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REGULARISATION OF SINGULAR 2D FORNASINI-MARCHESINI MODELS BY OUTPUT FEEDBACKS

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Necessary and sufficient conditions are established for regularisation of the singular 2D Fornasini-Marchesini model by an output feedback. Some procedures are presented for testing the regularity and regularisability of 2D models and for computation of the output feedback matrix. The procedures are illustrated by a numerical example.

1. Introduction

The regularisation of singular linear systems by state and output feedbacks has been considered in many papers (Bunse-Gestner *et al.*, 1992; 1994; Miminis, 1993; Ozcaldiran and Lewis, 1990). In (Bunse-Gestner *et al.*, 1994) it was shown that proportional and derivative output feedback controls can be constructed such that the closed-loop system is regular and has index at most one. The regularity guarantees the existence and uniqueness of solutions to singular linear systems (Campbell, 1980; Kaczorek, 1985; Ozcaldiran and Lewis, 1990). The regularisation problem by state feedbacks for the singular 2D Roesser model and the singular 2D first Fornasini-Marchesini model has been formulated and some necessary conditions and sufficient conditions have been established in (Kaczorek, 1985; 1997a; 1997b). The aim of this paper is to extend the result of (Kaczorek, 1997b) for the regularisation problem of the singular 2D first Fornasini-Marchesini model by an output feedback. Necessary and sufficient conditions will be established under which the singular 2D first Fornasini-Marchesini model can be regularised by an output feedback. Some procedures will be presented for testing the regularity and regularisability of the 2D model and for computation of the feedback matrix. The procedures are illustrated by a numerical example.

2. Statement of the Regularisation Problems

2.1. Regularisation Problem for the 2D Fornasini-Marchesini Model

Let $\mathbb{R}^{p\times q}$ be the set of $p\times q$ matrices with real entries and $\mathbb{R}^p := \mathbb{R}^{p\times 1}$. Consider

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the 2D Fornasini-Marchesini model (Fornasini and Marchesini, 1978; Kaczorek, 1993)

$$Ex_{i+1,j+1} = A_0 x_{ij} + A_1 x_{i+1,j} + A_2 x_{i,j+1} + Bu_{ij}, \qquad i,j \in \mathbb{Z}_+ \tag{1a}$$

$$y_{ij} = Cx_{ij} \tag{1b}$$

where $x_{ij} \in \mathbb{R}^n$ is a semistate vector, $y_{ij} \in \mathbb{R}^p$ denotes an output vector, $u_{ij} \in \mathbb{R}^m$ stands for an input vector, E, $A_k \in \mathbb{R}^{n\times n}$, $k = 0,1,2$, $B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{p\times n}$ with $\det E = 0$, $\mathbb{Z}_+$ is the set of nonnegative integers.

Definition 1. The model (1) is called *regular* if

$$\det\left[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2\right] \neq 0 \text{ for some } (z_1,z_2) \in \mathbb{C}^2 \tag{2}$$

where $\mathbb{C}^2 = \mathbb{C}\times\mathbb{C}$, $\mathbb{C}$ being the field of complex numbers. The model (1) is called *singular* if

$$\det\left[Ez_1z_2 - A_0 - A_1z_1 - A_2z_2\right] = 0 \text{ for all } (z_1,z_2) \in \mathbb{C}^2 \tag{3}$$

Let the output feedback have the form

$$u_{ij} = Fy_{ij} + v_{ij} \tag{4}$$

where $F \in \mathbb{R}^{m\times p}$ and v_{ij} is a new input vector.

Substituting (4) into (1a) and using (1b), we obtain

$$Ex_{i+1,j+1} = (A_0 + BFC)x_{ij} + A_1x_{i+1,j} + A_2x_{i,j+1} + Bv_{ij} \tag{5}$$

The regularisation problem for the model (1) can be formulated as follows:

Problem 1. Given matrices E, A_0, A_1, A_2, B, C of the singular model (1) with (3), find an output feedback matrix F of (4) such that the closed-loop system (5) is regular, i.e.

$$\det\left[Ez_1z_2 - A_0 - BFC - A_1z_1 - A_2z_2\right] \neq 0 \text{ for some } (z_1,z_2) \in \mathbb{C}^2 \tag{6}$$

2.2. Regularisation Problem for the 2D Roesser Model

Consider the 2D singular Roesser model (Kaczorek, 1993; Roesser, 1975)

$$\bar{E}x_{ij}^{(1)} = \bar{A}x_{i,j} + \bar{B}u_{i,j} \tag{7a}$$

$$y_{ij} = \bar{C}x_{ij} \tag{7b}$$

where $\bar{E}$, $\bar{A} \in \mathbb{R}^{n\times n}$, $\bar{B} \in \mathbb{R}^{n\times m}$, $\bar{C} \in \mathbb{R}^{p\times n}$ with $\det\bar{E} = 0$, $x_{ij} = \begin{bmatrix} x_{i,j}^h \\ x_{i,j}^v \end{bmatrix}$, $x_{ij}^{(1)} = \begin{bmatrix} x_{i+1,j}^h \\ x_{i,j+1}^v \end{bmatrix}$, $x_{ij}^h \in \mathbb{R}^{n_1}$ is a horizontal state vector, $x_{i,j}^v \in \mathbb{R}^{n_2}$ denotes a vertical state vector, $n_1 + n_2 = n$, $u_{i,j} \in \mathbb{R}^m$ stands for an input vector, and $y_{ij} \in \mathbb{R}^p$ is an output vector.

Definition 2. The model (7) is called *regular* if

$$\det\left[\bar{E}\operatorname{diag}\left[I_{n_1}z_1, I_{n_2}z_2\right] - \bar{A}\right] \neq 0 \text{ for some } (z_1,z_2) \in \mathbb{C}^2 \tag{8}$$

where I_k denotes the $k \times k$ identity matrix. Otherwise it is called *singular.*

Let the output feedback for (7) have the form

$$u_{ij} = \bar{F}y_{ij} + v_{ij} \tag{9}$$

where $\bar{F} \in \mathbb{R}^{m\times n}$ and v_{ij} is a new input vector. Substituting (9) into (7a) and using (7b) we obtain

$$\bar{E}x_{i,j}^{(1)} = (\bar{A} + \bar{B}\bar{F}\bar{C})x_{ij} + \bar{B}v_{ij} \tag{10}$$

The regularisation problem for (7) can be formulated as follows.

Problem 2. Given matrices $\bar{E}$, $\bar{A}$, $\bar{B}$, $\bar{C}$ of the singular model (7), find an output feedback matrix $\bar{F}$ of (9) such that the closed-loop model (10) is regular, i.e.

$$\det\left[\bar{E}\operatorname{diag}\left[I_{n_1}z_1, I_{n_2}z_2\right] - \bar{A} - \bar{B}\bar{F}\bar{C}\right] \neq 0 \text{ for some } (z_1,z_2) \in \mathbb{C}^2$$

It is easy to show that the model (7) is a particular case of (1) for

$$E := 0, \quad A_0 := \bar{A}, \quad A_1 := -\bar{E}\begin{bmatrix} I_{n_1} & 0 \\ 0 & 0 \end{bmatrix}, \quad A_2 := -\bar{E}\begin{bmatrix} 0 & 0 \\ 0 & I_{n_2} \end{bmatrix}, \quad B := \bar{B}$$

Therefore the regularisation problem of the Roesser model (7) is a particular case of the regularisation problem for (1).

In what follows, we shall consider only the details of the solution to Problem 1.

3. Preliminaries

Lemma 1. (Kaczorek, 1984) *The matrix equation* $AXB = C$ *is equivalent to the equation* $(A \otimes B^T)x = c$, *where* $x = [x_1, x_2, \dots, x_k]^T$, $c = [c_1, c_2, \dots, c_l]^T$, x_i, c_j, $i = 1,2,\dots,k$, $j = 1,2,\dots,l$ *are the rows of the matrices* X, C, *respectively. The upper index* T *denotes the transposition and* $\otimes$ *denotes the Kronecker product.*

Lemma 2. *Let the polynomials*

$$g_i(x) := \sum_j a_{ij}x^{k_{ij}}, \qquad i = 1,\dots,m \tag{11}$$

where a_{ij} *are given real coefficients and* $g_i(x_i) \neq 0$, $i = 1,\dots,m$, *for at least one point* $x_i \in \mathbb{R}$, $i = 1,\dots,m$. *Then there exists a point* $x_0 \in \mathbb{R}$ *such that* $g_i(x_0) \neq 0$ *for all* $i = 1,\dots,m$.

Proof. From the assumptions of the lemma, the polynomials $g_i(x)$ are not identicaly equal to zero. Then they can take zero values only at isolated points. Thus there always exist points at which they have nonzero values (simultaneously). ■

Lemma 3. *Let the polynomials*

$$g_i(x_1,\dots,x_n) := \sum_j a_{ij} x_1^{k_{ij}^{(1)}} \cdots x_n^{k_{ij}^{(n)}}, \qquad i=1,\dots,m \tag{12}$$

where a_{ij} are given real coefficients and $g_i(x_{1i},\dots,x_{ni}) \neq 0$, $i=1,\dots,m$, for at least one point $(x_{1i},\dots,x_{ni}) \in \mathbb{R}^n$, $i=1,\dots,m$. Then there exists a point $(x_{10},\dots,x_{n0}) \in \mathbb{R}^n$ such that $g_i(x_{10},\dots,x_{n0}) \neq 0$ for all $i=1,\dots,m$.

Proof. Let

$$g_i(x_{1i},\dots,x_{ni}) \neq 0, \quad i=1,\dots,m, \quad x_{ki} \in \mathbb{R} \tag{13}$$

Applying Lemma 2 for the polynomials $g_i(x_1,x_{2i},\dots,x_{ni})$ in one variable x_1 we can find $x_{10} \in \mathbb{R}$ such that $g_i(x_{10},x_{2i},\dots,x_{ni}) \neq 0$ and for the polynomials $g_i(x_{10},x_2,x_{3i},\dots,x_{ni})$ in variable x_2 we can find $x_{20} \in \mathbb{R}$ such that $g_i(x_{10},x_{20},x_{3i},\dots,x_{ni}) \neq 0$. Repeating this procedure, after n steps we can find $(x_{10},\dots,x_{n0}) \in \mathbb{R}$ such that $g_i(x_{10},x_{20},\dots,x_{n0}) \neq 0$. ■

Lemma 4. *The pencils*

$$G_i(z_1,z_2) := E_i z_1 z_2 - A_{0i} - A_{1i} z_1 - A_{2i} z_2, \qquad i=1,\dots,k \tag{14}$$

are regular if and only if the pencil

$$\operatorname{diag}\left\{G_1(z_1,z_2),\dots,G_k(z_1,z_2)\right\} \tag{15}$$

is regular.

Proof. The lemma will be proved only for $k=2$. The general case (for $k>2$) can be proved in a similar way.

Sufficiency is trivial.

Necessity. We will prove that if the pencils

$$\begin{cases} G_1(z_1,z_2) := E_1 z_1 z_2 - A_{01} - A_{11} z_1 - A_{21} z_2 \\ G_2(z_1,z_2) := E_2 z_1 z_2 - A_{02} - A_{12} z_1 - A_{22} z_2 \end{cases} \tag{16}$$

are regular, then there exists at least one pair $(z_{01},z_{02}) \in \mathbb{C}$ such that

$$\det G_1(z_{01},z_{02}) \neq 0, \qquad \det G_2(z_{01},z_{02}) \neq 0 \tag{17}$$

Let

$$g_1(z_1,z_2) := \det G_1(z_1,z_2), \qquad g_2(z_1,z_2) := \det G_2(z_1,z_2) \tag{18}$$

where $g_1(z_1, z_2)$ and $g_2(z_1, z_2)$ are some polynomial in variables z_1 and z_2. Define

$$z_1 = z_{11} + jz_{12}, \quad z_2 = z_{21} + jz_{22}, \quad z_{kl} \in \mathbb{R}, \ k = 1,2, \ l = 1,2 \tag{19}$$

$$g_1(z_1, z_2) = g_{11}(z_{11}, z_{12}, z_{21}, z_{22}) + jg_{12}(z_{11}, z_{12}, z_{21}, z_{22}) \tag{20}$$

$$g_2(z_1, z_2) = g_{21}(z_{11}, z_{12}, z_{21}, z_{22}) + jg_{22}(z_{11}, z_{12}, z_{21}, z_{22}) \tag{21}$$

where $j := \sqrt{-1}$.

It is assumed that

$$g_1(z_{11}, z_{21}) \neq 0 \ \text{ for } (z_{11}, z_{12}) \in \mathbb{C}, \ z_{11} = z_{11}^1 + jz_{11}^2, \ z_{21} = z_{21}^1 + jz_{21}^2 \tag{22}$$

$$g_2(z_{12}, z_{22}) \neq 0 \ \text{ for } (z_{12}, z_{22}) \in \mathbb{C}, \ z_{12} = z_{12}^1 + jz_{12}^2, \ z_{22} = z_{22}^1 + jz_{22}^2 \tag{23}$$

Without lost of generality we may assume that

$$g_{11}(z_{11}^1, z_{11}^2, z_{21}^1, z_{21}^2) \neq 0, \qquad g_{21}(z_{12}^1, z_{12}^2, z_{22}^1, z_{22}^2) \neq 0 \tag{24}$$

By Lemma 3 there exists $(z_{10}^1, z_{10}^2, z_{20}^1, z_{20}^2) \in \mathbb{R}^4$ such that

$$g_{11}(z_{10}^1, z_{10}^2, z_{20}^1, z_{20}^2) \neq 0, \qquad g_{21}(z_{10}^1, z_{10}^2, z_{20}^1, z_{20}^2) \neq 0 \tag{25}$$

and

$$\det G_1(z_{10}, z_{20}) \neq 0. \qquad \det G_2(z_{10}, z_{20}) \neq 0$$

for $z_{10} := z_{10}^1 + jz_{10}^2$, $z_{20} := z_{20}^1 + jz_{20}^2$. ∎

Lemma 5. *Consider the block matrix in the form*

$$A = \begin{bmatrix} A_1 + K & A_2 \\ A_3 & A_4 \end{bmatrix} \tag{26}$$

where $K \in \mathbb{R}^{n_1 \times n_2}$ *is arbitrary and* $A \in \mathbb{C}^{n \times n}$, $A_1 \in \mathbb{C}^{n_1 \times n_2}$, $A_2 \in \mathbb{C}^{n_1 \times (n-n_2)}$, $A_3 \in \mathbb{C}^{(n-n_1) \times n_2}$, $A_4 \in \mathbb{C}^{(n-n_1) \times (n-n_2)}$ *are given. There exists a matrix* $K \in \mathbb{R}^{n_1 \times n_2}$ *such that* A *is nonsingular if and only if the matrices* $[A_3, A_4]$ *and* $\left[\begin{smallmatrix} A_2 \\ A_4 \end{smallmatrix}\right]$ *have full row and column rank, respectively.*

Proof. Neccesity is trivial.

Sufficiency. It is assumed that the matrices $[A_3, A_4]$ and $\left[\begin{smallmatrix} A_2 \\ A_4 \end{smallmatrix}\right]$ has full row rank and full column rank, respectively. We will show that there always exists a matrix K such that A is nonsingular.

Let $M_4 \in \mathbb{C}^{(n-n_1) \times (n-n_1)}$ and $N_4 \in \mathbb{C}^{(n-n_2) \times (n-n_2)}$ be nonsingular matrices such that

$$M_4 A_4 N_4 = \begin{bmatrix} 0 & 0 \\ 0 & I_{r_4} \end{bmatrix} \tag{27}$$

$$\bar{A} := \begin{bmatrix} I & 0 \\ 0 & M_4 \end{bmatrix} A \begin{bmatrix} I & 0 \\ 0 & N_4 \end{bmatrix} = \left[\begin{array}{c|c:c} A_1 + K & A_{21} & A_{22} \\ \hline A_{31} & 0 & 0 \\ \hdashline A_{32} & 0 & I_{r_4} \end{array}\right] \tag{28}$$

where the matrices A_{21} and A_{31} have full column rank and full row rank, respectively. We have

$$\hat{A} := \begin{bmatrix} I & 0 & -A_{22} \\ 0 & I & 0 \\ 0 & 0 & I_{r_4} \end{bmatrix} \bar{A} \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -A_{32} & 0 & I_{r_4} \end{bmatrix} = \left[\begin{array}{c|c:c} A_1 - A_{22}A_{32} + K & A_{21} & 0 \\ \hline A_{31} & 0 & 0 \\ \hdashline 0 & 0 & I_{r_4} \end{array}\right] \tag{29}$$

Let $M_2 = \begin{bmatrix} M_{21} \\ M_{22} \end{bmatrix}$, $N_2 = [N_{21}\ N_{22}]$ be nonsingular matrices such that

$$M_2 A_{21} = \begin{bmatrix} A_{20} \\ 0 \end{bmatrix}, \qquad A_{31} N_2 = \begin{bmatrix} A_{30} & 0 \end{bmatrix}$$

where A_{20} and A_{30} are nonsingular matrices with appropriate dimensions. Let

$$\begin{aligned} M_2(A_1 - A_{22}A_{32})N_2 &= \begin{bmatrix} M_{21}(A_1 - A_{22}A_{32})N_{21} & M_{21}(A_1 - A_{22}A_{32})N_{22} \\ M_{22}(A_1 - A_{22}A_{32})N_{21} & M_{22}(A_1 - A_{22}A_{32})N_{22} \end{bmatrix} \\ &:= \begin{bmatrix} A'_{11} & A'_{12} \\ A'_{21} & A'_{22} \end{bmatrix} \end{aligned} \tag{30}$$

$$M_2 K N_2 = \begin{bmatrix} M_{21}KN_{21} & M_{21}KN_{22} \\ M_{22}KN_{21} & M_{22}KN_{22} \end{bmatrix} := \begin{bmatrix} K'_{11} & K'_{12} \\ K'_{21} & K'_{22} \end{bmatrix} \tag{31}$$

$$\tilde{A} := \begin{bmatrix} M_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} \hat{A} \begin{bmatrix} N_2 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix} = \begin{bmatrix} A'_{11} + K'_{11} & A'_{12} + K'_{12} & A_{20} & 0 \\ A'_{21} + K'_{21} & A'_{22} + K'_{22} & 0 & 0 \\ A_{30} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r_4} \end{bmatrix} \tag{32}$$

Note that there always exists a matrix $K \in \mathbb{R}^{n_1 \times n_2}$ such that $A'_{22} + K'_{22}$ is nonsingular, which guarantees that the matrix $\tilde{A}$ is nonsingular. Therefore there always exists $K \in \mathbb{R}^{n_1 \times n_2}$ such that the matrix A is nonsingular. ∎

Let $L_q \in \mathbb{R}^{q^2 \times p}$ be an arbitrary matrix with real entries and

$$\lambda_q(z_1, z_2) := \left[z_1^q z_2^q,\, z_1^q z_2^{q-1}, \ldots, z_1^q,\, z_1^{q-1} z_2^q,\, z_1^{q-1} z_2^{q-1}, \ldots, z_1,\, z_2,\, 1\right]^T$$

Lemma 6. *Let $E, A_0, A_1, A_2 \in \mathbb{R}^{s \times n}$, $s \le n$ be given matrices such that*

$$\det G(z_1, z_2) := \det\left[E z_1 z_2 - A_0 - A_1 z_1 - A_2 z_2\right] = 0 \quad \text{for all } (z_1, z_2) \in \mathbb{C}^2 \tag{33}$$

and $q \in \mathbb{Z}_+$ be the smallest degree of the vector $x(z_1, z_2) = L_{q+1}^T \lambda_q(z_1 z_2)$ such that

$$x^T(z_1, z_2) G(z_1, z_2) = 0 \quad \text{for all } (z_1, z_2) \in \mathbb{C}^2 \tag{34}$$

Then $q < s$ (where q is the smallest degree of z_1 then the smallest degree of z_2 is $q_2 < s^2$).

Proof. It is assumed that q is the smallest degree of the variable z_1. Let the smallest degree solution of (34) have the form

$$x(z_1, z_2) = x_q(z_2) z_1^q + x_{q-1}(z_2) z_1^{q-1} + \cdots + x_1(z_2) z_1 + x_0(z_2) \tag{35}$$

Define

$$\bar{E}(z_2) := E z_2 - A_1, \qquad \bar{A}(z_2) := A_2 z_2 + A_0 \tag{36}$$

From (34) we have

$$x_0^T \bar{A}(z_2) = 0 \tag{37a}$$

$$x_i^T \bar{E}(z_2) = x_{i+1}^T \bar{A}(z_2), \qquad i = 0, 1, \ldots, q-1 \tag{37b}$$

$$x_q^T \bar{E}(z_2) = 0 \tag{37c}$$

for all $z_2 \in \mathbb{C}$.

We shall show by contradiction that the vectors $x_1^T \bar{A}(z_2), x_2^T \bar{A}(z_2), \ldots, x_q^T \bar{A}(z_2)$ are linearly independent for all $z_2 \in \mathbb{C}$. Assume that

$$\begin{aligned} x_k^T \bar{A}(z_2) &= a_1 x_1^T \bar{A}(z_2) + a_2 x_2^T \bar{A}(z_2) + \cdots + a_{k-1} x_{k-1}^T \bar{A}(z_2) \\ &\quad + a_{k+1} x_{k+1}^T \bar{A}(z_2) + \cdots + a_q x_q^T \bar{A}(z_2) \end{aligned} \tag{38}$$

$$\begin{aligned} x_k^T \bar{A}(z_2) &= x_{k-1}^T \bar{E}(z_2) = a_1 x_0^T \bar{E}(z_2) + a_2 x_1^T \bar{E}(z_2) + \cdots + a_{k-1} x_{k-2}^T \bar{E}(z_2) \\ &\quad + a_{k+1} x_k^T \bar{E}(z_2) + \cdots + a_q x_q^T \bar{E}(z_2) \\ &= \left(a_1 x_0^T + a_2 x_1^T + \cdots + a_{k-1} x_{k-2} + a_{k+1} x_k^T + \cdots + a_q x_{q-1}^T\right) \bar{E}(z_2) \\ &= \Bigg(\underbrace{a_1 x_0^T + a_2 x_1^T + \cdots + a_{k-1} x_{k-2}^T - x_{k-1}^T + a_{k+1} x_k^T + \cdots + a_q x_{q-1}^T}_{\hat{x}_{q-1}^T}\Bigg) \bar{E}(z_2) \\ &=: \hat{x}_{q-1} \bar{E}(z_2) = 0 \end{aligned} \tag{39}$$

$$\hat{x}_{q-1}^T \bar{A}(z_2) = \left(a_1 x_0^T + a_2 x_1^T + \cdots + a_{k-1} x_{k-2}^T - x_{k-1}^T + a_{k+1} x_k^T + \cdots + a_q x_{q-1}^T\right) \bar{A}(z_2)$$
$$= \Big(\underbrace{a_2 x_0^T + \cdots + a_{k-1} x_{k-3}^T - x_{k-1}^T + a_{k+1} x_{k-1}^T + \cdots + a_q x_{q-2}^T}_{\hat{x}_{q-2}^T}\Big) \bar{E}(z_2)$$
$$=: \hat{x}_{q-2} \bar{E}(z_2)$$

....

$$\hat{x}_1^T \bar{A}(z_2) = a_q x_1^T \bar{A}(z_2) = \underbrace{a_q x_0^T}_{\hat{x}_0^T} \bar{E}(z_2) =: \hat{x}_0^T \bar{E}(z_2)$$
$$\hat{x}_0^T \bar{A}(z_2) = a_q x_0^T \bar{A}(z_2) = 0$$

From the comparison of (39) and (37) it follows that the vector

$$\hat{x}(z_1, z_2) = \hat{x}_{q-1}(z_2) z_1^{q-1} + \hat{x}_{q-2}(z_2) z_1^{q-2} + \cdots + \hat{x}_0(z_2) \tag{40}$$

is a solution of degree $q-1$ to (34), so we obtain a contradiction by assumption that q is the smallest degree of the solutions to (34). Therefore the vectors $x_1^T \bar{A}(z_2), x_2^T \bar{A}(z_2), \ldots, x_q^T \bar{A}(z_2)$ are linearly independent for all $z_2 \in \mathbb{C}$.

We shall show that the vectors $x_0^T(z_2), x_1^T(z_2), \ldots, x_q^T(z_2)$ are linearly independent for all $z_2 \in \mathbb{C}$. Assume that

$$b_0 x_0^T(z_2) + b_1 x_1^T(z_2) + \cdots + b_q x_q^T(z_2) = 0 \tag{41}$$

Then

$$b_1 x_1^T(z_2) \bar{A}(z_2) + b_2 x_2^T(z_2) \bar{A}(z_2) + \cdots + b_q x_q^T(z_2) \bar{A}(z_2) = 0 \tag{42}$$

since $x_0^T(z_2)\bar{A}(z_2) = 0$. The vectors $x_1^T \bar{A}(z_2), x_2^T \bar{A}(z_2), \ldots, x_q^T \bar{A}(z_2)$ are linearly independent and $b_i = 0$, $i = 1, 2, \ldots, q$. Since q is the smallest degree of vectors $x(z_1, z_2)$, it follows that $x_0(z_2) \neq 0$. From (41) we have $b_0 = 0$ and the vectors $x_0^T(z_2), x_1^T(z_2), \ldots, x_q^T(z_2)$ are linearly independent. Therefore $q < s$.

In the case when q is the smallest degree of the variable z_2, the proof is similar. ■

4. Solution to Problem 1

Let P be a row compression matrix of the matrix B, and Q be a column compression matrix of the matrix C. Define

$$B' := PB = \begin{bmatrix} B_1 \\ 0_{s,m} \end{bmatrix}, \qquad C' := CQ = \begin{bmatrix} C_1 & 0_{q,t} \end{bmatrix} \tag{43}$$

where B_1 has full row rank, C_1 has full column rank, $r_B \le \min(m, n)$ is the rank of the matrix B, $r_C \le \min(q, n)$ is the rank of C, $s = n - r_B$, $t = n - r_C$. Examples of such matrices are $P = U_B^T$ and $Q = V_C$, where U_B^T and U_C^T are the SVD matrices of B and C:

$$B = U_B \Sigma_B V_B^T, \qquad C = U_C \Sigma_C V_C^T \tag{44}$$

We define

$$\begin{cases} E' := PEQ = \begin{bmatrix} E_1 & E_2 \\ E_3 & E_4 \end{bmatrix}, & A_0' := PA_0Q = \begin{bmatrix} A_{01} & A_{02} \\ A_{03} & A_{04} \end{bmatrix} \\ A_1' := PA_1Q = \begin{bmatrix} A_{11} & A_{12} \\ A_{13} & A_{14} \end{bmatrix}, & A_2' := PA_2Q = \begin{bmatrix} A_{21} & A_{22} \\ A_{23} & A_{24} \end{bmatrix} \end{cases} \tag{45}$$

Theorem 1. *The singular model (1) with (3) is regularisable, i.e. there exists a matrix $F \in \mathbb{R}^{m\times p}$ such that the closed-loop model (5) is regular if and only if the matrix*

$$G^h(z_1, z_2) := \begin{bmatrix} E^h z_1 z_2 - A_0^h - A_1^h z_1 - A_2^h z_2 \end{bmatrix} \tag{46}$$

has full row rank for some $(z_1, z_2) \in \mathbb{C}$ and the matrix

$$G^v(z_1, z_2) := \begin{bmatrix} E^v z_1 z_2 - A_0^v - A_1^v z_1 - A_2^v z_2 \end{bmatrix} \tag{47}$$

has full column rank for some $(z_1, z_2) \in \mathbb{C}$, where

$$E^h := \begin{bmatrix} E_3 \; E_4 \end{bmatrix}, \; A_0^h := \begin{bmatrix} A_{03} \; A_{04} \end{bmatrix}, \; A_1^h := \begin{bmatrix} A_{13} \; A_{14} \end{bmatrix}, \; A_2^h := \begin{bmatrix} A_{23} \; A_{24} \end{bmatrix} \tag{48}$$

$$E^v := \begin{bmatrix} E_2 \\ E_4 \end{bmatrix}, \quad A_0^v := \begin{bmatrix} A_{02} \\ A_{04} \end{bmatrix}, \quad A_1^v := \begin{bmatrix} A_{12} \\ A_{14} \end{bmatrix}, \quad A_2^v := \begin{bmatrix} A_{22} \\ A_{24} \end{bmatrix} \tag{49}$$

Proof. We shall first prove that there exists a matrix F such that the closed-loop model is regular if and only if the matrices

$$\begin{cases} G^h(z_1, z_2) := \begin{bmatrix} E^h z_1 z_2 - A_0^h - A_1^h z_1 - A_2^h z_2 \end{bmatrix} \\ G^v(z_1, z_2) := \begin{bmatrix} E^v z_1 z_2 - A_0^v - A_1^v z_1 - A_2^v z_2 \end{bmatrix} \end{cases} \tag{50}$$

have simultaneously full row and column ranks for some $(z_1, z_2) \in \mathbb{C}^2$.

Necessity is trivial.

Sufficiency. We assume that the matrices $G^h(z_{10}, z_{20})$ and $G^v(z_{10}, z_{20})$ have full row and column ranks. By Lemma 5 there exists a matrix F_0 such that

$$\det \begin{bmatrix} E_1 z_{10} z_{20} - A_{11} z_{10} - A_{21} z_{20} - A_{01} - B_1 F_0 C_1 & E_2 z_{10} z_{20} - A_{12} z_{10} - A_{22} z_{20} - A_{02} \\ E_3 z_{10} z_{20} - A_{13} z_{10} - A_{23} z_{20} - A_{03} & E_4 z_{10} z_{20} - A_{14} z_{10} - A_{24} z_{20} - A_{04} \end{bmatrix} \neq 0$$

Hence for $F = F_0$ the closed-loop model is regular.

From Lemma 4 it follows that the matrices $G^h(z_1,z_2)$ and $G^v(z_1,z_2)$ have simultaneously full row and column ranks for some $(z_1,z_2)\in\mathbb{C}^2$ if and only if $G^h(z_1,z_2)$ has full row rank for some $(z_1,z_2)\in\mathbb{C}^2$ and $G^v(z_1,z_2)$ has full column rank for some $(z_1,z_2)\in\mathbb{C}^2$. ■

Define

$$M_E^i := \mathrm{diag}\left\{\underbrace{\begin{bmatrix} I_i \\ \hline 0_{1,i}\end{bmatrix},\dots,\begin{bmatrix} I_i \\ \hline 0_{1,i}\end{bmatrix}}_{i^2}\right\}$$

$$M_0^i := \Phi\{M_E^i, i+2\},\quad M_1^i := \Phi\{M_E^i, i+1\},\quad M_2^i := \Phi\{M_E^i, 1\} \tag{51}$$

where

$$\Phi\{x,k\} := \begin{bmatrix} 0 & I_k \\ I_{n-k} & 0 \end{bmatrix} X$$

for any matrix $X\in\mathbb{R}^{n\times m}$.

Theorem 2. *The model (1) is regularisable, i.e. there exists a matrix F of (4) such that the closed-loop model (5) is regular if and only if the matrices*

$$G_{s-1}^h := M_E^{s-1}\otimes(E^h)^T - M_0^{s-1}\otimes(A_0^h)^T - M_1^{s-1}\otimes(A_1^h)^T - M_2^{s-1}\otimes(A_2^h)^T \tag{52a}$$

$$G_{t-1}^v := M_E^{t-1}\otimes E^v - M_0^{t-1}\otimes A_0^v - M_1^{t-1}\otimes A_1^v - M_2^{t-1}\otimes A_2^v \tag{52b}$$

have full column ranks.

Proof. By Theorem 1 the regularisation problem has a solution if and only if the matrix

$$G^h(z_1,z_2) := \left[E^h z_1 z_2 - A_0^h - A_1^h z_1 - A_2^h z_2\right] \tag{53a}$$

has full row rank for some $(z_1,z_2)\in\mathbb{C}^2$ and the matrix

$$G^v(z_1,z_2) := \left[E^v z_1 z_2 - A_0^v - A_1^v z_1 - A_2^v z_2\right] \tag{53b}$$

has full column rank for some $(z_1,z_2)\in\mathbb{C}^2$.

Now we shall prove that

i) *The matrix (53a) has full row rank for some $(z_1,z_2)\in\mathbb{C}^2$ if and only if the matrix (52a) has full column rank.*

ii) *The matrix (53b) has full column rank for some $(z_1,z_2)\in\mathbb{C}^2$ if and only if the matrix (52b) has full column rank.*

i) Let $q \in \mathbb{Z}$ be a nonnegative integer and $L_q \in \mathbb{R}^{q^2 \times p}$. Define

$$\lambda_q(z_1, z_2) := \left[z_1^q z_2^q, z_1^q z_2^{q-1}, \ldots, z_1^q, z_1^{q-1} z_2^q, z_1^{q-1} z_2^{q-1}, \ldots, z_1 z_2, 1\right]^T \quad (54)$$

The matrix (53a) has full row rank for some $(z_1, z_2) \in \mathbb{C}^2$ if and only if the relation

$$\lambda_q^T(z_1, z_2) L_{q+1} G^h(z_1, z_2) = 0 \quad (55)$$

implies $L_{q+1} \equiv 0$ for any $q \in \mathbb{Z}_+$.

By Lemma 6, *if* $G^h(z_1, z_2)$ *does not have full row rank and* $q_{\min} = \min q$ *is the smallest of positive integers* q *such that* $\lambda_q^T(z_1, z_2) L_{q+1} G^h(z_1, z_2) = 0$, *then* $q_{\min} < s$. Therefore the matrix (53a) has full row rank for some $(z_1, z_2) \in \mathbb{C}^2$ if and only if the relation

$$\lambda_{s-1}^T(z_1, z_2) L_{s-1} G^h(z_1, z_2) = 0 \quad (56)$$

implies $L_{s-1} \equiv 0$.

From (53a) and Lemma 1, it follows that (56) is equivalent to

$$\left(M_E^{s-1} \otimes (E)^T - M_0^{s-1} \otimes (A_0^h)^T - M_1^{s-1} \otimes (A_1^h)^T - M_2^{s-1} \otimes (A_2^h)^T\right) l = 0 \quad (57)$$

where $l := [l_1, l_2, \ldots, l_{s-1}]^T$, l_{i-1}, $i = 1, 2, \ldots, s-1$ are rows of the matrix L_{s-1}. Hence (53) has a nonzero solution l if and only if the condition of the theorem is satisfied.

ii) It can be proved in a similar way. ■

5. Procedures for Testing the Regularity and Regularisability of the Model and Computation of the Output Feedback Matrix

We shall present a procedure for testing the regularity of a given 2D matrix pencil (2).

Procedure 1.

Step 1. For given matrices $E, A_0, A_1, A_2 \in \mathbb{R}^{n \times n}$ find the matrices $M_E^n, M_0^n, M_1^n, M_2^n$ from the relations (51).

Step 2. Compute the matrix $G_n = M_E^n \otimes E^T - M_0^n \otimes A_0^T - M_1^n \otimes A_1^T - M_2^n \otimes A_2^T$.

The pencil $Ez_1z_2 - A_0 - A_1z_1 - A_2z_2$ is regular if G_n has full column rank, otherwise it is singular. The 2D shuffle algorithm (Kaczorek, 1993) can be also used for testing regularity of the 2D pencil (2).

The following procedure can be used for testing regularisability of the singular 2D Fornasini-Marchesini model (1):

Procedure 2.

Step 1. Compute a row compression matrix of the matrix B and a column compression matrix of the matrix C satisfying (43).

Step 2. Compute the matrices (45).

Step 3. Compute the matrices (51).

Step 4. Compute the matrices (52).

The model (1) is regularisable if the matrices (52) have full column ranks, otherwise it is not regularisable.

If the conditions of Theorem 1 are satisfied, then the output feedback matrix can be calculated by using the following algorithm:

Procedure 3.

Step 1. Find row compression and column compression matrices P and Q satisfying (43), and by using (45) compute the matrices E', A'_0, A'_1, A'_2.

Step 2. Find a pair $(z_{10}, z_{20}) \in \mathbb{C}^2$ such that the matrices (46) and (47) have full row and column ranks. Compute the matrix

$$G_0 := \begin{bmatrix} E_1 z_{10} z_{20} - A_{11} z_{10} - A_{21} z_{20} - A_{01} & E_2 z_{10} z_{20} - A_{12} z_{10} - A_{22} z_{20} - A_{02} \\ E_3 z_{10} z_{20} - A_{13} z_{10} - A_{23} z_{20} - A_{03} & E_4 z_{10} z_{20} - A_{14} z_{10} - A_{24} z_{20} - A_{04} \end{bmatrix}$$

Step 3. Compute the matrix $\bar{G}_4 := E_4 z_{10} z_{20} - A_{14} z_{10} - A_{24} z_{20} - A_{04}$ and find nonsingular matrices P_3 and Q_3 such that $P_3 \bar{G}_4 Q_3 = \left[\begin{smallmatrix} 0 & 0 \\ 0 & \bar{G}'_4 \end{smallmatrix}\right]$, $\bar{G}'_4$ is nonsingular, and compute

$$G_3 := \begin{bmatrix} I & 0 \\ 0 & P_3 \end{bmatrix} G_0 \begin{bmatrix} I & 0 \\ 0 & Q_3 \end{bmatrix} = \left[\begin{array}{c|c:c} G_{11} & G_{21} & G_{22} \\ \hline G_{31} & 0 & 0 \\ \hdashline G_{32} & 0 & \bar{G}'_4 \end{array}\right]$$

Step 4. Compute

$$P_4 = \begin{bmatrix} I & 0 & -G_{22}(\bar{G}'_4)^{-1} \\ 0 & I & 0 \\ 0 & 0 & I \end{bmatrix}, \quad Q_4 = \begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ -(\bar{G}'_4)^{-1} G_{32} & 0 & I \end{bmatrix}$$

$$F_{11} = G_{22}(\bar{G}_4')^{-1}G_{32}, \quad G_4 := P_4 G_3 Q_4 = \left[\begin{array}{c|c:c} G_{11}' & G_{21} & 0 \\ \hline G_{31} & 0 & 0 \\ \hdashline 0 & 0 & \bar{G}_4' \end{array}\right]$$

where G_{31} and G_{21} have full row and column ranks, respectively.

Step 5. Find matrices L_{21} and R_{31} such that $L_{21}G_{21} = \begin{bmatrix} G_{21}' \\ 0 \end{bmatrix}$, $\det G_{21}' \neq 0$, $G_{31}R_{31} = \begin{bmatrix} G_{31}' & 0 \end{bmatrix}$, $\det G_{31}' \neq 0$,

$$P_5 = \begin{bmatrix} L_{21} & 0 \\ 0 & I \end{bmatrix}, \quad Q_5 = \begin{bmatrix} R_{31} & 0 \\ 0 & I \end{bmatrix}, \quad G_5 := P_5 G_4 Q_5 = \left[\begin{array}{c:c|c:c} \times & \times & G_{21}' & 0 \\ \hdashline \times & G_{11}'' & 0 & 0 \\ \hline G_{31}' & 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 & \bar{G}_4' \end{array}\right]$$

Step 6. Let

$$F_0 := \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 - G_{11}'' \end{bmatrix}$$

where V_1, V_2, V_3, V_4 are arbitrary matrices such that $\det V_4 \neq 0$. Then $F_1 := L_{21}^{-1}F_0R_{31}^{-1} + F_{11}$ and $F = (B_1)^{-1}F_1(C_1)^{-1}$ is a desired feedback matrix.

Note that there exist infinitely many desired matrices F, since there exist many pairs $(z_{10}, z_{20}) \in \mathbb{C}^2$ such that the matrices (50) have simultaneously full row and column ranks and the matrices V_1, V_2, V_3, V_4 are arbitrary.

6. Numerical Example

The proposed procedures have been implemented in the MATLAB language for testing the regularity and regularisability, and for computing the output feedback matrix. In the procedure for computation of the feedback matrix, we use an identity matrix in every matrix block which has to be nonsingular, and we use a zero matrix in every matrix block which can be arbitrary. For convenience, we shall make use of the MATLAB notation for matrices.

Consider the system (1) with

$$E = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_0 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

Using Procedure 1, we test the regularity:

Step 1.

$$M_E^{n-1} = \text{sparse}\Big([[1:6],[8:13],[15:20],[22:27],[29:34],[36:41]],\ [1:36],\ [1,1,\dots,1]\Big) \in \mathbb{R}^{49\times 36}$$

$$M_0^{n-1} = \text{sparse}\Big([[9:14],[16:21],[23:28],[30:35],[37:42],[44:49]],\ [1:36],\ [1,1,\dots,1]\Big) \in \mathbb{R}^{49\times 36}$$

$$M_1^{n-1} = \text{sparse}\Big([[2:7],[9:14],[16:21],[23:28],[30:35],[37:42]],\ [1:36],\ [1,1,\dots,1]\Big) \in \mathbb{R}^{49\times 36}$$

$$M_2^{n-1} = \text{sparse}\Big([[8:13],[15:20],[22:27],[29:34],[36:41],[43:48]],\ [1:36],\ [1,1,\dots,1]) \in \mathbb{R}^{49\times 36}$$

Step 2.

$$G_n = M_E^n \otimes E^T - M_0^n \otimes A_0^T - M_1^n \otimes A_1^T - M_2^n \otimes A_2^T \in \mathbb{R}^{294\times 216}$$

Step 3.

$\operatorname{rank} G_n = 196 < 216$, so the matrix G_n does not have full column rank. Hence the system (1) is singular.

Using Procedure 2, we test the regularisability:

Step 1. Note that the matrices B and C have already the row and column compressed forms $P = I$ and $Q = I$.

Step 2. The matrices (45) are: $E' = E$, $A_0' = A_0$, $A_1' = A_1$, $A_2' = A_2$, $B' = B$, $C' = C$.

Step 3. The matrices (51) are as follows:

$$M_E^2 = \text{sparse}\Big([1,2,3,5,6,7,9,10,11],\ [1:9],\ [1,1,\ldots,1]\Big)$$

$$M_0^2 = \text{sparse}\Big([6,7,8,10,11,12,14,15,16],\ [1:9],\ [1,1,\ldots,1]\Big)$$

$$M_1^2 = \text{sparse}\Big([2,3,4,6,7,8,10,11,12],\ [1:9],\ [1,1,\ldots,1]\Big)$$

$$M_2^2 = \text{sparse}\Big([5,6,7,9,10,11,13,14,15],\ [1:9],\ [1,1,\ldots,1]\Big)$$

Step 4. The matrix

$$\begin{aligned} G_2^h = \text{sparse}\Big(&[8,10,31,25,1,26,14,16,37,31,7,32,20,22,43,37,13,38,32,34,\\ &55,49,25,50,38,40,61,55,31,56,44,46,67,61,37,62,56,58,79,\\ &73,49,74,62,64,85,79,55,80,68,70,91,85,61,86,],\\ &[1,1,1,2,3,3,4,4,4,5,6,6,7,7,7,8,9,9,10,10,10,11,12,12,13,\\ &13,13,14,15,15,16,16,16,17,18,18,19,19,19,20,21,21,22,22,\\ &22,23,24,24,25,25,25,26,27,27],\\ &[[-1,-1,-1,-1],1,-\text{ones}(1,5),\ldots,-\text{ones}(1,5),1,-1]\Big) \end{aligned}$$

in (52a) has full column rank. Similarly, the matrix G_2^v in (52b) also has full column rank and so the model (1) is regularisable.

In order to find a feedback matrix, we shall use Procedure 3.

Step 1. The matrices B, C have already row and column compressed forms $P = I$, $Q = I$ and the matrices (45) are as follows: $E' = E$, $A_0' = A_0$, $A_1' = A_1$, $A_2' = A_2$, $B' = B$, $C' = C$.

Step 2. For $(z_{10}, z_{20}) = (2,2)$ we have

$$G_0 = \begin{bmatrix} 4 & 4 & -2 & -1 & -2 & 4 \\ 0 & -1 & 0 & -2 & 0 & -2 \\ -1 & -2 & 0 & 0 & 0 & 0 \\ -2 & -1 & -2 & -2 & 0 & 0 \\ 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Step 3.

$$\bar{G}_4 := E_4 z_{10} z_{20} - A_{14} z_{10} - A_{24} z_{20} - A_{04} = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$P_3 = \begin{bmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad Q_3 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$G_3 = \begin{bmatrix} 4 & 4 & -2 & -2 & 4 & 1 \\ 0 & -1 & 0 & 0 & -2 & -2 \\ -1 & -2 & 0 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 2 & 1 & 2 & 0 & 0 & 2 \end{bmatrix}$$

Step 4.

$$P_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0.5 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \qquad Q_4 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ -1 & -0.5 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$F_{11} = \begin{bmatrix} -1 & -0.5 & -1 \\ -2 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}, \qquad G_4 = \begin{bmatrix} 5 & 4.5 & -1 & -2 & 4 & 0 \\ 2 & 0 & 2 & 0 & -2 & 0 \\ -1 & -2 & 0 & 0 & 0 & 0 \\ 4 & -2 & 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

Step 5.

$$L_{21} = \begin{bmatrix} -0.9239 & 0.3827 & 0 \\ 0.3827 & 0.9239 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad R_{31} = \begin{bmatrix} 0.6154 & -0.7882 & 0 \\ -0.7882 & -0.6154 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Step 6.

$$F_0 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} -1 & -0.5 & -1 \\ -2 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and } F = \begin{bmatrix} -1 & -0.5 & -1 \\ -2 & -1 & -2 \\ 0 & 0 & 1 \end{bmatrix}$$

7. Concluding Remarks

The regularisation problem by output feedbacks has been formulated for the singular 2D first Fornasini-Marchesini model and the singular 2D Roesser model. Necessary and sufficient conditions have been established for regularisation of the singular 2D first Fornasini-Marchesini model and procedures for computation of a feedback matrix and for testing the regularity and regularisability have been given. The procedures have been illustrated by a numerical example.

A regularisation problem of the 2D singular model by state feedback is a particular case of the regularisation problem by output feedback when $C \equiv I_n$. With slight modifications the presented method can be also applied for regularisation of the singular general 2D model (Kaczorek, 1993; Kurek, 1985).

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