique characteristic is a series of sweeps, called passes, through a set of dynamics defined over a fixed finite duration known as the pass length. On each pass, an output, called the pass profile, is produced which acts as a forcing function on, and hence contributes to, the next pass profile. Industrial examples include long-wall coal cutting and metal-rolling operations (Smyth, 1992) and algorithmic examples include classes of iterative learning control schemes (Amann *et al.*, 1996).

In effect, the 2D systems structure of these processes arises from the need to use two co-ordinates to specify a variable, i.e. the pass number or index $k > 0$ and the position 't' along a given pass which is of finite duration by definition. Hence repetitive processes are a class of 2D systems where the duration of information propagation in one direction (along the pass) is finite and infinite in the other (pass to pass).

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The basic unique control problem is that the output sequence can contain oscillations that increase in amplitude in the pass-to-pass direction. Such behaviour is easily generated in simulation studies and in experiments on scaled models of industrial examples (Edwards, 1974; Smyth, 1992).

The most obvious approach to control this behaviour is to "join" successive pass profiles end to end to form the so-called infinite length single pass equivalent and then apply directly standard (1D) techniques. For example, if the dynamics are linear and the process is single-input/single-output (SISO), then the "classical" Nyquist theory could be applied (Edwards,1974).

In general, this strategy will fail (see (Smyth, 1992) for a complete treatment) since it ignores completely their inherent 2D systems structure. The previous work (Rogers and Owens, 1992) has developed a rigorous stability theory for linear constant pass length processes based on an abstract model in a Banach space setting which includes all such processes as special cases. Physically this theory can be interpreted (in terms of the underlying function space) as bounded-input/bounded-output (BIBO) stability independent of the pass length, also known as stability along the pass.

The results of applying this abstract theory to a wide range of sub-classes have been reported e.g. in (Rogers and Owens, 1992; Smyth, 1992). One of them is the subclass of so-called discrete processes which constitute the subject of this paper. Smyth (1992) details how one such set of tests can be implemented in a computer-aided analysis environment using, in effect, standard linear systems tests.

Discrete linear repetitive processes exhibit strong structural links with 2D linear systems described by the well-known Roesser (1975) model (or equivalents), see (Rocha *et al.*, 1996) for more details on this subject. This raises a possibility of using well-established 2D systems theory to answer basic systems theoretic (and controller design) questions for discrete linear repetitive processes for which few or no results are currently available. By analogy with standard (or conventional) linear systems, where strong structural links are also present, one such general question is: What (if anything) is meant by reachability/controllability and observability for these processes and how can such properties be characterised both theoretically and in form of computationally feasible tests?

The previous work (Rocha *et al.*, 1996) has shown that stability tests can be interchanged between these two (apparently distinct) areas. In particular, the stability conditions for these two classes of linear systems can be tested by applying the same tests. This, however, does not provide a suitable basis on which to study the application of 2D systems theory to currently open systems theoretic questions for discrete linear repetitive processes. The first major result of this paper removes this basic difficulty by showing that several equivalent 2D systems representations of the dynamics of these repetitive processes exist. These include both regular (or nonsingular) and singular models.

Based on this observation, it is shown that the stability theories for these two classes of systems are formally equivalent. Then using a regular Roesser model of the systems dynamics, a state transition matrix for discrete linear repetitive processes is developed. This leads to some fundamental results on reachability. Finally, some on-going research is briefly discussed.

2. Background

This section summarises the required results from the representation and stability theory based on the abstract model. A complete treatment, including proofs, can be found in (Rogers and Owens, 1992).

Suppose that E_α is a Banach space, W_α a linear subspace of E_α, and Y_k the pass profile produced on pass k by a linear repetitive process. Let us denote by $\alpha < +\infty$ the pass length and introduce a bounded linear operator $L_\alpha \in B(E_\alpha, E_\alpha)$. Then the dynamics of a linear repetitive process generated over a fixed pass length $\alpha < +\infty$, denoted by $S(E_\alpha, W_\alpha, L_\alpha)$, are described by linear recursion relations of the form

$$Y_{k+1} = L_\alpha Y_k + b_{k+1}, \quad k \geq 0 \tag{1}$$

Here $L_\alpha Y_k$ denotes the contribution of pass k to pass $k+1$ and $b_{k+1} \in W_\alpha$ represents initial conditions, disturbances and control input effects on pass $k+1$.

In order to avoid a source of confusion due to terminology, it is important to distinguish between repetitive processes and the general area of repetitive control systems. Here the term "repetitive" refers to the explicit interaction between successive pass profiles/outputs of an uncontrolled process to generate a sequence of pass profiles over the fixed finite pass length, i.e. the key unique feature of a repetitive process. The central idea of repetitive control (see e.g. Yamamoto, 1993) is that instead of teaching a system exactly how to behave, it is given a reference signal and learns the necessary control action using a suitable regulation mechanism.

The first form of stability for $S(E_\alpha, W_\alpha, L_\alpha)$ is termed asymptotic stability. In effect, it demands that "bounded disturbance sequences" produce bounded sequences of pass profiles (in the sense of the underlying function space norm) over a fixed finite pass length. One of several equivalent characterisations is the following:

Definition 1. $S(E_\alpha, W_\alpha, L_\alpha)$ is said to be *asymptotically stable* provided that there exist finite real scalars $M_\alpha > 0$ and $\lambda_\alpha \in (0,1)$ such that

$$\|L_\alpha^k\| \leq M_\alpha \lambda_\alpha^k, \quad k \geq 0 \tag{2}$$

where $\|\cdot\|$ denotes the norm on E_α.

Theorem 1. *$S(E_\alpha, W_\alpha, L_\alpha)$ is asymptotically stable if and only if*

$$r(L_\alpha) < 1 \tag{3}$$

where $r(\cdot)$ denotes the spectral radius.

To provide information on transient behaviour (in the pass to pass, i.e. k, direction), the limit profile is defined and characterised as follows.

Definition 2. Suppose that $S(E_\alpha, W_\alpha, L_\alpha)$ is asymptotically stable and let $\{b_k\}_{k\geq 1}$ be a disturbance sequence that converges strongly to a disturbance b_∞ in W_α. Then the strong limit

$$Y_\infty := \operatorname*{LIMIT}_{k \to +\infty} Y_k \tag{4}$$

is called the *limit profile* corresponding to $\{b_k\}_{k\geq 1}$.

Theorem 2. *Let the conditions of Definition 2 hold. Then the corresponding limit profile is the unique solution of the linear equation*

$$Y_\infty = L_\alpha Y_\infty + b_\infty \tag{5}$$

The existence of Y_∞ is always guaranteed by asymptotic stability but, since α is finite by definition, this property does not guarantee that the limit profile has "acceptable" (in a well-defined sense) dynamics along the pass. An example of this fact is given in Section 3. Applications do exist where asymptotic stability is all that is required (see (Amann *et al.*, 1996) for one such case). In general, however, asymptotic stability must be strengthened to avoid such problems, resulting in so-called stability along the pass. Basically this demands that bounded disturbance sequences produce bounded sequences of pass profiles independent of the pass length and can be characterised as follows:

Definition 3. $S(E_\alpha, W_\alpha, L_\alpha)$ is said to be *stable along the pass* provided that there exist finite real scalars $M_\infty > 0$ and $\lambda_\infty \in (0,1)$ independent of α such that

$$\left\|L_\alpha^k\right\| \leq M_\infty \lambda_\infty^k, \quad k \geq 0 \tag{6}$$

Theorem 3. $S(E_\alpha, W_\alpha, L_\alpha)$ *is stable along the pass if, and only if*

$$r_\infty := \sup_{\alpha>0} \ r(L_\alpha) < 1 \tag{7}$$

and

$$M_0 := \sup_{\alpha>0} \ \sup_{|z|=\lambda} \ \left\|(zI - L_\alpha)^{-1}\right\| < +\infty \tag{8}$$

for some real number $\lambda \in (r_\infty, 1)$.

The first condition here shows that asymptotic stability $\forall\, \alpha > 0$ is a necessary condition for stability along the pass. Also condition (8) does, of course, imply condition (7). In a large number of cases considered to-date, however, (7) has proved much easier (in relative terms) to interpret. This is the reason for retaining their separate identities here.

3. Discrete Linear Processes—2D Models and Stability

The state-space model of the sub-class of the so-called discrete unit memory linear repetitive processes has the structure

$$\begin{cases} X(k+1,t+1) = AX(k+1,t) + BU(k+1,t) + B_0Y(k,t) \\ Y(k+1,t) = CX(k+1,t) + DU(k+1,t) + D_0Y(k,t) \end{cases} \tag{9}$$

Here, on pass $k, X(k,t)$ is the $n{\times}1$ state vector at sample t along the pass length $\alpha < +\infty, Y(k,t)$ is the $m{\times}1$ vector pass profile, and $U(k,t)$ is the $l{\times}1$ vector of control inputs. It is a straightforward task to write (9) in form $S(E_\alpha, W_\alpha, L_\alpha)$, where E_α is the Banach space of bounded continuous mappings from the finite integer set $0 \le i \le \alpha$ into the vector space C^m of complex m-vectors with norm

$$\|Y\| := \max_{0 \le i \le \alpha} \|Y(i)\|_c \tag{10}$$

where $\|\cdot\|_c$ is any convenient norm in C^m. The defining equation for L_α is

$$(L_\alpha Y) = \begin{cases} D_0 Y(i): \quad i = 0 \\ D_0 Y(i) + \sum_{l=0}^{i-1} C A^{i-l-1} B_0 Y(l): \quad 0 \le t \le \alpha \end{cases} \tag{11}$$

When omitting the output equation (which has no role in this work), the Roesser model for 2D linear systems recursive in the positive quadrant has the structure

$$\begin{cases} X_h(k+1,t) = A_1 X_h(k,t) + A_2 X_v(k,t) + B_1 U(k,t) \\ X_v(k,t+1) = A_3 X_h(k,t) + A_4 X_v(k,t) + B_2 U(k,t) \end{cases} \tag{12}$$

Here k and t are respectively the (integer-valued) horizontal and vertical coefficients, X_h is the $n \times 1$ vector of horizontally transmitted information, X_v is the $m \times 1$ vector of vertically transmitted information, and U is the $l \times 1$ vector of control inputs. Also it is clear by inspection that (12) has strong structural links with the repetitive process state space model of (9) and this has led to the claim that (9) is, in fact, a Roesser model (Rocha *et al.*, 1996).

This claim is based on interpreting the state vector of (9) $\{X\}$ as horizontally transmitted information and the pass profile $\{Y\}$ as vertically transmitted information. In fact, however, a number of key structural differences exist between the repetitive process state-space model of (9) and the 2D Roesser state-space model of (12). They are discussed in detail in the work (Gałkowski *et al.*, 1995) which also proposes that the most appropriate point to start the development of a 2D systems state-space description for linear repetitive processes described by (9) is from a Fornasini-Marchesini type model (see e.g. Fornasini and Marchesini, 1978; Kaczorek, 1985).

Fornasini-Marchesini models of 2D linear systems of systems recursive in the positive quadrant do not split the state vector into horizontal and vertical components, i.e. X_h and X_v, respectively, in (12), which is the key feature of the Roesser model. Here again the output equation is not required and with $Z(k,t)$ denoting the state at point (k,t), $k \ge 0$, $t \ge 0$, the general version of this model has the structure

$$\begin{aligned} EZ(k+1,t+1) &= A_5 Z(k,t+1) + A_6 Z(k+1,t) + A_7 Z(k,t) \\ &+ B_3 U(k,t+1) + B_4 U(k+1,t) + B_5 U(k,t) \end{aligned} \tag{13}$$

As before, U is the (appropriately dimensioned) vector of control inputs. If $E = I$, (13) is called regular (or nonsingular) and singular if $\det(E) = 0$.

Return now to the repetitive process state-space model of (9) and introduce

$$Z(k,t) := \left[X^T(k,t) \mid Y^T(k,t)\right]^T \tag{14}$$

Then it follows immediately that (9) can be written in form

$$EZ(k+1,t+1) = A_8 Z(k+1,t) + A_9 Z(k,t) \;+ B_6 U(k+1,t) \tag{15}$$

where

$$E = \begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}, \quad A_8 = \begin{bmatrix} A & 0 \\ C & -I_m \end{bmatrix}, \quad A_9 = \begin{bmatrix} 0 & B_0 \\ 0 & D_0 \end{bmatrix}, \quad B_6 = \begin{bmatrix} B \\ D \end{bmatrix} \tag{16}$$

This is the singular version of (13) with $A_5 = 0,\ B_4 = 0,\ B_5 = 0$. (For a background for the structure and control of 2D linear systems described by singular state-space models we refer the reader to e.g. (Kaczorek, 1992) and the references given there, and (Lewis, 1992), as well as to numerous works published since then.)

In 2D linear systems theory (Lewis, 1992) it is possible to employ a change of state variable to write regular/singular Fornasini-Marchesini models in Roesser form. Consider first (13) when $A_7 = 0,\ A_5 \neq 0$, i.e. the first-order form. Then in this case the Roesser model results when

$$A_5 = \begin{bmatrix} * & * \\ 0 & 0 \end{bmatrix}, \quad A_6 = \begin{bmatrix} 0 & 0 \\ * & * \end{bmatrix}, \quad B_3 = \begin{bmatrix} * \\ 0 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 \\ * \end{bmatrix} \tag{17}$$

where $*$ denotes a non-zero matrix with compatible dimensions.

The 2D model of (15) is not of first-order form and hence the above analysis, i.e. based on (17), does not apply. Instead (see (Lewis, 1992) for the 2D systems case) introduce

$$\xi(k,t) = EZ(k,t+1) - A_8 Z(k,t) \tag{18}$$

$$\Phi(k,t) = \xi(k,t) - B_6 U(k,t) \tag{19}$$

where the role of (19) is to avoid the undesirable feature of a shift in the input vector. Using (18) and (19) the following is the singular Roesser model for linear repetitive processes described by (9):

$$\begin{bmatrix} I_{n+m} & 0 \\ 0 & E \end{bmatrix} \begin{bmatrix} \Phi(k+1,t) \\ Z(k,t+1) \end{bmatrix} = \begin{bmatrix} 0 & A_9 \\ I_{n+m} & A_8 \end{bmatrix} \begin{bmatrix} \Phi(k,t) \\ Z(k,t) \end{bmatrix} + \begin{bmatrix} 0 \\ B_6 \end{bmatrix} U(k,t) \tag{20}$$

In this model, both the state $\{X\}$ and the pass profile $\{Y\}$, as represented by the augmented vector Z, constitute vertically transmitted information. (Compare it with the earlier claim that the pass profile $\{Y\}$ is the vertically transmitted information.) Note also that the state dimension in this last model is twice that of (15).

The overall aim of constructing 2D systems models for discrete linear repetitive processes is to assist in answering basic systems theoretic questions (and related problems such as controller design) for these processes. In the next section it is shown that the singular model of (15) and (16) can be used to develop a transition matrix for (9) which then leads to a definition and some key basic results on reachability for these processes. These results extend and generalise the preliminary versions given in (Gałkowski *et al.*, 1996a; 1996c).

A key feature of the discrete linear repetitive process state-space model of (9) is that it is regular. This, in turn, suggests that a regular Roesser model for these processes can also be developed. The remainder of this section develops such a representation and uses it to establish stability equivalence.

Introduce the following vectors into (9):

$$\eta(k,t) = X(k,t+1) - AX(k,t) - BU(k,t) \tag{21}$$

$$\mu(k,t) = Y(k,t) - CX(k,t) - DU(k,t) \tag{22}$$

Then the result can be written as

$$\begin{bmatrix} \eta(k+1,t) \\ \mu(k+1,t) \\ X(k,t+1) \end{bmatrix} = \begin{bmatrix} 0 & B_0 & B_0C \\ 0 & D_0 & D_0C \\ I_n & 0 & A \end{bmatrix} \begin{bmatrix} \eta(k,t) \\ \mu(k,t) \\ x(k,t) \end{bmatrix} + \begin{bmatrix} B_0D \\ D_0D \\ B \end{bmatrix} U(k,t) \tag{23}$$

i.e. a regular Roesser model. Note also that the state dimension here is $n+2m$ in contrast to $2(n+m)$ for the singular Roesser model of (20).

The goal of the remainder of this section is to show an equivalence in terms of stability for (9) and its 2D Roesser model interpretation. Consider first the Roesser model of (23). Then application of the Huang (1972) test for stability of 2D linear systems described by the Roesser model gives the following proposition:

Proposition 1. *The 2D bounded-input/bounded-output (BIBO) stability test of Huang (1972) applied to (9) requires that:*

$$r(D_0) < 1, \quad r(A) < 1 \tag{24}$$

and all eigenvalues of the transfer function matrix

$$G(z_1) := C(z_1 I_n - A)^{-1} B_0 + D_0 \tag{25}$$

lie inside the unit circle in the complex plane $\forall\, |z_1| = 1$.

Consider now asymptotic stability of (9) which requires the computation of the spectral radius of the associated linear operator L_α. One method of doing this is to consider the equation

$$(zI - L_\alpha)Y = \eta \tag{26}$$

and construct necessary and sufficient conditions on the complex scalar z to ensure that a solution exists $\forall \eta \in E_\alpha$ and that this solution is bounded in the sense that

$||Y|| \leq K_0||\eta||$ for some real scalar K_0 and $\forall \eta \in E_\alpha$. The result of this is as follows with the proof in Chapter 3 of (Rogers and Owens, 1992):

Theorem 4. *Linear repetitive processes defined by (9) are asymptotically stable if and only if*

$$r(D_0) < 1 \tag{27}$$

Suppose now that Theorem 4 holds and also that the control input sequence applied converges strongly to U_∞. Then Theorem 2 applied to this case shows that the resulting limit profile dynamics are described by

$$\begin{aligned} X_\infty(t+1) &= \left(A + B_0\left(I_m - D_0\right)^{-1} C\right) X_\infty(t) \\ &\quad + \left(B + B_0\left(I_m - D_0\right)^{-1} D\right) U_\infty(t) \\ Y_\infty(t) &= \left(I_m - D_0\right)^{-1} C X_\infty(t) + \left(I_m - D_0\right)^{-1} D U_\infty(t) \end{aligned} \tag{28}$$

which is simply a proper standard (or 1D) linear time-invariant state space model. Hence if processes described by (9) are asymptotically stable, then, after a "sufficiently large" number of passes, their dynamics can be replaced by those of a standard linear system. Note, see also below, that asymptotic stability does not ensure that the resulting limit profile is stable in the standard sense, i.e. all eigenvalues of the matrix $(A + B_0(I_m - D_0)^{-1}C)$ have modulus strictly less than 1.

In the current context an immediate consequence of this result is that asymptotic stability of processes described by (9) is not equivalent to BIBO stability of their 2D linear systems (regular Roesser model) interpretations. This is a direct consequence of the fact that the pass length α is finite and hence the reason why this property is independent of, in particular, the eigenvalues of the matrix A which clearly govern the dynamics produced along any pass. As a simple example to demonstrate the potential weakness of this property, consider the following SISO single state process where $\beta > 1$, i.e. the case of (9) with $A = -0.5$, $B = 1$, $B_0 = 0.5 + \beta$, $C = 1$, $D_0 = D = 0$:

$$Y(k+1, t+1) = -0.5Y(k+1, t) + (0.5 + \beta)Y(k, t) + U(k+1, t) \tag{29}$$

This example is asymptotically stable but the limit profile dynamics

$$Y(\infty, t+1) = \beta Y(\infty, t) + U(\infty, t) \tag{30}$$

are unstable along the pass in the standard sense since $\beta > 1$!

In terms of stability along the pass (characterised by Theorem 3 here) note that in this case L_α is independent of α and hence (7) of this result holds if and only if $r(D_0) < 1$. Also the "boundedness condition" (8) of this result is equivalent to the existence of a $\lambda \in (r_\infty, 1)$ such that (26) has a uniformly bounded, with respect to α, solution $Y \in E_\alpha$ for all choices of $\eta \in E_\alpha$ satisfying $\sup_\alpha ||\eta|| < +\infty$ and $\forall |z| \geq \lambda$.

It is clear that, in general, (8) of Theorem 3 could prove very difficult to interpret. In the case of (9), however, the following interpretation is possible (for a proof see Chapter 3 of (Rogers and Owens, 1992)):

Theorem 5. *Linear repetitive processes defined by (9) are stable along the pass if and only if*

1. *the condition of Theorem 4 holds, and*
2. *$\exists$ real numbers $\varepsilon > 0$ and $\lambda \in (r_\infty, 1)$: all eigenvalues of the $n \times n$ matrix $A + B_0 (zI_m - D_0)^{-1} C$ have modulus strictly less than $1 - \varepsilon$ for all choices of $|z| \geq \lambda$.*

Using this last result, the following stability equivalence result can now be established:

Theorem 6. *Linear repetitive processes described by (9) are stable along the pass if and only if their 2D linear systems Roesser model interpretation as defined here by (23) is BIBO stable.*

Proof. To show the necessity, first note that (8) can be replaced by

$$M_0 := \sup_{\alpha > 0} \sup_{|z| \geq \lambda} \left\| (zI - L_\alpha)^{-1} \right\| < +\infty \tag{31}$$

for some $\lambda \in (r_\infty, 1)$. Hence

$$\Psi(z) := A + B_0 (zI_m - D_0)^{-1} C \tag{32}$$

must be a stability matrix at $|z| = +\infty$, i.e. A must be a stability matrix. Also write the 'characteristic polynomial' $|z_1 I_n - \Psi(z)|$ in form

$$|z_1 I_n - \Psi(z)| = \frac{|z_1 I_n - A| \, |zI_m - G(z_1)|}{|zI_m - D_0|} \tag{33}$$

Then stability along the pass holds provided that

$$|zI_m - G(z_1)| \neq 0, \quad \forall |z_1| \geq 1, \ |z| = \lambda \tag{34}$$

Note also that

$$\underset{|z_1| \to +\infty}{\text{LIMIT}} \ G(z_1) = D_0 \tag{35}$$

and considering the case of $|z| = 1$ yields that the eigenvalues of $G(z_1)$ with $|z_1| = 1$ lie in the open unit circle in the complex plane. Hence the 2D systems conditions for BIBO stability, as expressed by Proposition 1, are necessary for stability along the pass.

To show that they are also sufficient, suppose that D_0 is a stability matrix and hence $r_\infty < 1$. Also if A is a stability matrix and all eigenvalues of $G(z_1)$ with $|z_1| = 1$ lie in the open unit circle in the complex plane, it is possible to choose

$\lambda \in (r_\infty, 1)$: both sides of (33) are non-zero for $|z| = \lambda$ and $|z_1| = 1$ and such that all eigenvalues of A have modulus less than λ. Finally, considering the unit contour $\{z_1 : |z_1| = 1\}$ traversed clock-wise and applying standard encirclement theorems to the right-hand side of (33) yields the fact that all roots of this equation lie in the interior of the circle $|z_1| = 1$, $\forall |z| = \lambda$. Equivalently, (8) of Theorem 3 holds for this case. ■

It is a straightforward task to obtain a regular Fornasini-Marchesini model from the regular Roesser model of (23) and, of course, the stability equivalence result also transfers directly. Hence the details are omitted here. A key feature of Theorem 6 is that it releases the wealth of 2D systems stability tests for use with discrete linear repetitive processes and vice versa. For a detailed treatment of this general area, including (where appropriate) software development, see (Smyth, 1992).

To-date, little work has been reported on fundamental systems theoretic problems for discrete linear repetitive processes, such as controllability and observability. In the 2D linear systems area, a considerable volume of theory is now available for both the singular and regular cases. A background for this can be found in e.g. (Kaczorek, 1992), the relevant references given there, and numerous papers since this text. Also Rocha (1990) has studied aspects of this general area in a behavioural setting with some highly significant results.

One possible means of addressing these and related areas for discrete linear repetitive processes is, if possible, to extend/generalise the 2D linear systems approach. For 2D linear systems described by Roesser/Fornasini-Marchesini models the basic starting point has been a transition matrix description of the underlying system dynamics. The remainder of this paper develops a transition matrix for discrete linear repetitive processes and then uses it to develop some basic results on reachability/controllability.

4. Transition Matrix and Controllability/Reachability

Return to the regular Roesser model interpretation of the dynamics of (9) given by (23) and partition it as follows:

$$\begin{bmatrix} X_h(k+1,t) \\ X_v(k,t+1) \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} X_h(k,t) \\ X_v(k,t) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} U(k,t) \tag{36}$$

where

$$X_h(k,t) = \left[\eta^T(k,t) \mid \mu^T(k,t)\right]^T, \quad X_v(k,t) = X(k,t) \tag{37}$$

and

$$A_{11} = \begin{bmatrix} 0 & B_0 \\ 0 & D_0 \end{bmatrix}, \quad A_{12} = \begin{bmatrix} B_0 C \\ D_0 C \end{bmatrix}, \quad A_{21} = \begin{bmatrix} I & 0 \end{bmatrix}, \quad A_{22} = A$$

$$B_1 = \begin{bmatrix} B_0 D \\ D_0 D \end{bmatrix}, \qquad B_2 = B \tag{38}$$

Roesser (1975) has shown that the fundamental matrix sequence, or transition matrix, for a regular model of the form (36) is as follows:

$$\begin{aligned} &T_{p,q} = 0, \quad p < 0 \text{ and/or } q < 0 \\ &T_{0,0} = I \\ &T_{p,q} = T_{1,0}T_{p-1,q} + T_{0,1}T_{p,q-1} \end{aligned} \tag{39}$$

where

$$T_{1,0} = \begin{bmatrix} A_{11} & A_{12} \\ 0 & 0 \end{bmatrix}, \quad T_{0,1} = \begin{bmatrix} 0 & 0 \\ A_{21} & A_{22} \end{bmatrix} \tag{40}$$

In order to illustrate its structure for discrete linear processes, the first six terms are given below:

$$T_{1,0} = \begin{bmatrix} 0 & B_0 & B_0C \\ 0 & D_0 & D_0C \\ 0 & 0 & 0 \end{bmatrix}, \quad T_{0,1} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ I & 0 & A \end{bmatrix}, \quad T_{2,0} = \begin{bmatrix} 0 & B_0D_0 & B_0D_0C \\ 0 & D_0^2 & D_0^2C \\ 0 & 0 & 0 \end{bmatrix}$$

$$T_{0,2} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ A & 0 & A^2 \end{bmatrix}, \quad T_{1,2} = \begin{bmatrix} B_0CA & 0 & B_0CA^2 \\ D_0CA & 0 & D_0CA^2 \\ B_0C & AB_0 & AB_0C + B_0CA \end{bmatrix}$$

$$T_{2,1} = \begin{bmatrix} B_0D_0C & B_0CB_0 & B_0D_0CA + B_0CB_0C \\ D_0^2C & D_0CB_0 & D_0^2CA + D_0CB_0C \\ 0 & B_0D_0 & B_0D_0C \end{bmatrix} \tag{41}$$

Given appropriate boundary conditions, i.e. the initial pass profile $Y_0(t)$, $0 \le t \le \alpha$, and state initial conditions $X(k,0)$, $k \ge 1$, the transition matrix can be used to generate a general response formula for processes described by (9). The remainder of this section uses the transition matrix to introduce and characterise one concept of controllability/reachability for (9).

When continuing to treat the discrete linear repetitive processes of (9) as a 2D linear system with Roesser model description (36), the following definitions of local controllability and reachability can be introduced:

Definition 4. The discrete linear repetitive process (9) written in the 2D Roesser model form of (36) is said to be *locally controllable* in the rectangle $[0,F]\times[0,H]$, $0 \le H \le \alpha$, if for admissible boundary conditions $X_h(k,0)$, $k \ge 0$ and $X_v(0,t)$, $0 \le t \le \alpha, \exists$ a sequence of input vectors $U(k,t)$ on $(0,0) \le (k,t) \le (F,H)$ such that

$$X(F,H) := \begin{bmatrix} X_h(F,H) \\ X_v(F,H) \end{bmatrix} = 0 \tag{42}$$

Definition 5. The discrete linear repetitive process (9) written in the 2D Roesser model form of (36) is said to be *locally reachable* in the rectangle $[0, F] \times [0, H]$, $0 \leq H \leq \alpha$, if for admissible boundary conditions $X_h(k, 0)$, $k \geq 0$ and $X_v(0, t)$, $0 \leq t \leq \alpha$, and every vector $X_F, \exists$ a sequence of input vectors $U(k, t)$ on $(0, 0) \leq (k, t) \leq (F, H)$ such that

$$X(F, H) = X_F \tag{43}$$

where again $X(F, H)$ is defined by (42).

Conditions for the existence of these properties are known (Kaczorek, 1992). In particular, it is known that local reachability implies local controllability. The full characterisation of these notions for the general model of 2D systems is provided in (Kaczorek, 1994).

Given this last fact, the remainder of this paper will only deal with local reachability. In this context, it is appropriate to introduce the matrix

$$M_{i,j} = T_{i-1,j} \begin{bmatrix} B_1 \\ 0 \end{bmatrix} + T_{i,j-1} \begin{bmatrix} 0 \\ B_2 \end{bmatrix} \tag{44}$$

and the following theorem gives the necessary and sufficient conditions for local reachability:

Theorem 7. *The discrete linear repetitive process (9) written in the 2D Roesser model form of (36) is locally reachable in the rectangle* $[0, F] \times [0, H]$, $0 \leq H \leq \alpha$ *if and only if the matrix*

$$R_{F,H} := \begin{bmatrix} M_{01} & M_{10} & M_{11} & \cdots & M_{ij} & \cdots & M_{FH} \end{bmatrix} \tag{45}$$

has full-row rank.

Proof. This follows virtually identical arguments to those of (Kaczorek, 1992) for the 2D linear systems case. Hence it is omitted here. ■

Inspection of $R_{F,H}$ shows immediately that the following relationships hold, where $\subset$ means that the matrix on its left-hand side is a submatrix of the one on its right-hand side:

$$R_{F,H} \subset R_{F,H+1}, \qquad R_{F,H} \subset R_{F+1,H} \tag{46}$$

Hence (9) written in form (36) is locally reachable in all rectangles of pass F, i.e.

$$[0, F] \times [0, 1],\ [0, F] \times [0, 2],\ \cdots,\ [0, F] \times [0, \alpha]$$

if it is locally reachable in the rectangle $[0,F]\times[0,1]$. Also the matrix $R_{F,1}$ has the structure

$$R_{F,1} = \left[\begin{array}{cccccccc} B_0D & B_0D_0D & \cdots & B_0D_0^{F-1}D & 0 & B_0CB & B_0D_0CB + B_0CB_0D & \cdots \\ D_0D & D_0^2D & \cdots & D_0^FD & 0 & D_0CB & D_0^2CB + D_0CB_0D & \cdots \\ 0 & 0 & & 0 & B & B_0D & B_0D_0D & \end{array}\right.$$

$$\left.\begin{array}{c} B_0D_0^{F-2}CB_0D + \sum_{j=0}^{F-3} B_0D_0^{F-3-j}CB_0D_0^{j+1}D + B_0D_0^{F-1}CB \\ D_0^{F-1}CB_0D + \sum_{j=0}^{F-3} D_0^{F-2-j}CB_0D_0^{j+1}D + D_0^FCB \\ B_0D_0^{F-1}D \end{array}\right] \tag{47}$$

Consequently, all points on pass F would be reachable if

$$\operatorname{rank} R_{F1} = 2n + m \tag{48}$$

In fact, this condition never holds. Suppose, however, that the matrix D_0 is non-singular. Then the first block row in R_{F1} can be reduced to zero entries without changing its rank. The action required is to left multiply the second block row of R_{F1} by $B_0D_0^{-1}$ and subtract the result from the first block row.

This fact means that the 2D model for discrete linear repetitive processes given by (36) is never locally reachable. Note also that such a result is to be expected from the fact that the state sub-vectors η and μ depend on each other and hence arbitrary values for each of them cannot be achieved simultaneously. In fact, the maximal rank of R_{F1} is $n+m$ and, if this condition holds, then for arbitrary vectors $\mu(k,t)$ and $X(k,t)$, $\eta(k,t)$ can be computed as

$$\eta(k,t) = B_0D_0^{-1}\mu(k,t) \tag{49}$$

Also sufficient conditions for this limited reachability property are the same as those for local reachability of the general singular 2D model representation of (9). The details can be found in (Gałkowski *et al.*, 1996a; 1996c).

Continuing with the assumption that D_0 is a non-singular matrix, define the so-called restricted state vector for (9):

$$Z(k,t) = \begin{bmatrix} \mu(k,t) \\ X(k,t) \end{bmatrix} \tag{50}$$

Then the following restricted 2D state-space model of the Roesser type is obtained for the dynamics of (9):

$$\begin{bmatrix} \mu(k+1,t) \\ X(k,t+1) \end{bmatrix} = \begin{bmatrix} D_0 & D_0C \\ B_0D_0^{-1} & A \end{bmatrix} \begin{bmatrix} \mu(k,t) \\ X(k,t) \end{bmatrix} + \begin{bmatrix} D_0D \\ B \end{bmatrix} U(k,t) \tag{51}$$

Hence the subvector $\mu(k,t)$ propagates information in the vertical direction and the subvector $X(k,t)$ propagates information in the horizontal direction. For this model all points on pass F are locally reachable under the same conditions as for the general 2D singular model of (9), i.e. if and only if the following conditions hold:

$$\operatorname{rank} \Delta(F) = m \tag{52}$$

and

$$\operatorname{rank} \Big[B_0 \Delta(F-1) \mid B \Big] = n \tag{53}$$

where

$$\Delta(F) := \Big[D_0^{F-1} D \mid D_0^{F-2} D \mid \cdots \mid D \Big] \tag{54}$$

The details can again be found in (Gałkowski *et al.*, 1995).

It seems however that a natural approach to these problems is to embed their basic 2D structure in a classical 1D representation which leads to the so-called pass reachability/observability (Gałkowski *et al.*, 1996b).

5. Conclusions

Discrete linear repetitive processes exhibit strong structural links with 2D linear time-invariant systems by the well-known Roesser model (or equivalents). This raises a possibility of using well-established 2D linear systems theory to answer the basic systems theoretic (and controller design) questions for linear repetitive processes for which few or no results are currently available. By analogy to conventional (or standard) linear systems, where strong structural links are also present, one such question is as follows: What (if anything) is meant by reachability/controllability and observability for these processes and how can such properties be characterised both theoretically and in terms of computationally feasible tests?

Previous work, e.g. by (Rocha *et al.*, 1996), has shown that stability along the pass for discrete linear repetitive processes and BIBO stability of 2D linear systems described by the Roesser model can be tested using common tests. These tests are based on interpreting the state vector of a discrete linear repetitive process as horizontally transmitted information and the pass profile as vertically transmitted information. Equivalently, it was assumed that the state space model for discrete linear repetitive processes was in fact a Roesser model.

Indeed, a number of structural differences exist between the repetitive process state-space model and the 2D systems Roesser model. This, in turn, means that the approach employed in (Rocha *et al.*, 1996) cannot really be extended from the stability domain to general questions such as the one given above. The first set of major results in this paper consists of several new 2D linear systems models for the dynamics of discrete linear repetitive processes and attention has been paid to both standard and singular Roesser models.

In comparison with the previous approach, these models consider both the state and the pass profile as vertically transmitted information. It has been shown that stability along the pass is equivalent to BIBO stability in the 2D systems sense. Also a standard Roesser model interpretation has been used to develop some key results regarding the general question given above. These are a transition matrix plus a physically meaningful definition of reachability together with some basic characterisations of this property. Currently these results are being refined into computationally feasible tests. Also work is proceeding on extending these basic results to e.g. controllability, together with in-depth investigations of their overall role in understanding the basic dynamic behaviour of linear repetitive processes. Results from this research will be reported in due course.

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