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REGULARISATION OF SINGULAR 2D LINEAR MODELS BY STATE-FEEDBACKS

TADEUSZ KACZOREK*

New necessary and sufficient conditions are established under which the singular 2D first Fornasini-Marchesini model and the singular 2D Roesser model can be regularised by state feedbacks. Some procedures for computation of the respective feedback matrices are given and illustrated by numerical examples.

1. Introduction

The regularisation of singular (descriptor) linear systems by state and output feedbacks has been considered in many papers (Bunse-Gestner *et al.*, 1992; 1994; Ozcaldiran and Lewis, 1990). In (Bunse-Gestner *et al.*, 1994) it was shown that proportional and derivative output feedback controls can be constructed such that the closed-loop system is regular and has index at most one. The regularity guarantees the existence and uniqueness of solutions to singular linear systems (Campbell, 1980; Kaczorek, 1993; Klamka, 1991; Ozcaldiran and Lewis, 1990). The regularisation problem by state-feedbacks for singular 2D Roesser and singular 2D first Fornasini-Marchesini models has been formulated and necessary and some sufficient conditions have been established in (Kaczorek, 1997). The subject of the paper is to present new necessary and sufficient conditions under which the singular 2D first Fornasini-Marchesini model and the 2D singular Roesser model can be regularised by state feedbacks. Some procedure will be presented for computation of the appropriate feedback matrices.

2. Problem Statement

2.1. Singular 2D First Fornasini-Marchesini Model

Let $\mathbb{R}^{p\times q}$ be the set of real $p\times q$ matrices and $\mathbb{R}^p := \mathbb{R}^{p\times 1}$. Consider a 2D linear system described by the singular first Fornasini-Marchesini model (Fornasini and Marchesini, 1978; Kaczorek, 1985; 1993; Klamka, 1991; Kurek, 1985):

$$Ex_{i+1,j+1} = A_0x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1} + Bu_{ij}, \qquad i,j \in \mathbb{Z}_+ \tag{1}$$

where $x_{ij} \in \mathbb{R}^n$ is the semistate vector at the point (i,j), $u_{ij} \in \mathbb{R}^m$ stands for the input vector and $E, A_k \in \mathbb{R}^{n\times n}$, $k = 0,1,2$, $B \in \mathbb{R}^{n\times m}$. Moreover, $\mathbb{Z}_+$ is the set of nonnegative integers.

* Warsaw University of Technology, Institute of Control and Industrial Informatics, ul. Koszykowa 75, 00–662 Warsaw, Poland, e-mail: kaczorek@nov.isep.pw.edu.pl.

Definition 1. The model (1) is said to be *regular* if

$$\det\left[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2\right] \neq 0 \quad \text{for some } (z_1, z_2) \in \mathbb{C}^2 \tag{2}$$

where $\mathbb{C}$ is the field of complex numbers. The model (1) is called *singular* if

$$\det\left[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2\right] = 0 \quad \text{for all } (z_1, z_2) \in \mathbb{C}^2 \tag{3}$$

It is assumed that the model (1) is singular and

$$\operatorname{rank} B = m \tag{4a}$$

$$\det E = 0 \tag{4b}$$

Let a state feedback for (1) have the form

$$u_{ij} = Kx_{ij} + v_{ij} \tag{5}$$

where $K \in \mathbb{R}^{m\times n}$ and v_{ij} is a new input vector. Substitution of (5) into (1) yields

$$Ex_{i+1,j+1} = (A_0 + BK)x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1} + Bv_{ij} \tag{6}$$

The regularisation problem for (1) by (5) can be stated as follows.

Problem 1. Given matrices E, A_0, A_1, A_2 and B of (1), find a feedback matrix K of (5) such that the closed-loop system (6) is regular, i.e.

$$\det\left[Ez_1z_2 - A_0 - BK - A_1z_1 - A_2z_2\right] \neq 0 \quad \text{for some } (z_1, z_2) \in \mathbb{C}^2 \tag{7}$$

2.2. Singular 2D Roesser Model

Consider a 2D linear system described by the singular Roesser model (Kaczorek, 1985; 1993; Klamka, 1991; Roesser, 1975)

$$E\begin{bmatrix} x^h_{i+1,j} \\ x^v_{i,j+1} \end{bmatrix} = A\begin{bmatrix} x^h_{ij} \\ x^v_{ij} \end{bmatrix} + Bu_{ij}, \qquad i, j \in \mathbb{Z}_+ \tag{8}$$

where $x^h_{ij} \in \mathbb{R}^{n_1}$ is the horizontal semistate vector at the point (i, j), $x^v_{ij} \in \mathbb{R}^{n_2}$ denotes the vertical semistate vector at the point (i, j), $u_{ij} \in \mathbb{R}^m$ is the input vector,

$$E := \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}, \quad A := \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \quad E_{11}, A_{11} \in \mathbb{R}^{n_1\times n_1}, \quad E_{22}, A_{22} \in \mathbb{R}^{n_2\times n_2}$$

$$B := \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix}, \qquad B_{11} \in \mathbb{R}^{n_1\times m}, \qquad B_{22} \in \mathbb{R}^{n_2\times m}$$

Definition 2. The singular 2D Roesser model (8) is said to be *regular* if

$$\det \begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix} \neq 0 \quad \text{for some } (z_1, z_2) \in \mathbb{C}^2 \tag{9}$$

The model (8) is called *singular* if

$$\det \begin{bmatrix} E_{11}z_1 - A_{11} & E_{12}z_2 - A_{12} \\ E_{21}z_1 - A_{21} & E_{22}z_2 - A_{22} \end{bmatrix} = 0 \quad \text{for some } (z_1, z_2) \in \mathbb{C}^2 \tag{10}$$

It is assumed that the model (8) is singular and

$$\operatorname{rank} B = m \tag{11a}$$

$$\det E = 0 \tag{11b}$$

Let a state feedback for (8) have the form

$$u_{ij} = F \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + v_{ij} \tag{12}$$

where $F = [F_1, F_2]$, $F_1 \in \mathbb{R}^{m \times n_1}$, $F_2 \in \mathbb{R}^{m \times n_2}$ and v_{ij} is a new input vector. Substitution of (12) into (8) yields

$$E \begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix} = (A + BF) \begin{bmatrix} x_{ij}^h \\ x_{ij}^v \end{bmatrix} + Bv_{ij}, \quad i, j \in \mathbb{Z}_+ \tag{13}$$

where

$$A + BF = \begin{bmatrix} A_{11} + B_{11}F_1 & A_{12} + B_{11}F_2 \\ A_{21} + B_{22}F_1 & A_{22} + B_{22}F_2 \end{bmatrix} \tag{14}$$

The regularisation problem for (8) by (12) can be stated as follows.

Problem 2. Given matrices E, A, B of (8), find a feedback matrix F of (12) such that the closed-loop system (13) is regular, i.e.

$$\det \begin{bmatrix} E_{11}z_1 - A_{11} - B_{11}F_1 & E_{12}z_2 - A_{12} - B_{11}F_2 \\ E_{21}z_1 - A_{21} - B_{22}F_1 & E_{22}z_2 - A_{22} - B_{22}F_2 \end{bmatrix} \neq 0 \quad \text{for some } (z_1, z_2) \in \mathbb{C}^2 \tag{15}$$

Some new necessary and sufficient conditions will be established under which Problems 1 and 2 have solutions and appropriate procedures for computation of the feedback matrices K and F will be given.

3. Problem Solution

3.1. Testing Regularity and Singularity of the Model

The pencil $[Az + B]$ is said to be *regular* (Kaczorek, 1993) if

$$\det [Az + B] \neq 0 \quad \text{for some } z \in \mathbb{C} \tag{16}$$

Lemma 1. *The model (1) is regular if one of the following conditions is satisfied:*

i) *at least one of the matrices* E, A_0, A_1 *and* A_2 *is nonsingular,*

ii) *at least one of the pencils* $[Ez_1 - A_2]$, $[Ez_2 - A_1]$, $[A_1z_1 + A_0]$, $[A_2z_2 + A_0]$ *is regular.*

Proof. Suppose e.g. that A_1 is nonsingular. Then $\det[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] = \det A_1 \det[A_1^{-1}Ez_1z_2 - A_1^{-1}A_0 - I_nz_1 - A_1^{-1}A_2z_2] \neq 0$ for some $(z_1, z_2) \in \mathbb{C}^2$, since $\det A_1 \neq 0$ and $\det[A_1^{-1}Ez_1z_2 - A_1^{-1}A_0 - I_nz_1 - A_1^{-1}A_2z_2] \neq 0$ for some $(z_1, z_2) \in \mathbb{C}^2$. The proof of (i) in the remaining cases is similar.

From the equality

$$
\begin{aligned}
[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2] &= [z_2(Ez_1 - A_2) - (A_1z_1 + A_0)] \\
&= [z_1(Ez_2 - A_1) - (A_2z_2 + A_0)]
\end{aligned}
$$

and (16) it follows that if at least one of the pencils $[Ez_1 - A_2]$, $[Ez_2 - A_1]$, $[A_1z_1 + A_0]$, $[A_2z_2 + A_0]$ is regular, then (2) holds and the model (1) is regular. ■

For checking the regularity of the model (1), the 2D shuffle algorithm can be used (Kaczorek, 1993).

Lemma 2. *If the model (1) is regular, then*

$$
\operatorname{rank}\begin{bmatrix} E & A_0 & A_1 & A_2 \end{bmatrix} = n \tag{17a}
$$

and

$$
\operatorname{rank}\begin{bmatrix} E \\ A_0 \\ A_1 \\ A_2 \end{bmatrix} = n \tag{17b}
$$

Proof. From the equality

$$
\begin{aligned}
\begin{bmatrix} Ez_1z_2 - A_0 - A_1z_1 - A_2z_2 \end{bmatrix} &= \begin{bmatrix} E & A_0 & A_1 & A_2 \end{bmatrix} \begin{bmatrix} I_nz_1z_2 \\ -I_n \\ -I_nz_1 \\ -I_nz_2 \end{bmatrix} \\
&= \begin{bmatrix} I_nz_1z_2 & -I_n & -I_nz_1 & -I_nz_2 \end{bmatrix} \begin{bmatrix} E \\ A_0 \\ A_1 \\ A_2 \end{bmatrix}
\end{aligned}
$$

it follows that (2) implies (17a) and (17b). ■

From Lemmas 1 and 2 we have the following collloraries:

Collorary 1. *If the model (1) is singular, then* E, A_0, A_1, A_2 *are singular.*

Collorary 2. *If*

$$\mathrm{rank}\left[E\ A_0\ A_1\ A_2\right] < n$$

or

$$\mathrm{rank}\begin{bmatrix} E \\ A_0 \\ A_1 \\ A_2 \end{bmatrix} < n$$

then the model (1) is singular.

3.2. Singular 2D First Fornasini-Marchesini Model

Definition 3. The model (1) is called *regularisable by state feedback (5)* if there exists a feedback matrix K such that (7) holds.

Theorem 1. *The model (1) is regularisable by state feedback (5) only if*

$$\mathrm{rank}\left[E\ A_0\ A_1\ A_2\ B\right] = n \tag{18}$$

Proof. From the relation

$$\left[Ez_1z_2 - A_0 - BK - A_1z_1 - A_2z_2\right] = \left[E\ A_0\ A_1\ A_2\ B\right]\begin{bmatrix} I_nz_1z_2 \\ -I_n \\ -I_nz_1 \\ -I_nz_2 \\ -K \end{bmatrix} \tag{19}$$

it follows that there exists K such that (7) holds only if (18) is satisfied. ■

Theorem 2. *The model (1) is regularisable by state feedback (5) if at least one of the matrices* $[A_0\ B]$, $[A_1\ B]$, $[A_2\ B]$, $[E\ B]$ *has full row rank.*

Proof. The condition (7) is satisfied if

$$\det\left[A_0 + BK\right] \neq 0 \tag{20}$$

since

$$\det\left[Ez_1z_2 - A_0 - BK - A_1z_1 - A_2z_2\right]\Big|_{\substack{z_1=0\\ z_2=0}} = \det\left[-(A_0 + BK)\right]$$

■

It can be shown (Kaczorek, 1996) that there exists a feedback matrix K such that (20) holds iff

$$\mathrm{rank}\left[A_0\ B\right] = n \tag{21}$$

If (21) holds, then there exist nonsingular matrices $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$ of elementary row and columns operations such that

$$P\begin{bmatrix} A_0 & B \end{bmatrix} \begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \bar{A}_0 & 0 & \bar{B}_1 \\ 0 & \bar{B}_2 & 0 \end{bmatrix} \tag{22}$$

where $\bar{A}_0$ has full row rank, $\operatorname{rank} \bar{A}_0 = \operatorname{rank} A_0 = r$, $\bar{B}_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ is nonsingular, and $\bar{B}_1 \in \mathbb{R}^{r\times(m-n+r)}$. Note that if $n = r + m$, then the matrix $\bar{B}_1$ disappears.

Let $\bar{A}_1 \in \mathbb{R}^{(n-r)\times n}$ be a matrix such that the matrix $\begin{bmatrix} \bar{A}_0 \\ \bar{A}_1 \end{bmatrix}$ is nonsingular. Then for

$$K = Q\begin{bmatrix} \bar{B}_2^{-1}\bar{A}_1 \\ 0 \end{bmatrix} \tag{23}$$

we obtain

$$P\begin{bmatrix} A_0 + BK \end{bmatrix} = \begin{bmatrix} \bar{A}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & \bar{B}_1 \\ \bar{B}_2 & 0 \end{bmatrix} \begin{bmatrix} \bar{B}_2^{-1}\bar{A}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} \bar{A}_0 \\ \bar{A}_1 \end{bmatrix}$$

Let us assume that

$$\operatorname{rank}\begin{bmatrix} A_1 & B \end{bmatrix} = n \tag{24}$$

and $\operatorname{rank}[A_0\ B] < n$. It will be shown that if (24) holds, then there exists K such that the pencil $[A_1 z_1 + A_0 + BK]$ is regular.

If (24) holds, then there exist nonsingular matrices $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$ of elementary row and columns operations such that

$$\begin{aligned} P\begin{bmatrix} A_1 z_1 + A_0 + BK \end{bmatrix} &= P\begin{bmatrix} A_1 & A_0 & B \end{bmatrix} \begin{bmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ 0 & 0 & Q \end{bmatrix} \begin{bmatrix} I_n z_1 \\ I_n \\ Q^{-1}K \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_1 & \bar{A}_0 & \bar{B}_1 & 0 \\ 0 & 0 & 0 & \bar{B}_2 \end{bmatrix} \begin{bmatrix} I_n z_1 \\ I_n \\ Q^{-1}K \end{bmatrix} \\ &= \begin{bmatrix} \bar{A}_1 z_1 + \bar{A}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & \bar{B}_2 \end{bmatrix} Q^{-1}K \end{aligned} \tag{25}$$

where the matrix $[\bar{A}_1\ \bar{A}_0]$ has full row rank equal to $\operatorname{rank}[A_1\ A_0] = r$ and $\bar{B}_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ is nonsingular.

Let $\tilde{A} \in \mathbb{R}^{(n-r)\times n}$ be a matrix such that $\begin{bmatrix} \bar{A}_1 z_1 + \bar{A}_0 \\ \tilde{A} \end{bmatrix}$ is nonsingular for some $z_1 \in \mathbb{C}$. Then for

$$K = Q\begin{bmatrix} 0 \\ \bar{B}_2^{-1}\tilde{A} \end{bmatrix} \tag{26}$$

we obtain

$$P\big[A_1z_1 + A_0 + BK\big] = \begin{bmatrix} \bar{A}_1z_1 + \bar{A}_0 \\ 0 \end{bmatrix} + \begin{bmatrix} \bar{B}_1 & 0 \\ 0 & \bar{B}_2 \end{bmatrix} Q^{-1}K = \begin{bmatrix} \bar{A}_1z_1 + \bar{A}_0 \\ \tilde{A} \end{bmatrix} \tag{27}$$

Therefore, by Lemma 1, the model (1) is regularisable by (26) if (24) holds. If

$$\operatorname{rank}\big[A_2\ B\big] = n \tag{28}$$

the proof is similar.

Now it will be shown that if

$$\operatorname{rank}\big[E\ B\big] = n \tag{29}$$

and $\operatorname{rank}\big[A_0\ A_1\ A_2\ B\big] < n$ then there exists K such that (7) holds. If (29) is satisfied, then there exist nonsingular matrices $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$ such that

$$\begin{aligned} &P\big[Ez_1z_2 - A_0 - BK - A_1z_1 - A_2z_2\big] \\ &= P\big[E\ A_0\ B\ A_1\ A_2\big] \begin{bmatrix} I_n & 0 & 0 & 0 & 0 \\ 0 & I_n & 0 & 0 & 0 \\ 0 & 0 & Q & 0 & 0 \\ 0 & 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & 0 & I_n \end{bmatrix} \begin{bmatrix} I_nz_1z_2 \\ -I_n \\ -Q^{-1}K \\ -I_nz_1 \\ -I_nz_2 \end{bmatrix} \\ &= \begin{bmatrix} \hat{E} & \hat{A}_0 & \hat{B}_1 & 0 & \hat{A}_1 & \hat{A}_2 \\ 0 & 0 & 0 & \hat{B}_2 & 0 & 0 \end{bmatrix} \begin{bmatrix} I_nz_1z_2 \\ -I_n \\ -Q^{-1}K \\ -I_nz_1 \\ -I_nz_2 \end{bmatrix} \\ &= \begin{bmatrix} \hat{E}z_1z_2 - \hat{A}_1z_1 - \hat{A}_2z_2 - \hat{A}_0 \\ 0 \end{bmatrix} - \begin{bmatrix} \hat{B}_1 & 0 \\ 0 & \hat{B}_2 \end{bmatrix} Q^{-1}K \end{aligned} \tag{30}$$

where the matrix $[\hat{E}\ \hat{A}_0\ \hat{A}_1\ \hat{A}_2]$ has full row rank equal to $\operatorname{rank}[E\ A_0\ A_1\ A_2] = r$ and $\hat{B}_2 \in \mathbb{R}^{(n-r)\times(n-r)}$ is nonsingular.

Let $\hat{A} \in \mathbb{R}^{(n-r)\times n}$ be a matrix such that $\begin{bmatrix} \hat{E}z_1z_2 - \hat{A}_1z_1 - \hat{A}_2z_2 - \hat{A}_0 \\ -\hat{A} \end{bmatrix}$ is nonsingular for some $(z_1, z_2) \in \mathbb{C}^2$. Note that such $\hat{A}$ exists, since by (29) $\hat{B}_2$ is nonsingular. Substitution of

$$K = Q \begin{bmatrix} 0 \\ \hat{B}_2^{-1}\hat{A} \end{bmatrix} \tag{31}$$

into (30) yields

$$P\big[Ez_1z_2 - A_0 - BK - A_1z_1 - A_2z_2\big] = \begin{bmatrix} \hat{E}z_1z_2 - \hat{A}_1z_1 - \hat{A}_2z_2 - \hat{A}_0 \\ -\hat{A} \end{bmatrix}$$

Therefore the model (1) is regularisable by (31) if (29) holds. ■

Example 1. Consider the model (1) with

$$E = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 0 & 2 \\ 0 & -2 & 0 & -4 \\ 1 & -1 & 0 & -2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \\ 1 & -1 \end{bmatrix} \tag{32}$$

It is easy to check that the model (1) with (32) is not regular and the pair (A_0, B) satisfies the condition (21), since

$$\operatorname{rank}\,[A_0\ B] = \operatorname{rank}\begin{bmatrix} 0 & 1 & 0 & 2 & \vdots & 1 & 0 \\ -1 & 1 & 0 & 2 & \vdots & 0 & 1 \\ 0 & -2 & 0 & -4 & \vdots & -2 & 1 \\ 1 & -1 & 0 & -2 & \vdots & 1 & -1 \end{bmatrix} = 4$$

In this case we have

$$P = \begin{bmatrix} 1 & -1 & 0 & -1 \\ -2 & 1 & -1 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$PA_0 = \begin{bmatrix} \bar{A}_0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad PBQ = \begin{bmatrix} 0 \\ \bar{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \bar{B}_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Choosing $\bar{A}_1 = \left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right]$, from (23) we obtain

$$K = Q\begin{bmatrix} \bar{B}_2^{-1}\bar{A}_1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{33}$$

and

$$P[A_0 + BK] = \begin{bmatrix} \bar{A}_0 \\ \bar{A}_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

It is easy to check that the closed-loop system (6) with (33) is regular. ♦

Example 2. Consider the model (1) with

$$E = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & -1 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ 0 & -1 & 0 & -2 \\ 0 & -2 & 0 & -4 \\ 1 & 1 & 0 & 2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \\ 1 & -1 \end{bmatrix} \tag{34}$$

It is easy to check that the model (1) with (34) is not regular and the conditions (21), (24) and (28) are not satisfied, but

$$\operatorname{rank}\begin{bmatrix} E & B \end{bmatrix} = \operatorname{rank}\left[\begin{array}{cccc:cc} 1 & 0 & -1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ -2 & 0 & 2 & 0 & -2 & 1 \\ 0 & -1 & 0 & 0 & 1 & -1 \end{array}\right] = 4$$

Therefore, the condition (29) is satisfied and there exists K such that (7) holds. In this case we have

$$P = \begin{bmatrix} 1 & -1 & 0 & -1 \\ -1 & 0 & -1 & -1 \\ 2 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$P\begin{bmatrix} E & A_0 & A_1 & A_2 \end{bmatrix} = \begin{bmatrix} \hat{E} & \hat{A}_0 & \hat{A}_1 & \hat{A}_2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \left[\begin{array}{cccc:cccc:cccc:cccc} 1 & 0 & -1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$PBQ = \begin{bmatrix} 0 \\ \hat{B}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ \cdots & \cdots \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Choosing $\hat{A} = \left[\begin{smallmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{smallmatrix}\right]$ and using (31), we obtain

$$K = Q \begin{bmatrix} 0 \\ \hat{B}_2^{-1}\hat{A}_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{35}$$

Hence

$$P\big[Ez_1z_2 - A_0 - BK - A_1z_1 - A_2z_2\big] = \begin{bmatrix} \hat{E}z_1z_2 - \hat{A}_1z_1 - \hat{A}_2z_2 - \hat{A}_0 \\ -\hat{A} \end{bmatrix}$$

$$= \begin{bmatrix} z_1z_2 - z_1 & -z_2 - 1 & -z_1z_2 & -z_1 - 2 \\ z_1z_2 - z_1 & z_1z_2 & -z_1z_2 & -z_1 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Therefore the model (1) with (34) is regularisable by (35).

If (4a) holds and $n > m$, then there exists a nonsingular matrix $\bar{P} \in \mathbb{R}^{n\times n}$ of elementary row operations such that

$$\bar{P}B = \begin{bmatrix} \bar{B} \\ 0 \end{bmatrix} \tag{36}$$

where $\bar{B} \in \mathbb{R}^{m\times m}$ is nonsingular.

Using (36), we obtain

$$\bar{P}\big[Ez_1z_2 - A_0 - BK - A_1z_1 - A_2z_2\big] = \bar{P}\big[E\ A_0\ A_1\ A_2\ B\big] \begin{bmatrix} I_nz_1z_2 \\ -I_n \\ -I_nz_1 \\ -I_nz_2 \\ -K \end{bmatrix}$$

$$= \begin{bmatrix} \bar{E}_1 & \bar{A}_{01} & \bar{A}_{11} & \bar{A}_{21} & \bar{B} \\ \bar{E}_2 & \bar{A}_{02} & \bar{A}_{12} & \bar{A}_{22} & 0 \end{bmatrix} \begin{bmatrix} I_nz_1z_2 \\ -I_n \\ -I_nz_1 \\ -I_nz_2 \\ -K \end{bmatrix} \tag{37}$$

where

$$\begin{bmatrix} \bar{E}_1 & \bar{A}_{01} & \bar{A}_{11} & \bar{A}_{21} \\ \bar{E}_2 & \bar{A}_{02} & \bar{A}_{12} & \bar{A}_{22} \end{bmatrix} = \bar{P}\big[E\ A_0\ A_1\ A_2\big], \qquad \begin{matrix} \bar{E}_1, \bar{A}_{01}, \bar{A}_{11}, \bar{A}_{21} \in \mathbb{R}^{m\times n} \\ \bar{E}_2, \bar{A}_{02}, \bar{A}_{12}, \bar{A}_{22} \in \mathbb{R}^{(n-m)\times n} \end{matrix}$$

♦

Theorem 3. *The model (1) is regularisable by state feedback (5) if and only if*

$$\mathrm{rank}\left[\bar{E}_2 z_1 z_2 - \bar{A}_{02} - \bar{A}_{12} z_1 - \bar{A}_{22} z_2\right] = n - m \quad \textit{for some} \quad (z_1, z_2) \in \mathbb{C}^2 \tag{38}$$

Proof. From (37) we have

$$\bar{P}\left[E z_1 z_2 - A_0 - BK - A_1 z_1 - A_2 z_2\right] = \begin{bmatrix} \bar{E}_1 z_1 z_2 - \bar{A}_{01} - \bar{B}K - \bar{A}_{11} z_1 - \bar{A}_{21} z_2 \\ \bar{E}_2 z_1 z_2 - \bar{A}_{02} - \bar{A}_{12} z_1 - \bar{A}_{22} z_2 \end{bmatrix} \tag{39}$$

Let the condition (38) be satisfied for $(z_1^0, z_2^0) \in \mathbb{C}^2$. Then it follows from (39) that there exists a matrix K such that (7) holds for $(z_1^0, z_2^0) \in \mathbb{C}^2$, since $\bar{B}$ is nonsingular. It is easy to see that if (38) does not hold, then there does not exist any matrix K such that (7) holds. ■

The condition (38) can be checked with the use of Lemma 1 or the 2-D shuffle algorithm (Kaczorek, 1993).

Example 3. Consider the model (1) with

$$E = \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 1 & 0 & 2 \\ -1 & 1 & 0 & 2 \\ 0 & -2 & 0 & -4 \\ 1 & -1 & 0 & -2 \end{bmatrix}$$

$$A_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ -2 & 0 & 0 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -2 & 1 \\ 0 & -1 \end{bmatrix} \tag{40}$$

It is easy to check that the model (1) with (40) is not regular and it does not satisfy the conditions of Theorem 2. In this case for $n = 4$, $m = 2$ and

$$\bar{P} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ -2 & 1 & -1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}$$

we have

$$\bar{P}\left[E\ A_0\ A_1\ A_2\ B\right] = \begin{bmatrix} \bar{E}_1 & \bar{A}_{01} & \bar{A}_{11} & \bar{A}_{21} & \bar{B} \\ \bar{E}_2 & \bar{A}_{02} & \bar{A}_{12} & \bar{A}_{22} & 0 \end{bmatrix}$$

$$= \left[\begin{array}{cccc:cccc:cccc:cccc:cc} 1 & 0 & -1 & 0 & 0 & 1 & 0 & 2 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \tag{41}$$

and

$$\operatorname{rank}\left[\bar{E}_2 z_1 z_2 - \bar{A}_{02} - \bar{A}_{12} z_1 - \bar{A}_{22} z_2\right] = \operatorname{rank}\begin{bmatrix} 1 & -z_1 - 1 & -z_2 & -2 \\ 0 & -z_1 & 0 & 0 \end{bmatrix} = 2$$

for some $(z_1, z_2) \in \mathbb{C}^2$. Therefore the condition (38) is satisfied and the model can be regularised by state feedback.

From (41) we have

$$\left[\bar{E}_1 z_1 z_2 - \bar{A}_{01} - \bar{A}_{11} z_1 - \bar{A}_{21} z_2\right] = \begin{bmatrix} z_1 z_2 - z_1 & -z_2 - 1 & -z_1 z_2 & -z_1 - 2 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{42}$$

and

$$\bar{B} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Choosing e.g.

$$K = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{43}$$

we obtain

$$\bar{P}\left[E z_1 z_2 - A_0 - BK - A_1 z_1 - A_2 z_2\right] = \begin{bmatrix} z_1 z_2 - z_1 & -z_2 - 1 & -z_1 z_2 & -z_1 - 2 \\ 0 & 0 & -1 & 0 \\ 1 & -z_1 - 1 & -z_2 & -2 \\ 0 & -z_1 & 0 & 0 \end{bmatrix}$$

and

$$\begin{aligned} &\det\left[E z_1 z_2 - A_0 - BK - A_1 z_1 - A_2 z_2\right] \\ &= \det \bar{P}^{-1} \det \begin{bmatrix} z_1 z_2 - 2 & -z_2 - 1 & -z_1 z_2 & -z_1 - 2 \\ 0 & 0 & -1 & 0 \\ 1 & -z_1 - 1 & -z_2 & -2 \\ 0 & -z_1 & 0 & 0 \end{bmatrix} \\ &= z_1(z_2 + 2) + 2(z_1 z_2 - 1) \end{aligned}$$

Therefore the closed-loop system (6) with (40) and (43) is regular. ♦

3.3. Singular 2D Roesser Model

Definition 4. The model (8) is called *regularised by state feedback (12)* if there exists a feedback matrix F such that (15) holds.

Theorem 4. *The model (8) is regularisable by state feedback (12) only if*

$$\operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_{11} \\ E_{21} & E_{22} & A_{21} & A_{22} & B_{22} \end{bmatrix} = n, \quad n := n_1 + n_2 \tag{44}$$

Proof. From the relation

$$\begin{bmatrix} E_{11}z_1 - A_{11} - B_{11}F_1 & E_{12}z_2 - A_{12} - B_{11}F_2 \\ E_{21}z_1 - A_{21} - B_{22}F_1 & E_{22}z_2 - A_{22} - B_{22}F_2 \end{bmatrix}$$

$$= \begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_{11} \\ E_{21} & E_{22} & A_{21} & A_{22} & B_{22} \end{bmatrix} \begin{bmatrix} I_{n_1}z_1 & 0 \\ 0 & I_{n_2}z_2 \\ -I_{n_1} & 0 \\ 0 & -I_{n_2} \\ -F_1 & -F_2 \end{bmatrix} \tag{45}$$

it follows that there exists $F = [F_1 \, , F_2]$ such that (15) holds only if (44) is satisfied. ■

Theorem 5. *The model (8) is regularisable by state feedback if*

$$\operatorname{rank}\begin{bmatrix} A_{11} & A_{12} & B_{11} \\ A_{21} & A_{22} & B_{22} \end{bmatrix} = n \tag{46a}$$

and/or

$$\operatorname{rank}\begin{bmatrix} E_{11} & E_{12} & B_{11} \\ E_{21} & E_{22} & B_{22} \end{bmatrix} = n \tag{46b}$$

Proof. The condition (15) is satisfied if the matrix (14) is nonsingular, since

$$\det \begin{bmatrix} E_{11}z_1 - A_{11} - B_{11}F_1 & E_{12}z_2 - A_{12} - B_{11}F_2 \\ E_{21}z_1 - A_{21} - B_{22}F_1 & E_{22}z_2 - A_{22} - B_{22}F_2 \end{bmatrix}\Bigg|_{\substack{z_1=0 \\ z_2=0}}$$

$$= \det \begin{bmatrix} -A_{11} - B_{11}F_1 & -A_{12} - B_{11}F_2 \\ -A_{21} - B_{22}F_1 & -A_{22} - B_{22}F_2 \end{bmatrix}$$

If (46a) holds, then there exist nonsingular matrices $P \in \mathbb{R}^{n\times n}$ and $Q \in \mathbb{R}^{m\times m}$ of elementary row and column operations such that

$$P[A\ B]\begin{bmatrix} I_n & 0 \\ 0 & Q \end{bmatrix} = \begin{bmatrix} \bar{A}_1 & \vdots & 0 & 0 \\ 0 & \vdots & \bar{B}_2 & 0 \\ \hat{A}_1 & \vdots & 0 & 0 \\ 0 & \vdots & 0 & \hat{B}_2 \end{bmatrix}$$

where $\begin{bmatrix} \bar{A}_1 \\ \hat{A}_1 \end{bmatrix}$ has full row rank and $\bar{B}_2$, $\hat{B}_2$ are nonsingular.

Let $\bar{A}_2$ and $\hat{A}_2$ be any matrices such that the matrix

$$\begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \hat{A}_1 \\ \hat{A}_2 \end{bmatrix}$$

is nonsingular. Then for

$$F = Q\begin{bmatrix} \bar{B}_2^{-1} & 0 \\ 0 & \hat{B}_2^{-1} \end{bmatrix}\begin{bmatrix} \bar{A}_2 \\ \hat{A}_2 \end{bmatrix} \tag{47}$$

we obtain

$$P[A+BF] = \begin{bmatrix} \bar{A}_1 \\ 0 \\ \hat{A}_1 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ \bar{B}_2 & 0 \\ 0 & 0 \\ 0 & \hat{B}_2 \end{bmatrix}\begin{bmatrix} \bar{B}_2^{-1} & 0 \\ 0 & \hat{B}_2^{-1} \end{bmatrix}\begin{bmatrix} \bar{A}_2 \\ \hat{A}_2 \end{bmatrix} = \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \\ \hat{A}_1 \\ \hat{A}_2 \end{bmatrix}$$

Now it will be shown that if (46b) holds, then there exists F such that (15) is satisfied. If (46b) holds, then there exist nonsingular matrices $P \in \mathbb{R}^{n\times n}$, $Q \in \mathbb{R}^{m\times m}$ such that

$$\bar{P}\begin{bmatrix} E_{11}z_1 - A_{11} - B_{11}F_1 & E_{12}z_2 - A_{12} - B_{11}F_2 \\ E_{21}z_1 - A_{21} - B_{22}F_1 & E_{22}z_2 - A_{22} - B_{22}F_2 \end{bmatrix}$$

$$= \bar{P}\begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_{11} \\ E_{21} & E_{22} & A_{21} & A_{22} & B_{22} \end{bmatrix}\begin{bmatrix} I_{n_1} & 0 & 0 & 0 & 0 \\ 0 & I_{n_2} & 0 & 0 & 0 \\ 0 & 0 & I_{n_1} & 0 & 0 \\ 0 & 0 & 0 & I_{n_2} & 0 \\ 0 & 0 & 0 & 0 & Q \end{bmatrix}\begin{bmatrix} I_{n_1}z_1 & 0 \\ 0 & I_{n_2}z_2 \\ -I_{n_1} & 0 \\ 0 & -I_{n_2} \\ -Q^{-1}F_1 & -Q^{-1}F_2 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & \bar{A}_{11} & \bar{A}_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & \bar{B}_1 & 0 \\ \bar{E}_{21} & \bar{E}_{22} & \bar{A}_{21} & \bar{A}_{22} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \bar{B}_2 \end{bmatrix} \begin{bmatrix} I_{n_1} z_1 & 0 \\ 0 & I_{n_2} z_2 \\ -I_{n_1} & 0 \\ 0 & -I_{n_2} \\ -Q^{-1}F_1 & -Q^{-1}F_2 \end{bmatrix}$$

$$= \begin{bmatrix} \bar{E}_1 z_1 - \bar{A}_{11} & \bar{E}_{12} z_2 - \bar{A}_{12} \\ 0 & 0 \\ \bar{E}_{21} z_1 - \bar{A}_{21} & \bar{E}_{22} z_2 - \bar{A}_{22} \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ \bar{B}_1 & 0 \\ 0 & 0 \\ 0 & \bar{B}_2 \end{bmatrix} \begin{bmatrix} Q^{-1}F_1 & Q^{-1}F_2 \end{bmatrix} \tag{48}$$

where the matrices $[\bar{E}_{11}\ \bar{E}_{12}\ \bar{A}_{11}\ \bar{A}_{12}]$ and $[\bar{E}_{21}\ \bar{E}_{22}\ \bar{A}_{21}\ \bar{A}_{22}]$ have full ranks equal to r_1 and r_2, respectively, and the matrices $\bar{B}_1$ and $\bar{B}_2$ are nonsingular.

Let $\bar{A}_1 \in \mathbb{R}^{(n_1 - r_1)\times n}$, $\bar{A}_2 \in \mathbb{R}^{(n_2 - r_2)\times n}$ be some matrices such that the matrix

$$D(z_1, z_2) := \begin{bmatrix} \bar{E}_1 z_1 - \bar{A}_{11} & \bar{E}_{12} z_2 - \bar{A}_{12} \\ \multicolumn{2}{c}{\bar{A}_1} \\ \bar{E}_{21} z_1 - \bar{A}_{21} & \bar{E}_{22} z_2 - \bar{A}_{22} \\ \multicolumn{2}{c}{\bar{A}_2} \end{bmatrix}$$

is nonsingular. Note that such matrices $\bar{A}_1$ and $\bar{A}_2$ exist, since by (46b)

$$\operatorname{rank} \begin{bmatrix} \bar{E}_{11} & \bar{E}_{12} & 0 & 0 \\ 0 & 0 & \bar{B}_1 & 0 \\ \bar{E}_{21} & \bar{E}_{22} & 0 & 0 \\ 0 & 0 & 0 & \bar{B}_2 \end{bmatrix} = n$$

For

$$[F_1\ F_2] = Q \begin{bmatrix} \bar{B}_1^{-1} & 0 \\ 0 & \bar{B}_2^{-1} \end{bmatrix} \begin{bmatrix} \bar{A}_1 \\ \bar{A}_2 \end{bmatrix} \tag{49}$$

from (48) we obtain

$$P \begin{bmatrix} E_{11} z_1 - A_{11} - B_{11} F_1 & E_{12} z_2 - A_{12} - B_{11} F_2 \\ E_{21} z_1 - A_{21} - B_{22} F_1 & E_{22} z_2 - A_{22} - B_{22} F_2 \end{bmatrix} = D(z_1, z_2)$$

∎

Example 4. Consider the model (8) with

$$
\begin{aligned}
E &= \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \left[\begin{array}{ccc:cc} 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 2 & 0 \\ 0 & 1 & 1 & 2 & 1 \\ \hdashline 1 & 0 & 1 & 0 & -1 \\ 2 & 0 & 2 & 0 & -2 \end{array}\right], \\
A &= \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{ccc:cc} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 \\ 1 & 1 & 1 & 1 & 0 \\ \hdashline 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 2 \end{array}\right], \quad B = \begin{bmatrix} B_{11} \\ B_{12} \end{bmatrix} = \left[\begin{array}{cc} 1 & 0 \\ 0 & 1 \\ 2 & -1 \\ \hdashline 1 & 1 \\ 2 & 3 \end{array}\right]
\end{aligned} \tag{50}
$$

It is easy to check that the model (8) with (50) is not regular and the pair (A, B) satisfies the condition (46a) for $n_1 = 3$, $n_2 = 2$. In this case for

$$
P = \begin{bmatrix} 2 & 1 & -1 & 0 & 0 \\ 0 & 1 & 0 & 2 & -1 \\ -1 & -1 & 1 & 0 & 0 \\ 1 & 1 & -1 & 3 & -1 \\ 0 & 0 & 0 & -2 & 1 \end{bmatrix}, \quad Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}
$$

we have

$$
PA = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad PBQ = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}
$$

and

$$
\bar{A}_1 = \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \\ 0 & 1 & 2 & 1 & -1 \end{bmatrix}, \quad \hat{A}_1 = [0\,1\,0\,0\,1], \quad \bar{B}_2 = [1], \quad \hat{B}_2 = [1]
$$

Choosing

$$
\bar{A}_2 = [0\,0\,1\,0\,0], \qquad \hat{A}_2 = [0\,0\,0\,1\,0]
$$

from (47) we obtain

$$F = Q \begin{bmatrix} \bar{B}_2^{-1} & 0 \\ 0 & \hat{B}_2^{-1} \end{bmatrix} \begin{bmatrix} \bar{A}_2 \\ \hat{A}_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{51}$$

It is easy to check that the matrix $A + BF$ is nonsingular and the closed-loop system (13) with (51) is regular.

If (11a) holds, then there exists a nonsingular matrix $\hat{P} \in \mathbb{R}^{n\times n}$ of elementary row operations such that

$$\hat{P} \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} = \begin{bmatrix} \hat{B}_1 \\ 0 \\ \hat{B}_2 \\ 0 \end{bmatrix} \tag{52}$$

where $\hat{B}_1 \in \mathbb{R}^{m_1\times m}$ and $\hat{B}_2 \in \mathbb{R}^{m_2\times m}$ have full row ranks and $m_1 + m_2 = m$.

Using (52), we obtain

$$\hat{P} \begin{bmatrix} E_{11}z_1 - A_{11} - B_{11}F_1 & E_{12}z_2 - A_{12} - B_{11}F_2 \\ E_{21}z_1 - A_{21} - B_{22}F_1 & E_{22}z_2 - A_{22} - B_{22}F_2 \end{bmatrix}$$

$$= \hat{P} \begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_{11} \\ E_{21} & E_{22} & A_{21} & A_{22} & B_{22} \end{bmatrix} \begin{bmatrix} I_{n_1}z_1 & 0 \\ 0 & I_{n_2}z_2 \\ -I_{n_1} & 0 \\ 0 & -I_{n_2} \\ -F_1 & -F_2 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{E}_{11} & \hat{E}_{12} & \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \tilde{E}_{11} & \tilde{E}_{12} & \tilde{A}_{11} & \tilde{A}_{12} & 0 \\ \hat{E}_{21} & \hat{E}_{22} & \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \tilde{E}_{21} & \tilde{E}_{22} & \tilde{A}_{21} & \tilde{A}_{22} & 0 \end{bmatrix} \begin{bmatrix} I_{n_1}z_1 & 0 \\ 0 & I_{n_2}z_2 \\ -I_{n_1} & 0 \\ 0 & -I_{n_2} \\ -F_1 & -F_2 \end{bmatrix} \tag{53}$$

where

$$\begin{bmatrix} \hat{E}_{11} & \hat{E}_{12} & \hat{A}_{11} & \hat{A}_{12} \\ \tilde{E}_{11} & \tilde{E}_{12} & \tilde{A}_{11} & \tilde{A}_{12} \\ \hat{E}_{21} & \hat{E}_{22} & \hat{A}_{21} & \hat{A}_{22} \\ \tilde{E}_{21} & \tilde{E}_{22} & \tilde{A}_{21} & \tilde{A}_{22} \end{bmatrix} = \hat{P} \begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} \\ E_{21} & E_{22} & A_{21} & A_{22} \end{bmatrix}$$

where $\hat{E}_{11}, \hat{A}_{11} \in \mathbb{R}^{m_1\times n_1}$, $\hat{E}_{12}, \hat{A}_{12} \in \mathbb{R}^{m_1\times n_2}$, $\hat{E}_{21}, \hat{A}_{21} \in \mathbb{R}^{m_2\times n_1}$, $\hat{E}_{22}, \hat{A}_{22} \in \mathbb{R}^{m_2\times n_2}$. ♦

Theorem 5. *The model (8) is regularisable by state feedback (12) if and only if*

$$\operatorname{rank} \tilde{D}(z_1, z_2) = n - m \quad \textit{for some} \quad (z_1, z_2) \in \mathbb{C}^2$$

where

$$\tilde{D}(z_1, z_2) := \begin{bmatrix} \tilde{E}_{11} z_1 - \tilde{A}_{11} & \tilde{E}_{12} z_2 - \tilde{A}_{12} \\ \tilde{E}_{21} z_1 - \tilde{A}_{21} & \tilde{E}_{22} z_2 - \tilde{A}_{22} \end{bmatrix} \tag{54}$$

Proof. From (53) we have

$$\hat{P} \begin{bmatrix} E_{11} z_1 - A_{11} - B_{11} F_1 & E_{12} z_2 - A_{12} - B_{11} F_2 \\ E_{21} z_1 - A_{21} - B_{22} F_1 & E_{22} z_2 - A_{22} - B_{22} F_2 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{E}_{11} z_1 - \hat{A}_{11} - \hat{B}_1 F_1 & \hat{E}_{12} z_2 - \hat{A}_{12} - \hat{B}_1 F_2 \\ \tilde{E}_{11} z_1 - \tilde{A}_{11} & \tilde{E}_{12} z_2 - \tilde{A}_{12} \\ \hat{E}_{21} z_1 - \hat{A}_{21} - \hat{B}_2 F_1 & \hat{E}_{22} z_2 - \hat{A}_{22} - \hat{B}_2 F_2 \\ \tilde{E}_{21} z_1 - \tilde{A}_{21} & \tilde{E}_{22} z_2 - \tilde{A}_{22} \end{bmatrix} \tag{55}$$

In a similar way as in the proof of Theorem 3, it can be easily shown that there exists a matrix F such that (15) holds if and only if (54) is satisfied. ■

Remark 1. The matrix F can be computed as follows: Choose a pair $(z_1^0, z_2^0) \in \mathbb{C}^2$ such that $\tilde{D}(z_1^0, z_2^0)$ has full row rank. Then choose a matrix $\hat{A} \in \mathbb{R}^{m \times n}$ such that the matrix $\begin{bmatrix} \tilde{D}(z_1^0, z_2^0) \\ \hat{A} \end{bmatrix}$ is nonsingular. From (55) we have

$$F = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix} \left[\hat{D}(z_1^0, z_2^0) - \hat{A} \right] \tag{56}$$

$$\hat{D}(z_1, z_2) := \begin{bmatrix} \hat{E}_{11} z_1 - \hat{A}_{11} & \hat{E}_{12} z_2 - \hat{A}_{12} \\ \hat{E}_{21} z_1 - \hat{A}_{21} & \hat{E}_{22} z_2 - \hat{A}_2 \end{bmatrix}$$

Remark 2. Instead of the transformation (53), the following equivalent one can also be used:

$$\hat{P} \begin{bmatrix} E_{11} z_1 - A_{11} - B_{11} F_1 & E_{12} z_2 - A_{12} - B_{11} F_2 \\ E_{21} z_1 - A_{21} - B_{22} F_1 & E_{22} z_2 - A_{22} - B_{22} F_2 \end{bmatrix}$$

$$= \begin{bmatrix} \hat{E}_1 & \hat{A}_1 & \hat{B} \\ \tilde{E}_2 & \tilde{A}_2 & 0 \end{bmatrix} \begin{bmatrix} I_{n_1} z_1 & 0 \\ 0 & I_{n_2} z_2 \\ -I_{n_1} & 0 \\ 0 & -I_{n_2} \\ -F_1 & -F_2 \end{bmatrix} \tag{53'}$$

where

$$\begin{bmatrix} \hat{E}_1 & \hat{A}_1 \\ \tilde{E}_2 & \tilde{A}_2 \end{bmatrix} = \hat{P} \begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} \\ E_{21} & E_{22} & A_{21} & A_{22} \end{bmatrix}, \quad \hat{E}_1, \hat{A}_1 \in \mathbb{R}^{m\times n}, \quad \tilde{E}_2, \tilde{A}_2 \in \mathbb{R}^{(n-m)\times n}$$

$\hat{P} \in \mathbb{R}^{n\times n}$ and $\hat{B} \in \mathbb{R}^{m\times m}$ are nonsingular. In this case the condition (52) is equivalent to the condition

$$\operatorname{rank}\left\{ [\tilde{E}_2 \; \tilde{A}_2] \begin{bmatrix} I_{n_1} z_1 & 0 \\ 0 & I_{n_2} z_2 \\ -I_{n_1} & 0 \\ 0 & -I_{n_2} \end{bmatrix} \right\} = n - m \quad \text{for some } (z_1, z_2) \in \mathbb{C}^2 \quad (54')$$

Example 5. Consider the model (8) with

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix} = \left[\begin{array}{ccc:cc} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 2 \\ \hdashline 0 & 1 & 0 & 0 & 1 \\ 0 & 2 & 0 & 0 & 3 \end{array}\right]$$

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} = \left[\begin{array}{ccc:cc} 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \hdashline 0 & 1 & 0 & -1 & -1 \\ 0 & 2 & 0 & -2 & 0 \end{array}\right], \quad B = \begin{bmatrix} B_{11} \\ B_{22} \end{bmatrix} = \left[\begin{array}{cc} 0 & 0 \\ 0 & 1 \\ 0 & 1 \\ \hdashline 1 & 0 \\ 2 & 0 \end{array}\right] \qquad (57)$$

It is easy to check that the model (8) with (57) is not regular and the conditions (46) are not satisfied. In this case $n_1 = 3$, $n_2 = 2$, $m = 2$ and for

$$\hat{P} = \left[\begin{array}{ccc:cc} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ -1 & -1 & 1 & 0 & 0 \\ \hdashline 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 \end{array}\right]$$

we obtain

$$\hat{P}\begin{bmatrix} E_{11} & E_{12} & A_{11} & A_{12} & B_{11} \\ E_{21} & E_{22} & A_{21} & A_{22} & B_{22} \end{bmatrix} = \begin{bmatrix} \hat{E}_{11} & \hat{E}_{12} & \hat{A}_{11} & \hat{A}_{12} & \hat{B}_1 \\ \tilde{E}_{11} & \tilde{E}_{12} & \tilde{A}_{11} & \tilde{A}_{12} & 0 \\ \hat{E}_{21} & \hat{E}_{22} & \hat{A}_{21} & \hat{A}_{22} & \hat{B}_2 \\ \tilde{E}_{21} & \tilde{E}_{22} & \tilde{A}_{21} & \tilde{A}_{22} & 0 \end{bmatrix}$$

$$= \left[\begin{array}{ccc:cc:ccc:cc:cc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hdashline 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & -1 & -1 & 1 & 0 \\ \hdashline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \end{array}\right] \tag{58}$$

Hence

$$\operatorname{rank}\tilde{D}(z_1,z_2) = \operatorname{rank}\left[\begin{array}{ccc:cc} 0 & 0 & 0 & 1 & z_2 \\ 0 & 0 & z_1 & 0 & z_2 \\ \hdashline 0 & 0 & 0 & 0 & z_2-2 \end{array}\right] = 3 \quad \text{for some} \quad (z_1,z_2)\in\mathbb{C}^2$$

Therefore the condition (52) is satisfied and the model can be regularised by state feedback. From (56) we have

$$\hat{D}(z_1,z_2) := \left[\begin{array}{ccc:cc} 0 & 0 & -1 & 0 & 0 \\ \hdashline 0 & z_1-1 & 0 & 1 & z_2+1 \end{array}\right]$$

and $\hat{B}_1 = [0\ 1]$, $\hat{B}_2 = [1\ 0]$. Choosing e.g. $z_1^0 = z_2^0 = 1$ and $\hat{A} = \left[\begin{smallmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{smallmatrix}\right]$, from (56) we obtain

$$F = \begin{bmatrix} \hat{B}_1 \\ \hat{B}_2 \end{bmatrix}^{-1} \left[\hat{D}(z_1^0,z_2^0) - \hat{A}\right] = \begin{bmatrix} 0 & -1 & 0 & 1 & 2 \\ -1 & 0 & -1 & 0 & 0 \end{bmatrix} \tag{59}$$

and

$$\begin{bmatrix} \hat{E}_{11}z_1 - \hat{A}_{11} - \hat{B}_1F_1 & \hat{E}_{12}z_2 - \hat{A}_{12} - \hat{B}_1F_2 \\ \tilde{E}_{11}z_1 - \tilde{A}_{11} & \tilde{E}_{12}z_2 - \tilde{A}_{12} \\ \hat{E}_{21}z_1 - \hat{A}_{21} - \hat{B}_2F_1 & \hat{E}_{22}z_2 - \hat{A}_{22} - \hat{B}_2F_2 \\ \tilde{E}_{21}z_1 - \tilde{A}_{21} & \tilde{E}_{22}z_2 - \tilde{A}_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & z_2 \\ 0 & 0 & z_1 & 0 & z_2 \\ 0 & z_1 & 0 & 0 & z_2-1 \\ 0 & 0 & 0 & 0 & z_2-2 \end{bmatrix} \tag{60}$$

The matrix (60) is nonsingular for some $(z_1, z_2) \in \mathbb{C}^2$. Therefore, the model (8) with (57) is state-feedback regularised by (59). ♦

4. Concluding Remarks

The regularisation problem by state-feedbacks has been formulated for the singular 2D first Fornasini-Marchesini model and the singular 2D Roesser model. New necessary and sufficient conditions have been established for regularisation of the models and procedures for computation of the feedback matrices have been given. The procedures have been illustrated by numerical examples. With slight modifications the approach can be extended to the singular 2D general model (Kaczorek, 1993; Kurek, 1985).

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