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IMPLICIT STATE ESTIMATORS AND THEIR APPLICATION TO POLE ASSIGNMENT CONTROLLERS FOR SYSTEMS WITH UNCERTAINTIES

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This paper presents implicit robust state observers for SISO minimum phase dynamical systems with arbitrarily relative degrees (with respect to the relation between the disturbance and the output). For systems with relative degree one, the state is expressed by the filters of the input and the output. No *a-priori* knowledge of the disturbance is required in this case. For systems with higher relative degrees, by first estimating the disturbance, the state vector is asymptotically expressed by the filters of the input, the filters of the output and the estimates of the first-order filters of the disturbance. Then the state observer and the estimated disturbance are applied to a controller to place desired poles and to cancel the disturbance. Finally, examples and simulation results show that the proposed algorithms are effective.

1. Introduction

The problem of controlling uncertain dynamical systems subject to external disturbances has been one of the topics of interest recently. Many of the proponents of the associated theoretical developments have found it convenient to assume that the system state vector is available for use by the control scheme. In practice, it is not always possible to measure the state vector. In such cases, either a design method based solely upon the input and output information is required, or a suitable estimate of the state vector has to be constructed for use in the original control law. This paper considers the latter approach.

As for the state estimation problem for the systems with uncertainties, relatively few authors have considered it. It is known that the VSS theory has many advantages in solving the problems with uncertainties (DeCarlo *et al.*, 1988; Utkin, 1992). But about its application to the state and disturbance estimation problems, very few theoretical works have been reported.

Utkin (1987) presents a discontinuous observer by forcing the error between the estimated and measured outputs to exhibit a sliding mode. And it is pointed out that the proposed method finds a difficulty in selecting the switched gain owing to the

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uncertainty of the initial condition. Walcott *et al.* (1987) and Walcott and Zak (1988) use a Lyapunov-based approach to formulate an observer in the presence of bounded disturbances. Edwards and Spurgeon (1994) effectively consider the problem first proposed by Walcott *et al.* (1987). However, all these results are subject to MIMO minimum phase systems with relative degree one (with respect to the relation of disturbance-output). For uncertain systems (even for SISO uncertain systems) with higher relative degrees, very few authors have discussed the design problems of the state observers.

For SISO systems with relative degree two, the state observer is constructed by using the estimated disturbance and the filters of the input and the output (Chen and Minamide, 1996). Further, Chen (1996) gives a robust observer for third-order systems with arbitrarily relative degrees. In this work, the disturbance is estimated recursively by using the VSS equivalent control theory.

This paper deals with the robust observer design problems for SISO minimum phase dynamical systems with arbitrarily relative degrees (with respect to the relation between the disturbance and the output). In Section 2, the problem is formulated. In Section 3, by introducing a filter with n different poles, a new implicit observer is formulated by employing first-order filters of the input, output and disturbance. In Section 4, by using the VSS approach, the disturbance is estimated. For plants with relative degree one, an observer is constructed without *a-priori* knowledge of the disturbance. For plants with higher relative degrees, an observer is constructed by the estimates of the first-order filters of the disturbance and the filters of the input and the output. In Section 5, the obtained observer and the estimated disturbance are applied to a pole assignment controller which also has a function to cancel the disturbance. Finally, numerical examples are given to illustrate the proposed algorithms.

2. Problem Formulation

Consider the system described by

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) + kv(x,u,t), \qquad x(t_0) = x_0 \\ y(t) = c^T x(t) \end{cases} \tag{1}$$

where $x(t)$ is an unknown state vector with known dimension n, t_0 stands for the starting time, x_0 denotes the unknown initial state, $u(t)$ and $y(t)$ are respectively the scalar input and output. Furthermore, $v(x,u,t)$ is a signal composed of the model uncertainties, the nonlinear parts of the system and the disturbances. It is bounded:

$$\left|v(x,u,t)\right| \le \rho(y,u,t) \tag{2}$$

where $\rho(y,u,t)$ is a known function. Finally, A, b, k, c are known matrices given in the observable canonical form

$$A = \left[\begin{array}{c|c} -a_1 & I \\ \vdots & \text{---} \\ -a_n & 0 \end{array}\right] \triangleq \left[\begin{array}{c|c} -a & \begin{array}{c} I \\ \text{---} \\ 0 \end{array} \end{array}\right], \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}, \quad k = \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix}, \quad c = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \tag{3}$$

Only the case where

$$k(s) = k_1 s^{n-1} + \cdots + k_{n-1}s + k_n \tag{4}$$

is a Hurwitz polynomial will be discussed.

This paper attempts to construct a robust state observer and a state feedback pole-assignment controller for system (1). For simplicity, in the sequel the signal $v(x,u,t)$ will be called the disturbance of the system and denoted by $v(t)$.

3. Implicit Observers

3.1. The Traditional Implicit State Observer

To begin with, define a stable $n \times n$ matrix F by

$$F = \left[\begin{array}{c|c} -f & \begin{array}{c} I \\ \hline 0 \end{array} \end{array}\right] \tag{5}$$

Then (1) can be rewritten as

$$\begin{aligned} \dot{x}(t) &= Fx(t) + (f-a)y(t) + bu(t) + kv(t) \\ &\triangleq Fx(t) + h_a y(t) + h_b u(t) + h_c v(t), \quad x(t_0) = x_0 \end{aligned} \tag{6}$$

Now, let us define the following three matrices:

$$L(h) = \begin{bmatrix} h_1 & h_2 & \cdots & h_{n-1} & h_n \\ h_2 & h_3 & \cdots & h_n & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ h_{n-1} & h_n & \cdots & 0 & 0 \\ h_n & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{7}$$

$$U(f) = \begin{bmatrix} 0 & f_1 & \cdots & f_{n-2} & f_{n-1} \\ 0 & 0 & \cdots & f_{n-3} & f_{n-2} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & f_1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \tag{8}$$

$$H(f,h) = L(h)\Big(I + U(f)\Big) - L(f)U(h) \tag{9}$$

Some useful properties of the matrix $H(f,h)$ are stated in the next lemma.

Lemma 1. *$H(f,h)$ is a symmetric matrix satisfying*

$$H(f,h)\frac{\xi(s)}{\det(sI-F)} = (sI-F)^{-1}h \tag{10}$$

where $\xi(s) = \left[s^{n-1}, \ldots, s, 1\right]^T$. *Further, if the polynomials*

$$f(s) \triangleq \det(sI - F) = s^n + f_1 s^{n-1} + \cdots + f_{n-1}s + f_n \tag{11}$$

and

$$h(s) = h_1 s^{n-1} + \cdots + h_{n-1}s + h_n \tag{12}$$

are coprime, then $H(f, h)$ *is non-singular.*

Proof. See (Minamide *et al.*, 1983). ■

It is worth mentioning that the initial conditions for all the filters of the input, output and disturbance are assumed to be zero in this paper. Fortunately, this treatment does not lose any generality since non-zero initial conditions only contribute to the state some additive terms which decay exponentially to zero.

In this paper, s denotes, as the case may be, the Laplace-transform variable or the differential operator $\mathrm{d}(\cdot)/\mathrm{d}t$. Taking the Laplace transform of (6) gives

$$X(s) = (sI - F)^{-1}\Big\{h_a Y(s) + h_b U(s) + h_c V(s) + x_0\Big\} \tag{13}$$

By Lemma 1, from (13) the state vector can be reconstructed as

$$\begin{aligned} x(t) \;=\; & H(f, h_a)\frac{\xi(s)}{f(s)}y(t) + H(f, h_b)\frac{\xi(s)}{f(s)}u(t) \\ & + H(f, h_c)\frac{\xi(s)}{f(s)}v(t) + H(f, x_0)z(t) \end{aligned} \tag{14}$$

where $z(t)$ is an exponentially decreasing vector defined as

$$\dot{z}(t) = F^T z(t), \qquad z(t_0) = \left[1, 0, \ldots, 0\right]^T \tag{15}$$

Remark 1. It should be pointed out that s denotes the differential operator and $(\xi(s)/f(s))v(t)$ is not available in (14).

As the state can be reconstructed by (14), the traditional implicit state observer is constructed by the following result.

Theorem 1. *The implicit state observer* $\hat{x}(t)$ *can be formulated as*

$$\hat{x}(t) = H(f, h_a)\frac{\xi(s)}{f(s)}y(t) + H(f, h_b)\frac{\xi(s)}{f(s)}u(t) + H(f, h_c)\frac{\xi(s)}{f(s)}v(t) \tag{16}$$

Proof. From (14)–(16), it is obvious that $x(t) - \hat{x}(t) \to 0$ as $t \to \infty$, where the roots of $f(s)$ determine the rate convergence.

3.2. A New Implicit State Observer

In this section, some operations will be carried out on the state observer introduced in (16). Now, let us consider the Hurwitz polynomial $f(s)$ in (16) defined as

$$f(s) = s^n + f_1 s^{n-1} + \cdots + f_n = (s+\lambda_1)(s+\lambda_2)\cdots(s+\lambda_n) \tag{17}$$

where $\lambda_i \neq \lambda_j$ as $i \neq j$, for $i, j = 1, \ldots, n$.

Pre-multiplying (16) by the vector $\left[\lambda_i^{n-1}, \lambda_i^{n-2}, \ldots, (-1)^{n-1}\right]$ yields

$$\begin{aligned}[\lambda_i^{n-1}, -\lambda_i^{n-2}, \ldots, (-1)^{n-1}]\hat{x}(t) &= [\lambda_i^{n-1}, -\lambda_i^{n-2}, \ldots, (-1)^{n-1}]H(f, h_a)\frac{\xi(s)}{f(s)}y(t) \\ &+ [\lambda_i^{n-1}, -\lambda_i^{n-2}, \ldots, (-1)^{n-1}]H(f, h_b)\frac{\xi(s)}{f(s)}u(t) \\ &+ [\lambda_i^{n-1}, -\lambda_i^{n-2}, \ldots, (-1)^{n-1}]H(f, h_c)\frac{\xi(s)}{f(s)}v(t)\end{aligned} \tag{18}$$

Lemma 2. *For the matrix $H(f,h)$ defined in (7)–(9), the following equation is valid:*

$$\left[\lambda^{n-1}, -\lambda^{n-2}, \ldots, (-1)^{n-1}\right] H(f,h) = \chi\,[1 \;\; g_1 \;\; \cdots \;\; g_{n-1}] \tag{19}$$

where $g_1, \ldots, g_{n-1}$ and χ are respectively described by

$$f(s) = \left(s^{n-1} + g_1 s^{n-2} + \cdots + g_{n-1}\right)(s+\lambda) \tag{20}$$

and

$$\chi = h_1\lambda^{n-1} - h_2\lambda^{n-2} + \cdots + (-1)^{n-1}h_n \tag{21}$$

Proof. See Appendix 1. ■

Therefore, by applying Lemma 2, we have

$$\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \ldots, (-1)^{n-1}\right] H(f,h_a) = \chi_{ia}\,[1 \;\; g_{i,1} \;\; \cdots \;\; g_{i,n-1}] \tag{22}$$

$$\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \ldots, (-1)^{n-1}\right] H(f,h_b) = \chi_{ib}\,[1 \;\; g_{i,1} \;\; \cdots \;\; g_{i,n-1}] \tag{23}$$

$$\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \ldots, (-1)^{n-1}\right] H(f,h_c) = \chi_{ic}\,[1 \;\; g_{i,1} \;\; \cdots \;\; g_{i,n-1}] \tag{24}$$

where

$$\chi_{ia} = (f_1 - a_1)\lambda_i^{n-1} - (f_2 - a_2)\lambda_i^{n-2} + \cdots + (-1)^{n-1}(f_n - a_n) \tag{25}$$

$$\chi_{ib} = b_1\lambda_i^{n-1} - b_2\lambda_i^{n-2} + \cdots + (-1)^{n-1}b_n \tag{26}$$

$$\chi_{ic} = k_1\lambda_i^{n-1} - k_2\lambda_i^{n-2} + \cdots + (-1)^{n-1}k_n \tag{27}$$

and $g_{i,1}, \ldots, g_{i,n-1}$ are determined by

$$f(s) = \left(s^{n-1} + g_{i,1}s^{n-2} + \cdots + g_{i,n-1}\right)(s + \lambda_i) \tag{28}$$

Thus, by (22)–(28), eqn. (18) can be simplified as

$$\left[\lambda_i^{n-1}, -\lambda_i^{n-2}, \ldots, (-1)^{n-1}\right] \hat{x}(t) = \chi_{ic}\frac{v(t)}{s + \lambda_i} + \chi_{ia}\frac{y(t)}{s + \lambda_i} + \chi_{ib}\frac{u(t)}{s + \lambda_i} \tag{29}$$

Now, for $i = 1, 2, \ldots, n$, writing the n equations in (29) in a compact form yields

$$\begin{bmatrix} \lambda_1^{n-1} & -\lambda_1^{n-2} & \cdots & (-1)^{n-2}\lambda_1 & (-1)^{n-1} \\ \lambda_2^{n-1} & -\lambda_2^{n-2} & \cdots & (-1)^{n-2}\lambda_2 & (-1)^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_n^{n-1} & -\lambda_n^{n-2} & \cdots & (-1)^{n-2}\lambda_n & (-1)^{n-1} \end{bmatrix} \hat{x}(t)$$

$$= \begin{bmatrix} \dfrac{\chi_{1c}}{s + \lambda_1} v(t) \\ \vdots \\ \dfrac{\chi_{nc}}{s + \lambda_n} v(t) \end{bmatrix} + \begin{bmatrix} \dfrac{\chi_{1a}}{s + \lambda_1} y(t) \\ \vdots \\ \dfrac{\chi_{na}}{s + \lambda_n} y(t) \end{bmatrix} + \begin{bmatrix} \dfrac{\chi_{1b}}{s + \lambda_1} u(t) \\ \vdots \\ \dfrac{\chi_{nb}}{s + \lambda_n} u(t) \end{bmatrix} \tag{30}$$

On the other hand, it is well-known that the Vandermonde matrix

$$\Lambda \triangleq \begin{bmatrix} \lambda_1^{n-1} & -\lambda_1^{n-2} & \cdots & (-1)^{n-2}\lambda_1 & (-1)^{n-1} \\ \lambda_2^{n-1} & -\lambda_2^{n-2} & \cdots & (-1)^{n-2}\lambda_2 & (-1)^{n-1} \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \lambda_n^{n-1} & -\lambda_n^{n-2} & \cdots & (-1)^{n-2}\lambda_n & (-1)^{n-1} \end{bmatrix} \tag{31}$$

is nonsingular when $\lambda_i \neq \lambda_j$ for $i \neq j$ $(i, j = 1, \ldots, n)$.

Therefore, by pre-multiplying (30) with Λ^{-1}, the state can be reconstructed by the first-order filters of $v(t)$, $y(t)$ and $u(t)$.

Theorem 2. *A new implicit observer of $x(t)$ can be formulated as*

$$\hat{x}(t) = \Lambda^{-1} \left\{ \begin{bmatrix} \dfrac{\chi_{1c}}{s + \lambda_1} v(t) \\ \vdots \\ \dfrac{\chi_{nc}}{s + \lambda_n} v(t) \end{bmatrix} + \begin{bmatrix} \dfrac{\chi_{1a}}{s + \lambda_1} y(t) \\ \vdots \\ \dfrac{\chi_{na}}{s + \lambda_n} y(t) \end{bmatrix} + \begin{bmatrix} \dfrac{\chi_{1b}}{s + \lambda_1} u(t) \\ \vdots \\ \dfrac{\chi_{nb}}{s + \lambda_n} u(t) \end{bmatrix} \right\} \tag{32}$$

Proof. As the expression of $\hat{x}(t)$ in (32) is just an algebraic transform of (16), the proof is the same as that of Theorem 1. ∎

Remark 2. In the above implicit observer, the first-order filters of the disturbance are not available. They will be estimated in the next section.

4. Description of the Robust Observers

In what follows, the systems are divided into the following two cases:
Case 1. $k_1 \neq 0$,
Case 2. $k_i = 0$ $(i = 1, 2, \ldots, r-1)$, but $k_r \neq 0$ $(r > 1)$.

By defining

$$a(s) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \tag{33}$$

$$b(s) = b_1 s^{n-1} + \cdots + b_{n-1}s + b_n \tag{34}$$

$$k(s) = k_1 s^{n-1} + \cdots + k_{n-1}s + k_n \tag{35}$$

the differential equation (1) can be rewritten as

$$a(s)y(t) = b(s)u(t) + k(s)v(t) \tag{36}$$

Case 1. Choose n different Hurwitz polynomials as

$$\hat{f}_i(s) = \frac{1}{k_1}k(s)(s+\lambda_i) \tag{37}$$

where λ_i $(i = 1, \ldots, n)$ are defined in (17). Then dividing (36) by $\hat{f}_i(s)$ yields

$$\frac{1}{s+\lambda_i}v(t) = \frac{1}{k_1}\left\{\frac{a(s)}{\hat{f}_i(s)}y(t) - \frac{b(s)}{\hat{f}_i(s)}u(t)\right\} \tag{38}$$

So $(1/(s+\lambda_i))v(t)$ can be expressed by available signals. Therefore, by using Theorem 2, the state observer $\hat{x}(t)$ can be constructed by the known signals composed of $y(t)$ and the filters of $y(t)$ and $u(t)$.

Theorem 3. *In case $k_1 \neq 0$, the robust observer can be formulated as*

$$\hat{\hat{x}}(t) = \Lambda^{-1}\left\{\frac{1}{k_1}\begin{bmatrix} \chi_{1c}\left(\dfrac{a(s)}{\hat{f}_1(s)}y(t) - \dfrac{b(s)}{\hat{f}_1(s)}u(t)\right) \\ \vdots \\ \chi_{nc}\left(\dfrac{a(s)}{\hat{f}_n(s)}y(t) - \dfrac{b(s)}{\hat{f}_n(s)}u(t)\right)\end{bmatrix} + \begin{bmatrix}\dfrac{\chi_{1a}}{s+\lambda_1}y(t) \\ \vdots \\ \dfrac{\chi_{na}}{s+\lambda_n}y(t)\end{bmatrix} + \begin{bmatrix}\dfrac{\chi_{1b}}{s+\lambda_1}u(t) \\ \vdots \\ \dfrac{\chi_{nb}}{s+\lambda_n}u(t)\end{bmatrix}\right\} \tag{39}$$

Proof. The theorem is obvious by replacing the terms $(1/(s+\lambda_i))v(t)$ in (32) by their available expressions described in (38). ∎

Remark 3. It should be noted that no *a-priori* information of the disturbance is needed in this case, and there is no necessity to estimate the disturbance. The state observer is formulated by the filters of the input and output. Furthermore, discontinuous formulations as in (Edwards and Spurgeon, 1994) can be avoided.

Case 2. In the following, the disturbance will be estimated by using the VSS theory. First of all, the upper bounds of the filters of the disturbance must be estimated. For a positive constant λ, by employing the definition

$$\frac{1}{s+\lambda}v(t) = \int_{t_0}^{t} e^{-\lambda(t-\tau)}v(\tau)\,\mathrm{d}\tau \tag{40}$$

the next result can inductively be obtained.

Lemma 3. *An upper bound of* $(1/(s+\lambda)^i)v(t)$ *can be estimated as*

$$\left|\frac{1}{(s+\lambda)^i}v(t)\right| \le \frac{1}{(s+\lambda)^i}\rho\Big(y(t),u(t),t\Big) \triangleq \omega_i(t) \tag{41}$$

Proof. The proof is omitted. ∎

Remark 4. By the definition in (41), it is obvious that $\omega_0(t) \doteq \rho(y,u,t)$.

Now, we introduce a Hurwitz polynomial $l(s)$ as

$$l(s) = s^n + l_1 s^{n-1} + \cdots + l_n = \frac{1}{k_r}k(s)(s+\lambda)^r \tag{42}$$

Dividing both sides of (36) by $l(s)$ yields

$$y(t) = k_r\left\{\frac{l(s)-a(s)}{k(s)(s+\lambda)^r}y(t) + \frac{b(s)}{k(s)(s+\lambda)^r}u(t)\right\} + \frac{k_r}{(s+\lambda)^r}v(t) \tag{43}$$

Then multiplying both the sides of (43) with $s+\lambda$ gives

$$\begin{aligned}\dot{y}(t)+\lambda y(t) &= k_r\left\{\frac{l(s)-a(s)}{k(s)(s+\lambda)^{r-1}}y(t) + \frac{b(s)}{k(s)(s+\lambda)^{r-1}}u(t)\right\} \\ &\quad + \frac{k_r}{(s+\lambda)^{r-1}}v(t)\end{aligned} \tag{44}$$

Based on (44), we get the next theorem.

Theorem 4. *Construct the differential equations*

$$\begin{aligned}\dot{\hat{y}}(t) + \lambda\hat{y}(t) &= k_r\left\{\frac{l(s)-a(s)}{k(s)(s+\lambda)^{r-1}}y(t) + \frac{b(s)}{k(s)(s+\lambda)^{r-1}}u(t)\right\}\\ &\quad + k_r w_1(t), \qquad \hat{y}(t_0) = 0\end{aligned} \tag{45}$$

$$\dot{\hat{w}}_{i-1}(t) + \lambda\hat{w}_{i-1}(t) = \hat{w}_i(t), \qquad \hat{w}_{i-1}(t_0) = 0 \ \textit{(for } 2 \le i \le r) \tag{46}$$

where $w_1(t)$ and $w_i(t)$ (for $2 \le i \le r$) are determined as

$$w_1(t) = \omega_{r-1}(t)\,\mathrm{sign}\left\{k_r\{y(t) - \hat{y}(t)\}\right\} \tag{47}$$

$$w_i(t) = \omega_{r-i}(t)\,\mathrm{sign}\left\{w_{i-1}(t) - \hat{w}_{i-1}(t)\right\} \tag{48}$$

$\hat{y}(t)$ and $\hat{w}_{i-1}(t)$ (for $2 \le i \le r$) are the signals generated by (45) and (46), respectively. Then $w_i(t)$ are the corresponding estimates of $(1/(s+\lambda)^{r-i})v(t)$ for $i = 1, 2, \ldots, r$.

Proof. See Appendix 2. ■

Remark 5. It can be seen that the parameter λ determines the rates of convergence of $w_i(t) - (1/(s+\lambda)^{r-i})v(t)$ for $i = 1, 2, \ldots, r$.

Remark 6. Theorem 4 is also valid for Case 1, in which $w_1(t)$ can be regarded as an estimate of the disturbance $v(t)$.

From Theorems 2 and 4, an observer for Case 2 can be constructed by the following theorem.

Theorem 5. *For Case 2, the robust state observer of (1) can be constructed as*

$$\hat{\hat{x}}(t) = \Lambda^{-1}\left\{\begin{bmatrix}\chi_{c1}w_{1,r-1}(t)\\ \vdots\\ \chi_{cn}w_{n,r-1}(t)\end{bmatrix} + \begin{bmatrix}\dfrac{\chi_{1a}}{s+\lambda_1}y(t)\\ \vdots\\ \dfrac{\chi_{na}}{s+\lambda_n}y(t)\end{bmatrix} + \begin{bmatrix}\dfrac{\chi_{1b}}{s+\lambda_1}u(t)\\ \vdots\\ \dfrac{\chi_{nb}}{s+\lambda_n}u(t)\end{bmatrix}\right\} \tag{49}$$

where $w_{i,r-1}(t)$ are the corresponding estimates of $(1/(s+\lambda_i))v(t)$ for $i = 1, 2, \ldots, n$.

Proof. From (14), (32) and (49), we have

$$\begin{aligned}x(t) - \hat{\hat{x}}(t) &= \{x(t) - \hat{x}(t)\} + \left\{\hat{x}(t) - \hat{\hat{x}}(t)\right\}\\ &= H(f, x_0)z(t) + \Lambda^{-1}\begin{bmatrix}\chi_{c1}\left\{\dfrac{1}{s+\lambda_1}v(t) - w_{1,r-1}(t)\right\}\\ \vdots\\ \chi_{cn}\left\{\dfrac{1}{s+\lambda_n}v(t) - w_{n,r-1}(t)\right\}\end{bmatrix}\end{aligned} \tag{50}$$

From (15) and Theorem 4, it can be easily concluded that $x(t) - \hat{x}(t) \to 0$ as $t \to \infty$. Thus $\hat{x}(t)$ defined in (49) is an estimate of the state $x(t)$. ■

Remark 7. The signals $w_{i,r-1}(t)$ (for $i = 1, 2, \ldots, n$) can be either individually generated by a procedure similar to that of Theorem 4, or calculated by $w_{r-1}(t)\{(s+\lambda)/(s+\lambda_i)\}$, where $w_{r-1}(t)$ is generated in Theorem 4.

Remark 8. From Theorem 2 it can be seen that the state can be asymptotically expressed by the first filters of the input, output and disturbance. This is a reason why we do not employ the estimate $w_r(t)$ of the disturbance to generate the state observer directly by a differential equation.

5. A Pole-Assignment Controller

For simplicity, in this section we assume that the disturbance is not directly related to the control input. We also assume that the disturbance is matched, i.e. $b = k$.

Let the desired closed-loop transfer function be represented by

$$G_d(s) = \frac{b(s)}{d(s)} \tag{51}$$

where the zeros of the Hurwitz polynomial

$$d(s) = s^n + d_1 s^{n-1} + \cdots + d_n \tag{52}$$

determine the desired closed-loop poles.

Consider the state-feedback control law defined by

$$u(t) = -\kappa^T \hat{x}(t) - w_r(t) + \gamma(t) \tag{53}$$

where κ is an $n \times 1$ feedback gain vector, $\gamma(t)$ denotes a uniformly bounded external input and the disturbance estimate $w_r(t)$ obtained in Theorem 4 is employed to cancel the disturbance $v(t)$. With an appropriate choice of the feedback gain vector κ, the characteristic equation of the closed-loop system becomes

$$\det(sI - A + b\kappa^T) = d(s) \tag{54}$$

The calculation method of κ can be found in (Minamide *et al.*, 1983).

For the system (1) controlled by (53), we obtain the following result.

Theorem 6. *With the pole assignment controller (53), the global system is uniformly bounded, and the overall system output $y(t)$ tracks asymptotically the desired output $y_d(t) = \{b(s)/d(s)\}\gamma(t)$.*

Proof. By using the control law (53), the system (1) will be described by

$$\dot{x}(t) = (A - b\kappa^T)x(t) + b\gamma(t) + b\kappa^T\Big\{x(t) - \hat{x}(t)\Big\} + b\Big\{v(t) - w_r(t)\Big\} \tag{55}$$

Since $\gamma(t)$ is a uniformly bounded signal and $A - b\kappa^T$ is a stable matrix, by applying the results $\{x(t) - \hat{x}(t)\} \to 0$ and $\{v(t) - w_r(t)\} \to 0$ (as $t \to \infty$), it can be easily concluded that the state $x(t)$ is uniformly bounded. Then, by Theorem 5, the estimated state $\hat{x}(t)$ is also uniformly bounded. So, the input determined in (53) is uniformly bounded. Therefore, from (55), we have

$$y(t) = \frac{b(s)}{d(s)}\gamma(t) + \frac{b(s)}{d(s)}\left\{\kappa^T\left[x(t) - \hat{x}(t)\right] + \left[v(t) - w_r(t)\right]\right\} \tag{56}$$

Let $y_d(t) \overset{\Delta}{=} \{b(s)/d(s)\}\gamma(t)$. Thus (56) gives

$$y(t) - y_d(t) = \frac{b(s)}{d(s)}\left\{\kappa^T\left[x(t) - \hat{x}(t)\right] + \left[v(t) - w_r(t)\right]\right\} \tag{57}$$

Since $x(t) - \hat{x}(t) \to 0$, $v(t) - w_r(t) \to 0$, and $d(s)$ is a Hurwitz polynomial, it can be easily concluded that $y(t) - y_d(t) \to 0$ (as $t \to \infty$), i.e. the pole assignment can be achieved as $t \to \infty$. ■

6. Design Examples

In this section, for the two possible cases discussed in Section 4, examples will be presented to show the design procedure and simulation results.

Example 1. Consider a stable system with relative degree one described by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 2 \end{bmatrix} v(t) \\ y(t) = \begin{bmatrix} 1, & 0 \end{bmatrix} x(t) = x_1(t) \end{cases}$$

where the disturbance is governed by $v(t) = (\sin 2t)\, 0.5y(t)\{y(t) + 2u(t)\}/(|y(t)| + 0.5)$ and the input is assumed to be $u(t) = \sin t$. Suppose that the starting time is $t_0 = 0$. The unknown initial condition is assumed to be $x_0 = \begin{bmatrix} -1, & 2 \end{bmatrix}^T$. The purpose of this example is to estimate the state $x(t)$. As $x_1(t)$ is the output, we only need to estimate $x_2(t)$.

We choose the parameters λ_1 and λ_2 as $\lambda_1 = 1$ and $\lambda_2 = 2$, i.e. the Hurwitz polynomial in (17) is chosen as $f(s) = s^2 + 3s + 2$. Then we have

$$\chi_{1a} = 1, \quad \chi_{2a} = 3, \quad \chi_{1b} = 1, \quad \chi_{2b} = 2, \quad \chi_{1c} = -1, \quad \chi_{2c} = 0$$

As $k(s) = s + 2$, the Hurwitz polynomials defined in (37) are chosen as

$$\hat{f}_1(s) = (s+2)(s+1), \qquad \hat{f}_2(s) = (s+2)(s+2)$$

From Theorem 3, the implicit observer can be constructed as

$$\hat{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -\left\{ \dfrac{s^2+s+1}{(s+2)(s+1)} y(t) - \dfrac{s}{(s+2)(s+1)} u(t) \right\} \\ 0 \end{bmatrix} + \begin{bmatrix} \dfrac{1}{s+1} y(t) \\ \dfrac{3}{s+2} y(t) \end{bmatrix} + \begin{bmatrix} \dfrac{1}{s+1} u(t) \\ \dfrac{2}{s+2} u(t) \end{bmatrix} \right\}.$$

Computer simulation results for $x_2(t)$ and $\hat{x}_2(t)$ are shown in Fig. 1, where the sampling period is set to 0.001 s. The difference at the beginning is due to the initial conditions. ♦

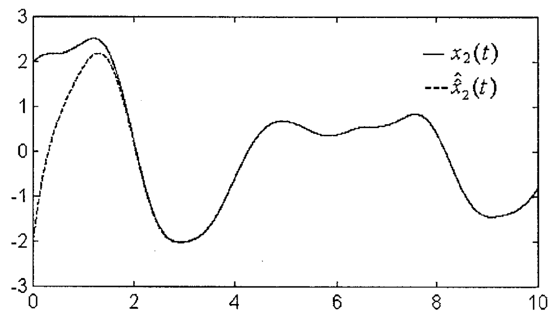

Fig. 1. The genuine state $x_2(t)$ and its estimate $\hat{x}_2(t)$ for Example 1.

Example 2. Consider an unstable system with relative degree two, described by

$$\begin{cases} \dot{x}(t) = \begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \big(u(t) + v(t)\big) \\ y(t) = \begin{bmatrix} 1, & 0 \end{bmatrix} x(t) = x_1(t) \end{cases}$$

The disturbance $v(t)$ is governed by $v(t) = (0.5\cos t + 0.25\sin 2t)0.5y(t)x_2(t) / (|x_2(t)| + 0.5)$, its upper bound is known as $\rho(y(t), t) = 0.5|y(t)|$. Suppose that the starting time is $t_0 = 0$. The unknown initial state x_0 is assumed to equal $[1,\ 1]^T$. The external reference input is adopted in the form

$$\gamma(t) = 3\sin t$$

The desired closed-loop poles are supposed to be the roots of the polynomial

$$d(s) = s^2 + 6s + 9$$

The purpose of this example is to estimate the state $x(t)$ and to synthesize a pole-assignment controller to achieve this goal.

From (54) the feedback gain κ can be calculated as

$$\kappa = [15,\ 7]^T$$

Choose the parameters $\lambda_1 = 1$ and $\lambda_2 = 2$, i.e. the Hurwitz polynomial in (17) is of the form

$$f(s) = (s+1)(s+2)$$

Then we have $\chi_{1a} = 3$, $\chi_{2a} = 7$, $\chi_{1b} = \chi_{1c} = -1$, $\chi_{2b} = \chi_{2c} = -1$. From Theorem 2, the implicit state observer can be constructed as

$$\hat{x}(t) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} \frac{-1}{s+1}v(t) \\ \frac{-1}{s+2}v(t) \end{bmatrix} + \begin{bmatrix} \frac{3}{s+2}y(t) \\ \frac{7}{s+2}y(t) \end{bmatrix} + \begin{bmatrix} \frac{-1}{s+1}u(t) \\ \frac{-1}{s+2}u(t) \end{bmatrix} \right\}$$

where $(1/(s+1))v(t)$ and $(1/(s+2))v(t)$ are unknown.

Now, let us consider the first-order filters of the disturbance and the disturbance. As $k(s) = 1$, choose the Hurwitz polynomial $l(s)$ in (42) as $l(s) = (s+1)^2$. From (44) we have

$$\dot{y}(t) + y(t) = \frac{3s}{s+1}y(t) + \frac{1}{s+1}u(t) + \frac{1}{s+1}v(t)$$

By Theorem 4, the following differential equations are constructed:

$$\dot{\hat{y}}(t) + \hat{y}(t) = \frac{3s}{s+1}y(t) + \frac{1}{s+1}u(t) + w_1(t)$$

$$\dot{\hat{w}}_1(t) + \hat{w}_1(t) = w_2(t)$$

where

$$w_1(t) = \frac{0.5}{s+1}|y(t)|\,\mathrm{sign}\left\{y(t) - \hat{y}(t)\right\}$$

$$w_2(t) = 0.5|y(t)|\,\mathrm{sign}\left\{w_1(t) - \hat{w}(t)\right\}$$

Therefore $w_1(t)$ and $w_2(t)$ can be regarded as estimates of $(1/(s+1))v(t)$ and $\dot{v}(t)$, respectively.

Accordingly, from Theorem 5, the state observer is formed as

$$\hat{\hat{x}}(t) = \begin{bmatrix} 1 & -1 \\ 2 & -1 \end{bmatrix}^{-1} \left\{ \begin{bmatrix} -w_1(t) \\ -\frac{s+1}{s+2}w_1(t) \end{bmatrix} + \begin{bmatrix} \frac{3}{s+2}y(t) \\ \frac{7}{s+2}y(t) \end{bmatrix} + 1 \begin{bmatrix} \frac{-1}{s+1}u(t) \\ \frac{-1}{s+2}u(t) \end{bmatrix} \right\}$$

Therefore the state-feedback pole-assignment controller can be constructed as

$$u(t) = -[15,\ 7]\hat{\hat{x}}(t) + \gamma(t) - w_2(t)$$

In digital implementations, the discontinuous function $\text{sign}(\eta)$ is approximated by the differentiable function $\eta/(|\eta|+\delta)$, where $\delta > 0$ is very small. If $\delta \to 0$, it is easy to see that $\eta/(|\eta|+\delta) \to \text{sign}(\eta)$. Thus, the discontinuous functions $w_1(t)$ and $w_2(t)$ can be approximately smoothed. The approximation error can be made as small as we want by choosing δ to be sufficiently small. In the presented computer simulation process, δ is chosen as 0.001 and the sampling period is set to 0.001 s. The simulation results are shown in Figs. 2–5. ♦

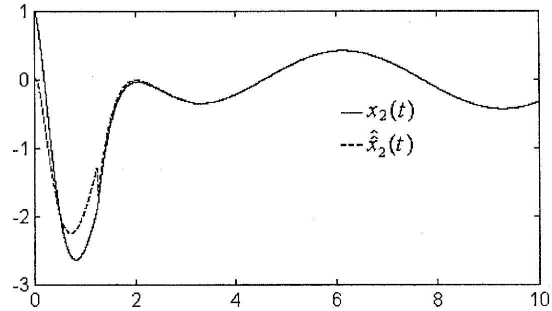

Fig. 2. The genuine state $x_2(t)$ and its estimate $\hat{x}_2(t)$ of Example 2.

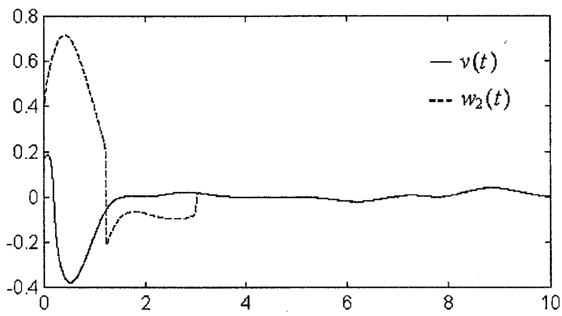

Fig. 3. The disturbance $v(t)$ and its estimate $w_2(t)$ of Example 2.

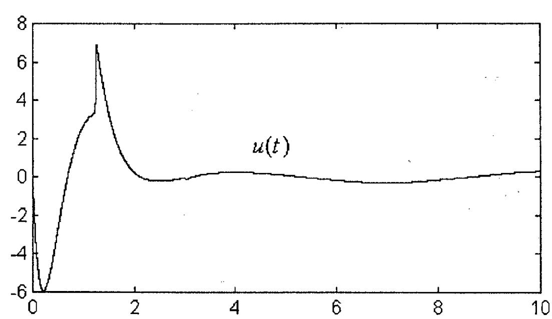

Fig. 4. The pole-assignment control $u(t)$ of Example 2.

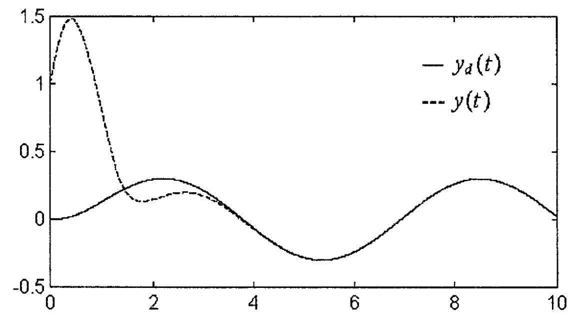

Fig. 5. The controlled output $y(t)$ and the desired output $y_d(t)$ of Example 2.

Remark 9. When implemented on a digital computer, the parameter δ should not be much smaller than the sampling period.

7. Conclusions

In this paper, based on implicit observer techniques, the state is mathematically expressed by first-order filters of the input, output and disturbance for SISO systems. By appealing to the VSS equivalent control method, the filters of the disturbance (eventually the disturbance) are estimated for SISO systems with arbitrarily relative degrees. The estimated first-order filters of the disturbance are used to generate a state observer of the system. Then the estimated disturbance and generated state observer are employed to construct a state-feedback controller to place the desired poles and to cancel the disturbance. Examples and simulation results show that the proposed algorithms are effective for practical applications.

In order to implement the proposed formulation on a digital computer, the discontinuous functions are approximated by differentiable functions in the simulation process. The approximation error can be controlled to be very small by choosing a small parameter δ and a small sampling period.

The proposed method is expected to be extended to multi-input multi-output (MIMO) systems with uncertainties.

Appendices

Appendix 1. *Proof of Lemma 2.*

From the relation

$$f(s) = s^n + f_1 s^{n-1} + \cdots + f_n = (s^{n-1} + g_1 s^{n-2} + \cdots + g_{n-1})(s + \lambda) \quad \text{(A1)}$$

the following two equations are obtained:

$$\begin{bmatrix} 1 & f_1 & f_2 & \cdots & f_{n-1} \\ 0 & 1 & f_1 & \cdots & f_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & f_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \begin{bmatrix} 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ 0 & 1 & g_1 & \cdots & g_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & g_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{A2}$$

$$\begin{bmatrix} f_1 & f_2 & f_3 & \cdots & f_n \\ f_2 & f_3 & f_4 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ f_{n-1} & f_n & 0 & \cdots & 0 \\ f_n & 0 & 0 & \cdots & 0 \end{bmatrix} = \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix} \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ g_1 & g_2 & g_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ g_{n-2} & g_{n-1} & 0 & \cdots & 0 \\ g_{n-1} & 0 & 0 & \cdots & 0 \end{bmatrix} \tag{A3}$$

Consequently,

$$[\lambda^{n-1}, -\lambda^{n-2}, \ldots, (-1)^{n-1}]H(f,h)$$

$$= [\lambda^{n-1}, -\lambda^{n-2}, \ldots, (-1)^{n-1}] \begin{bmatrix} h_1 & h_2 & \cdots & h_{n-1} & h_n \\ h_2 & h_3 & \cdots & h_n & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ h_{n-1} & h_n & \cdots & 0 & 0 \\ h_n & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & f_1 & \cdots & f_{n-2} & f_{n-1} \\ 0 & 1 & \cdots & f_{n-3} & f_{n-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & f_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$- [\lambda^{n-1}, -\lambda^{n-2}, \ldots, (-1)^{n-1}] \begin{bmatrix} f_1 & f_2 & \cdots & f_{n-1} & f_n \\ f_2 & f_3 & \cdots & f_n & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ f_{n-1} & f_n & \cdots & 0 & 0 \\ f_n & 0 & \cdots & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & h_1 & \cdots & h_{n-2} & h_{n-1} \\ 0 & 0 & \cdots & h_{n-3} & h_{n-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & h_1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}$$

$$= [h_1, \ldots, h_n] \begin{bmatrix} \lambda^{n-1} & 0 & 0 & \cdots & 0 \\ -\lambda^{n-2} & \lambda^{n-1} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ (-1)^{n-2}\lambda & (-1)^{n-3}\lambda^2 & (-1)^{n-4}\lambda^3 & \cdots & 0 \\ (-1)^{n-1} & (-1)^{n-2}\lambda & (-1)^{n-3}\lambda^2 & \cdots & \lambda^{n-1} \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & \lambda & 0 & \cdots & 0 \\ 0 & 1 & \lambda & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ 0 & 1 & g_1 & \cdots & g_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & g_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$
-\left[\lambda^{n-1}, -\lambda^{n-2}, \ldots, (-1)^{n-1}\right] \begin{bmatrix} \lambda & 1 & 0 & \cdots & 0 \\ 0 & \lambda & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & 0 & \cdots & \lambda \end{bmatrix}
$$

$$
\times \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ g_1 & g_2 & g_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ g_{n-2} & g_{n-1} & 0 & \cdots & 0 \\ g_{n-1} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & h_1 & \cdots & h_{n-2} & h_{n-1} \\ 0 & 0 & \cdots & h_{n-3} & h_{n-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & h_1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
$$

$$
= [h_1, \ldots, h_n] \begin{bmatrix} \lambda^{n-1} & \lambda^n & 0 & \cdots & 0 \\ -\lambda^{n-2} & 0 & \lambda^n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ (-1)^{n-2}\lambda & 0 & 0 & \cdots & \lambda^n \\ (-1)^{n-1} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ 0 & 1 & g_1 & \cdots & g_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & g_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}
$$

$$
- [\lambda^n, 0, \ldots, 0] \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ g_1 & g_2 & g_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ g_{n-2} & g_{n-1} & 0 & \cdots & 0 \\ g_{n-1} & 0 & 0 & \cdots & 0 \end{bmatrix} \begin{bmatrix} 0 & h_1 & \cdots & h_{n-2} & h_{n-1} \\ 0 & 0 & \cdots & h_{n-3} & h_{n-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & h_1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
$$

$$
= [\chi_c, h_1\lambda^n, \ldots, h_{n-1}\lambda^n] \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ 0 & 1 & g_1 & \cdots & g_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & g_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}
$$

$$
-\lambda^n [1, g_1, \ldots, g_{n-1}] \begin{bmatrix} 0 & h_1 & \cdots & h_{n-2} & h_{n-1} \\ 0 & 0 & \cdots & h_{n-3} & h_{n-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & h_1 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}
$$

$$= [\chi_c, h_1\lambda^n, \ldots, h_{n-1}\lambda^n] \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ 0 & 1 & g_1 & \cdots & g_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & g_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

$$-\lambda^n [0, h_1, \ldots, h_{n-1}] \begin{bmatrix} 1 & g_1 & \cdots & g_{n-2} & g_{n-1} \\ 0 & 1 & \cdots & g_{n-3} & g_{n-2} \\ \vdots & \vdots & \ldots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & g_1 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix}$$

$$= [\chi_c, 0, \ldots, 0] \begin{bmatrix} 1 & g_1 & g_2 & \cdots & g_{n-1} \\ 0 & 1 & g_1 & \cdots & g_{n-2} \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \cdots & g_1 \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} = \chi_c [1, g_1, \ldots, g_{n-1}] \tag{A4}$$

where

$$\chi_c = h_1\lambda^{n-1} - h_2\lambda^{n-2} + \cdots + (-1)^{n-1}h_n \tag{A5}$$

Therefore Lemma 2 is proved. ■

Appendix 2. *Proof of Theorem 4.*

The mathematical-induction principle is employed to prove the theorem.

Step 1. Taking into account (44), we consider the next system (45) together with (47), where $\hat{y}(t)$ is the signal generated by eqn. (45). Combining (44) and (45) yields

$$\dot{\bar{y}}(t) + \lambda\bar{y}(t) = k_r \left\{ \frac{1}{(s+\lambda)^{r-1}} v(t) - w_1(t) \right\} \tag{A6}$$

where $\bar{y}(t) = y(t) - \hat{y}(t)$. From (A6), differentiating $(\bar{y}(t))^2$ gives

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}(\bar{y}(t))^2 &= -2\lambda(\bar{y}(t))^2 + 2\bar{y}(t)k_r \left\{ \frac{1}{(s+\lambda)^{r-1}} v(t) - w_1(t) \right\} \\ &= -2\lambda(\bar{y}(t))^2 + 2\bar{y}(t)k_r \frac{1}{(s+\lambda)^{r-1}} v(t) - 2\left|\bar{y}(t)k_r\right| \omega_{r-1}(t) \\ &\le -2\lambda(\bar{y}(t))^2 \end{aligned}$$

Thus, it is obvious that $\bar{y}(t)$ converges exponentially to zero.

In order to derive the sliding equations through the equivalent control method, it is necessary to solve

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{y}(t) = 0 \tag{A7}$$

from (A6) with respect to $w_1(t)$. This yields

$$w_{1eq}(t) = \frac{1}{(s+\lambda)^{r-1}}v(t) \tag{A8}$$

So $w_1(t)$ can be regarded as an estimate of $(1/(s+\lambda)^{r-1})v(t)$.

Step 2. We will use $w_1(t)$ to estimate $(1/(s+\lambda)^{r-2})v(t)$ by appealing to the following trivial differential equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{(s+\lambda)^{r-1}}v(t)\right\} + \frac{\lambda}{(s+\lambda)^{r-1}}v(t) = \frac{1}{(s+\lambda)^{r-2}}v(t) \tag{A9}$$

Consider the corresponding differential equation

$$\dot{\hat{w}}_1(t) + \lambda\hat{w}_1(t) = w_2(t), \qquad \hat{w}_1(t_0) = 0 \tag{A10}$$

where $w_2(t)$ is the input determined by

$$w_2(t) = \omega_{r-2}(t)\,\mathrm{sign}\left\{w_1(t) - \hat{w}_1(t)\right\} \tag{A11}$$

and $\hat{w}_1(t)$ is generated by (A10). Let $\bar{w}_1(t) = (1/(s+\lambda)^{r-1})v(t) - \hat{w}_1(t)$. Then from (A9) and (A10) we have

$$\dot{\bar{w}}_1(t) + \lambda\bar{w}_1(t) = \frac{1}{(s+\lambda)^{r-2}}v(t) - w_2(t) \tag{A12}$$

It can be proved that

$$\bar{w}_1(t) \to 0 \quad \text{as} \quad t \to \infty \tag{A13}$$

The proof of (A13) is given in Appendix 3.

Similarly, by the equivalent control method, $w_2(t)$ can be regarded as an estimate of $(1/(s+\lambda)^{r-2})v(t)$.

Step i ($3 \le i \le r$). Based on the trivial differentiation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{1}{(s+\lambda)^{r-i+1}}v(t)\right\} + \frac{\lambda}{(s+\lambda)^{r-i+1}}v(t) = \frac{1}{(s+\lambda)^{r-i}}v(t) \tag{A14}$$

we can construct the corresponding differential equation

$$\dot{\hat{w}}_{i-1}(t) + \lambda\hat{w}_{i-1}(t) = w_i(t), \qquad \hat{w}_{i-1}(t_0) = 0 \tag{A15}$$

where $w_i(t)$ is determined as

$$w_i(t) = \omega_{r-i}(t)\,\mathrm{sign}\left\{w_{i-1}(t) - \hat{w}_{i-1}(t)\right\} \tag{A16}$$

and $\hat{w}_{i-1}(t)$ is the signal generated by (A15). In much the same way as in Appendix 3, it can be proved that

$$\frac{1}{(s+\lambda)^{r-i+1}}v(t) - \hat{w}_{i-1}(t) \to 0 \quad \text{as} \quad t \to \infty \tag{A17}$$

Thus $w_i(t)$ can be regarded as estimates of $(1/(s+\lambda)^{r-i})v(t)$ for $i = 3, \ldots, r$, respectively. By the mathematical-induction principle, the theorem is proved. ■

Appendix 3. *Proof of relation (A13).*

From (A12), we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bar{w}_1^2(t) &= -2\lambda\bar{w}_1^2(t) + 2\bar{w}_1(t)\left\{\frac{1}{(s+\lambda)^{r-2}}v(t) - w_2(t)\right\}\\ &= -2\lambda\bar{w}_1^2(t) + 2\bar{w}_1(t)\frac{1}{(s+\lambda)^{r-2}}v(t) - 2\bar{w}_1(t)\omega_{r-2}(t)\\ &\quad \times \operatorname{sign}\left\{\bar{w}_1(t) + w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t)\right\}\end{aligned} \tag{A18}$$

As regards the relation between the functions $\bar{w}_1(t)$ and $w_1(t) - (1/(s+\lambda)^{r-1})v(t)$, we will divide the derivations into three cases.

Case 1. There exists a positive constant T_1 such that

$$|\bar{w}_1(t)| \geq \left|w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t)\right| \tag{A19}$$

for all $t > T_1$.

Case 2. There exists a positive constant T_2 such that

$$|\bar{w}_1(t)| < \left|w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t)\right| \tag{A20}$$

for all $t > T_2$.

Case 3. It corresponds to neither Case 1 nor Case 2.

Now, a detailed analysis is outlined for each case:

Case 1. (A18) gives

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\bar{w}_1^2(t) &= -2\lambda\bar{w}_1^2(t) + 2\bar{w}_1(t)\frac{1}{(s+\lambda)^{r-2}}v(t)\\ &\quad - 2\bar{w}_1(t)\omega_{r-2}(t)\operatorname{sign}\{\bar{w}_1(t)\} \leq -2\lambda\bar{w}_1^2(t)\end{aligned} \tag{A21}$$

It can be concluded that $\bar{w}_1(t)$ approaches exponentially zero as $t \to \infty$.

Case 2. Since $w_1(t)$ is an estimate of $(1/(s+\lambda)^{r-1})v(t)$, i.e.

$$w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t) \to 0 \tag{A22}$$

it can be easily concluded from (A20) that $\bar{w}_1(t) \to 0$ as $t \to \infty$.

Case 3. If the relation

$$|\bar{w}_1(t_0)| \geq \left| w_1(t_0) - \frac{1}{(s+\lambda)^{r-1}}v(t_0) \right| \tag{A23}$$

holds for time instant t_0, then from (A18) we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\bar{w}_1^2(t_0) \leq -2\lambda \bar{w}_1^2(t_0) \tag{A24}$$

i.e. as t increases from t_0, $\bar{w}_1^2(t)$ decreases until the relation

$$|\bar{w}_1(t)| \leq \left| w_1(t) - \frac{1}{(s+\lambda)^{r-1}}v(t) \right| \tag{A25}$$

is satisfied, otherwise this contradicts the assumption of Case 3.

Thus, when (A23) holds for some instant, sooner or later (A25) will hold as t increases from this instant. From the assumption of Case 3, it can be seen that there is an infinite number of such instants (at least a denumerable set), and the values of the instants approach infinity. Therefore, making use of the fact that $w_1(t) - (1/(s+\lambda)^{r-1})v(t) \to 0$, we conclude that

$$\bar{w}_1(t) \to 0 \quad \text{as} \quad t \to \infty \tag{A26}$$

By combining the above three cases, relation (A13) is proved. ■

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