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MULTISINE APPROXIMATION OF MULTIVARIATE ORTHOGONAL RANDOM PROCESSES

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An approach to the synthesis and simulation of wide-sense stationary multivariate orthogonal random processes defined by their power spectral density matrices is presented. The approach is based on approximating the non-parametric power spectral density representation by the periodogram matrix of a multivariate orthogonal multisine random time-series. This periodogram matrix is used to construct the corresponding spectrum of the multivariate orthogonal multisine random time-series (synthesis). Application of the inverse finite discrete Fourier transform to this spectrum results in a multivariate orthogonal multisine random time-series with the predefined periodogram matrix (simulation). The properties of multivariate orthogonal multisine random process approximations obtained in this way are discussed. Attention is paid to asymptotic gaussianess.

Keywords: simulation random processes, multivariate orthogonal random processes, simulated identification, multisine random time-series, fast Fourier transform.

1. Introduction

Multisine time-series have been known for a long time. They are sums of many discrete-time sines with amplitudes and phase shifts determined by a variety of methods, depending upon the purpose for which the multisine time-series are to serve. Recently, their popularity has increased thanks to the possibility of generating them by numerically efficient Fast Fourier Transform (FFT) algorithms.

Multisine time-series may be used as basic building blocks for synthesising and simulating various deterministic and random processes with predetermined spectral or correlation properties. It seems that a theoretical foundation for such a synthesis is given by the famous Gauss sum (Schroeder, 1990). Its individual complex terms with the period length equal to any prime number exhibit the interesting property of whiteness. Their correlation function is equal to zero for some non-zero shifts. This idea has attracted no attention for a long time and the interesting potential of multisine time-series seems to be largely unexplored. Some short discussion of multisine time-series can be found in the books of Kay (1986), Marple (1987) and Godfrey (1993). Recently they have been applied to synthesise white noise of scalar (Figwer

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and Niederliński, 1992), bivariate (Niederliński and Figwer, 1995) and multivariate (Figwer and Niederliński, 1995) type, as well as random processes given by their power spectral densities (Figwer, 1997; Schinozuka and Deodatis, 1991).

The paper presents an application of multisine time-series to the synthesis and simulation of wide-sense stationary multivariate orthogonal random processes defined by their power spectral density matrices, where:

- The synthesis means the determination of the spectrum (the finite discrete Fourier transform) of a multivariate orthogonal multisine random time-series on the basis of the power spectral density matrix of the wide-sense stationary multivariate orthogonal random process to be simulated.
- Simulation means the generation of the corresponding multivariate orthogonal multisine random process approximation by performing the inverse discrete Fourier transform of the synthesised spectrum.

The problem of synthesising and simulating random processes defined by their power spectral densities given in an analytic form has been solved satisfactorily for rational power spectral densities (Åström, 1970). This approach is applied as an approximation for non-rational cases (Pillai and Shim, 1993). The resulting time-series are both synthesised and simulated as outputs of a discrete-time linear filter excited by white noise. Spectral and correlation properties of the obtained time-series highly depend on:

- The quality of white noise used as a driving input. A recent comparison of some Gaussian white noise generators can be found in L'Ecuyer (1990) and Bosq and Smili (1991). The generators currently used for simulation purposes belong as a rule to the class of linear congruential recursive generators. This scheme has been generalised to non-linear generators and generators by inversion. There also exist congruential linear and non-linear generators producing sequences of multivariate random white noise (Niederreiter, 1992) but they do not posses a mechanism providing, important in the multivariate case, orthogonality of its elements; no systematic approach to deal with this problem for congruential generators has been known so far.
- The filter parameters' accuracy obtainable for any given rational power spectral density by using spectral factorisation. In the non-rational case, the corresponding rational approximation can be calculated using the minimax or least-squares error criteria (DeFatta *et al.*, 1992), applied to the power spectral density.
- The rounding errors accumulating in recursive calculations.

However, the power spectral density parametric representation is hardly ever available. Very often the power spectral density of the random process to be simulated is given only by a non-parametric representation, e.g. as a diagram.

It follows from Doob's Spectral Representations Theorem (Christensen, 1991; Priestley, 1981) that any wide-sense stationary random process can be approximated arbitrarily close by a sum of sines and cosines with amplitudes being zero mean, independent random variables and deterministic phase shifts equal to zero.

In the presented approach the power spectral density matrix of a wide-sense stationary multivariate orthogonal random process is approximated by the periodogram matrix of a multivariate orthogonal multisine random time-series with deterministic amplitudes chosen so that for a given number of equally spaced frequencies from the range $[0, 2\pi)$, the periodogram matrix is equal to the original power spectral density matrix. The periodogram matrix may be used in turn to construct the corresponding finite discrete Fourier transform vector (the spectrum) provided that the phase shifts for each sine component are chosen. It is well-known that any periodogram matrix corresponds to infinitely many different time-series which differ by the choices of the phase shifts. It is demonstrated in the paper that in order to get ergodic random processes, the phase shifts should be chosen with some well-defined random properties. The spectrum with the chosen phase shifts is transformed into the time-domain by the inverse finite discrete Fourier transform. Using this approach, a broad range of multivariate orthogonal random processes can be synthesised and simulated provided that their power spectral density matrices are available.

Multisine approximations of wide-sense stationary multivariate orthogonal random processes obtained by this approach have discrete spectra. However, the original processes have continuous power spectral densities. It turns out that by fulfilling certain conditions on sampling in the frequency domain, the approximation of continuous power spectral densities by discrete spectra is not creating any loss of information.

Additionally, original random processes have autocorrelation functions converging to zero for large lags. This property holds for multisine time-series provided that the number of sines is sufficiently large. For practical random process simulation, it is usually possible to choose the necessary number of the sine components.

The attractiveness of the proposed multisine approach to the synthesis and simulation of wide-sense stationary multivariate orthogonal random processes is due to many factors:

- There is no need to solve the spectral factorisation problem for a given power spectral density to calculate the corresponding parametric approximation needed for simulation.
- Time-series can be precisely defined in the frequency-domain, which is of importance for a number of applications, e.g. optimal input design for identification (Godfrey, 1993; Yuan and Ljung, 1985) and data encryption (Niederliński and Figwer, 1998).
- Frequency-domain definitions are directly used to generate, by means of the inverse finite discrete Fourier transform, the simulated random processes which satisfy the ergodic hypothesis and are asymptotically Gaussian.
- Particular realisations of the simulated random processes can be obtained by transforming the realisations of the synthesised spectrum back into the time-domain by the inverse finite discrete Fourier transform implemented on the basis of FFT algorithms.

- The approach can be used for non-parametrically defined wide-sense stationary random, rational and non-rational, scalar and multivariate random processes, for which only the power spectral densities are available.
- It can be used to generate various types of scalar and multivariate white noise (Figwer and Niederliński, 1992; 1995; Niederliński and Figwer, 1995), which turn out to have interesting properties when compared with standard approaches, e.g. congruential generators.
- It gives an opportunity to reduce radically the simulation effort by a simulation time-scale contraction, which is a new technique in the simulation of Gaussian random processes.

2. Multivariate Orthogonal Multisine Random Time-Series

The basic N-sample multivariate orthogonal multisine random time-series (MOMRS) (Figwer and Niederliński, 1995) is defined in the time-domain by the p-dimensional multivariate time-series:

$$\mathbf{u}^N(i) = \left[u_1^N(i), u_2^N(i), \dots, u_p^N(i)\right]^T \tag{1}$$

where the r-th ($r = 1, 2, \dots, p$) MOMRS element $u_r^N(i)$ is a sum of some discrete-time sine components with the constraint that the same frequency may not appear in more than one MOMRS element. The r-th element is given by

$$u_r^N(i) = \sum_{\Omega n \in N_r} A_n \sin(\Omega n i + \phi_n) \tag{2}$$

N_r is the set of all frequencies Ωn present in the r-th MOMRS element $u_r^N(i)$, and

$$N_1 \cup N_2 \cup \dots \cup N_p = \{0, \Omega, \dots, \pi\} \tag{3}$$

These sets are pairwise disjoint:

$$N_s \cap N_t = \emptyset \tag{4}$$

for $s \neq t$, $s, t = 1, 2, \dots, p$. $\Omega = 2\pi/N$ denotes the fundamental relative frequency, $n = 0, 1, \dots, N/2$ denotes the consecutive harmonics of this frequency in the range $[0, \pi]$, $i = 0, 1, \dots, N-1$ denotes the consecutive discrete time instants, A_n are deterministic amplitudes of the sine components ($A_n \in \mathcal{R}$), ϕ_n are phase shifts, of which ϕ_0 is deterministic and the remaining phase shifts are random, independent and:

- uniformly distributed on $[0, 2\pi)$ for $n = 1, 2, \dots, N/2 - 1$,
- Bernoulli distributed $\mathcal{B}(1/2, \{\alpha, \pi + \alpha\})$ for $n = N/2$, i.e.

$$P\left\{\phi_{\frac{N}{2}} = \alpha\right\} = P\left\{\phi_{\frac{N}{2}} = \pi + \alpha\right\} = \frac{1}{2} \tag{5}$$

where $P\{X\}$ denotes the probability of an event X.

When the time range is changed from $i = 0, 1, \ldots, N-1$ up to $i = 0, 1, \ldots, \infty$, an extended MOMRS $\mathbf{u}(i)$ is obtained. It follows from the above choice of random phase shifts that the extended MOMRS $\mathbf{u}(i)$ is an ergodic multivariate time-series (Figwer and Niederliński, 1995) because it is a stationary random process for which time-averaged results obtained for any time-series realisation are equal to the corresponding ensemble averaged results over a collection of the time-series.

The fact that elements of the MOMRS have no common frequencies under the Parseval theorem implies the orthogonality of its elements for the ensemble averaging:

$$\mathcal{E}\{u_r(i)u_s(i)\} = 0 \tag{6}$$

as well as for the time-domain averaging:

$$\frac{1}{qN}\sum_{i=0}^{qN-1} u_r(i)u_s(i) = 0 \tag{7}$$

where $r \neq s$, $r, s = 1, 2, \ldots, p$ and $q = 1, 2, \ldots, \infty$.

The elements of the extended MOMRS expected value vector (or the mean-value vector) $\mathcal{E}\{\mathbf{u}(i)\} = [\mathcal{E}\{u_1(i)\}, \mathcal{E}\{u_2(i)\}, \ldots, \mathcal{E}\{u_p(i)\}]^T$ are given for $r = 1, 2, \ldots, p$ by

$$\mathcal{E}\{u_r(i)\} = \begin{cases} A_0 \sin\phi_0 & \text{if} \quad 0 \in N_r \\ 0 & \text{otherwise} \end{cases} \tag{8}$$

The corresponding ensemble averaged ($\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\}$) or time-domain averaged ($\mathbf{R_{uu}}(\tau)$; $\mathbf{R_{uu}}(\tau) = \mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\}$) correlation function matrix is given by

$$\begin{aligned} &\mathcal{E}\{\mathbf{u}(i)\mathbf{u}^T(i-\tau)\} \\ &\quad = \operatorname{diag}\Big\{\mathcal{E}\{u_1(i)u_1(i-\tau)\}, \mathcal{E}\{u_2(i)u_2(i-\tau)\}, \ldots, \mathcal{E}\{u_p(i)u_p(i-\tau)\}\Big\} \end{aligned} \tag{9}$$

where $\mathcal{E}\{u_r(i)u_r(i-\tau)\}$ is the autocorrelation function of the r-th MOMRS element:

$$\mathcal{E}\{u_r(i)u_r(i-\tau)\} = \sum_{\Omega n \in N_r \setminus \{0,\pi\}} \frac{A_n^2}{2}\cos(\Omega n\tau) + \mathcal{E}\{u_{0,\pi}(i)u_{0,\pi}(i-\tau)\} \tag{10}$$

and

$$\mathcal{E}\{u_{0,\pi}(i)u_{0,\pi}(i-\tau)\} = \begin{cases} A_0^2 \sin^2\phi_0 + (-1)^\tau A_{\frac{N}{2}}^2 \sin^2\alpha & \text{if} \quad (0 \in N_r) \wedge (\pi \in N_r) \\ A_0^2 \sin^2\phi_0 & \text{if} \quad (0 \in N_r) \wedge (\pi \notin N_r) \\ (-1)^\tau A_{\frac{N}{2}}^2 \sin^2\phi_{\frac{N}{2}} & \text{if} \quad (0 \notin N_r) \wedge (\pi \in N_r) \\ 0 & \text{if} \quad (0 \notin N_r) \wedge (\pi \notin N_r) \end{cases} \tag{11}$$

Any change in the assumption about the distributions of the random phase shifts ϕ_n in the MOMRS definition results in an extended MOMRS for which the expected value vector and autocorrelation function matrix are time-dependent. For instance, the choice of all random phase shifts ϕ_n as Bernoulli distributed $\mathcal{B}(1/2, \{\alpha, \pi + \alpha\})$ leads to non-stationary extended MOMRS's which exhibit the following symmetries:

- for $\alpha = 0$ and additionally $A_0 = 0$ or $\phi_0 = 0$, the resulting extended MOMRS's are odd sequences:

$$\mathbf{u}(i + qN) = -\mathbf{u}(qN - i) \tag{12}$$

- for $\alpha = \pi/2$, the resulting extended MOMRS's are even sequences:

$$\mathbf{u}(i + qN) = \mathbf{u}(qN - i) \tag{13}$$

where $i = 1, 2, \ldots, N - 1$ and $q = 1, 2, \ldots, \infty$.

The spectrum of the basic N-sample MOMRS is given in the frequency-domain for the (relative) frequency range $[0, 2\pi)$ by its p-dimensional vector of finite discrete Fourier transforms $\mathbf{U}^N(j\Omega m) = [U_1^N(j\Omega m), U_2^N(j\Omega m), \ldots, U_p^N(j\Omega m)]^T$ with the r-th element:

$$\begin{aligned} U_r^N(j\Omega m) &= \sum_{i=0}^{N-1} u_r(i) e^{-j\Omega m} \\ &= \frac{N}{2j} \sum_{\Omega n \in N_r} A_n \left[e^{j\phi_n}\delta(m - n) - e^{-j\phi_n}\delta(m - (N - n))\right] \end{aligned} \tag{14}$$

where $m = 0, 1, \ldots, N-1$ indicates consecutive harmonics of the fundamental relative frequency Ω in the range $[0, 2\pi)$.

This frequency-domain representation allows us to efficiently generate particular realisations of the basic N-sample MOMRS $\mathbf{u}^N(i)$ by transforming realisations of the spectrum $\mathbf{U}^N(j\Omega m)$ back into the time-domain by the inverse finite discrete Fourier transform implemented on the basis of FFT algorithms.

By finite Fourier transform techniques (Bendat and Piersol, 1986), the periodogram matrix of the basic N-sample MOMRS is given by

$$\begin{aligned} \Phi_{\mathbf{uu}}^N(j\Omega m) &= \frac{T}{N}\mathbf{U}^N(j\Omega m)\left(\mathbf{U}^N(-j\Omega m)\right)^T \\ &= \operatorname{diag}\left\{\Phi_{11}^N(\Omega m) + j0, \Phi_{22}^N(\Omega m) + j0, \ldots, \Phi_{rr}^N(\Omega m) + j0\right\} \end{aligned} \tag{15}$$

where $m = 0, 1, \ldots, N - 1$, T is the sampling interval, $\Phi_{rr}^N(\Omega m)$ $(r = 1, 2, \ldots, p)$ is the periodogram of the r-th MOMRS element:

$$\Phi_{rr}^N(\Omega m) = \frac{TN}{4} \sum_{\Omega n \in N_r \setminus \{0,\pi\}} A_n^2 \Big[\delta(m - n) + \delta\big(m - (N - n)\big)\Big] + \Phi_{0,\pi}^N(\Omega m) \tag{16}$$

and

$$\Phi^N_{0,\pi}(\Omega m)=\begin{cases} TN\Big(A_0^2\sin^2\phi_0\delta(m)+A_{\frac{N}{2}}^2\sin^2\alpha\delta\Big(m-\dfrac{N}{2}\Big)\Big) & \text{if } (0\in N_r)\wedge(\pi\in N_r)\\ TNA_0^2\sin^2\phi_0\delta(m) & \text{if } (0\in N_r)\wedge(\pi\notin N_r)\\ TNA_{\frac{N}{2}}^2\sin^2\phi_{\frac{N}{2}}\delta\Big(m-\dfrac{N}{2}\Big) & \text{if } (0\notin N_r)\wedge(\pi\in N_r)\\ 0 & \text{if } (0\notin N_r)\wedge(\pi\notin N_r)\end{cases} \tag{17}$$

It should be noticed that in spite of random phase shifts, the periodogram matrix of the MOMRS is a real-valued matrix with deterministic elements which are uniquely defined by the set of amplitudes $\{A_0, A_1, \ldots, A_{N/2}\}$ and two phase shifts $\{\phi_0, \alpha\}$. This implies that the shapes of MOMRS periodogram matrix elements can be fitted to any given power spectral density function matrix elements of a wide-sense stationary multivariate orthogonal random process. This is the main idea behind the proposed synthesis and simulation method of multivariate orthogonal random processes.

3. Power Spectral Density Defined Time-Series

3.1. Synthesis

The power spectral density matrix of a causal, wide-sense stationary multivariate orthogonal random process with finite powers of its elements may be approximated by the periodogram matrix of an MOMRS with the amplitudes of the sine components chosen so as to make the values of the MOMRS element periodograms equal to the power spectral densities of the original random process for some equally spaced frequencies from the range $[0, 2\pi)$. This equal spacing between the frequency lines can be achieved by ordering the consecutive frequencies circularly to consecutive elements of the MOMRS. Such an ordering will be called the consecutively circular ordering and denoted by the upper index c in the symbols N_r^c $(r = 1, 2, \ldots, p)$ describing the sets of frequencies. The frequency Ωn is a member of N_r^c when

$$r = n \bmod p + 1 \tag{18}$$

If $N/(2p)$ is an integer number, then the zero- and Nyquist-frequencies are elements of the set N_1^c. This set consists of $n_1 = N/(2p) + 1$ elements. Other sets N_r^c $(r = 2, 3, \ldots, p)$ have $n_r = N/(2p)$ elements. When $N/(2p)$ is not an integer number, the MOMRS elements $u_r(i)$ $(r = 1, 2, \ldots, p)$ have different numbers of sine components n_r. For a large N $(N \gg p)$ the number n_r for all MOMRS elements can be approximated by $N/(2p)$.

Let $\mathbf{v}(i)$ be a wide-sense stationary, real-valued multivariate orthogonal random process with the power spectral density matrix

$$\Phi_{\mathbf{vv}}(j\omega T) = \operatorname{diag}\{\Phi_{v_1v_1}(\Omega m) + j0, \Phi_{v_2v_2}(\Omega m) + j0, \ldots, \Phi_{v_pv_p}(\Omega m) + j0\} \tag{19}$$

which satisfies, for $\omega T \in [0, 2\pi)$, the following conditions:

$$\Phi_{\mathbf{vv}}(j\omega T) = \Phi_{\mathbf{vv}}\big(j(2\pi - \omega T)\big) \tag{20}$$

and

$$\left\|\Phi_{\mathbf{vv}}(j\omega T)\right\| < \infty \tag{21}$$

where

$$\left\|\Phi_{\mathbf{vv}}(j\omega T)\right\| = \sqrt{\sum_{r=1}^{p}\sum_{s=1}^{p}\left|\Phi_{v_r v_s}(j\Omega m)\right|^2} \tag{22}$$

It is assumed that the autocorrelation function matrix $\mathbf{R}_{\mathbf{vv}}(\tau)$ of $\mathbf{v}(i)$ for lags $|\tau| > N/(2p) - 1$ satisfies

$$\mathbf{R}_{\mathbf{vv}}(\tau) = \mathbf{0} \tag{23}$$

This implies that $\mathbf{v}(i)$ is a random process with the finite correlation time $N/(2p)-1$.

The power spectral density $\Phi_{v_r v_r}(\omega T)$ $(r = 1, 2, \dots, p)$ is sampled in the frequency-domain by choosing the n_r sample points (approximation nodes) along the ωT axis at relative, equidistant frequencies from the set N_r^c. It does not produce aliasing if the spacing Δ_r between the samples along the frequency axis is such that

$$\Delta_r \leq \frac{2\pi}{n_r} \tag{24}$$

where $r = 1, 2, \dots, p$. When the maximum spacing

$$\Delta = \max_{r=1,2,\dots,p} \Delta_r = p\frac{2\pi}{N} = p\Omega \tag{25}$$

is chosen, the original power spectral densities $\Phi_{v_r v_r}(\omega T)$ $(r = 1, 2, \dots, p)$ can be recovered from its sampled values (periodograms of approximating multisine random time-series) by using the sinc interpolation (Jerri, 1977).

The approximation criterion

$$\Phi_{v_r v_r}(\omega T)\,\big|_{\omega T \in N_r^c} = \Phi_{rr}^N(\Omega m)\,\big|_{\Omega m \in N_r^c} \tag{26}$$

for $r = 1, 2, \dots, p$ allows us to synthesise the r-th element $U_r(j\Omega m)$ of the MOMRS discrete Fourier transform $\mathbf{U}^N(j\Omega m)$ as follows:

- for $m = 0$ and $\Omega m \in N_r^c$:

$$U_r^N(j0) = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(0)} + j0 \tag{27}$$

- for $m = 1, 2, \dots, N/2 - 1$ $\Omega m \in N_r^c$:

$$\mathrm{Re}\left\{U_r^N(j\Omega m)\right\} = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(\Omega m)}\,\sin\phi_m \tag{28}$$

$$\mathrm{Im}\left\{U_r^N(j\Omega m)\right\} = -\sqrt{\frac{N}{T}\Phi_{v_r v_r}(\Omega m)}\,\cos\phi_m \tag{29}$$

where ϕ_m are random, independent and uniformly distributed on $[0, 2\pi)$;

- for $m = N/2$ and $\Omega m \in N_r^c$:

$$U_r^N(j\pi) = \sqrt{\frac{N}{T}\Phi_{v_r v_r}(\pi)}\ \sin\phi_{\frac{N}{2}} + j0 \tag{30}$$

where the quantity $\phi_{\frac{N}{2}}$ is random, independent and Bernoulli distributed $\mathcal{B}\left(1/2, \{\alpha, \pi + \alpha\}\right)$;

- for $m = 0, 1, \ldots, N/2$ and $\Omega m \notin N_r^c$:

$$U_r^N(j\Omega m) = 0 + j0 \tag{31}$$

- for $N - m = N - 1, N - 2, \ldots, N - (N/2 - 1)$:

$$U_r^N\left(j\Omega(N-m)\right) = \mathrm{Re}\left\{U_r^N(j\Omega m)\right\} - j\mathrm{Im}\left\{U_r^N(j\Omega m)\right\} \tag{32}$$

The inverse discrete Fourier transform of the spectrum

$$\mathbf{U}^N(j\Omega m) = \left[U_1^N(j\Omega m), U_2^N(j\Omega m), \ldots, U_p^N(j\Omega m)\right]^T \tag{33}$$

gives a real-valued MOMRS $\mathbf{u}^N(i)$.

The assumption (23) can be interpreted as a lower bound on the number N of samples for multivariate orthogonal random time-series to be simulated. When it is satisfied, the original power spectral density matrix may be reconstructed uniquely without producing aliasing.

For asymptotically uncorrelated random processes ($\lim_{\tau\to\infty}\mathbf{R}_{\mathbf{vv}}(\tau) = \mathbf{0}$) the assumption (23) can be satisfied only asymptotically for $n_r \to \infty$ ($r = 1, 2, \ldots, p$). In this case, a finite number of approximation nodes implies aliasing in the shift-domain of the corresponding autocorrelation function. This aliasing can be made insignificant by selecting a sufficiently large N such that for all $\tau > N/(2p) - 1$ it is reasonably to assume that $\mathbf{R}_{\mathbf{vv}}(\tau)$ is a zero matrix.

3.2. Asymptotic Properties

The extended MOMRS obtained from application of the approximation criterion (26) to the power spectral density matrix $\Phi_{\mathbf{vv}}(\omega T)$ ($\omega T \in [0, 2\pi)$) of a wide-sense stationary multivariate orthogonal random process $\mathbf{v}(i)$ turns asymptotically for $N \to \infty$ into a Gaussian multivariate multisine orthogonal random time-series:

Lemma 1. *Assuming that:*

1. $\Phi_{\mathbf{vv}}(j\omega T)$ *($\|\Phi_{\mathbf{vv}}(j\omega T)\| < \infty$ for $\omega T \in [0, 2\pi)$) is the power spectral density matrix*

$$\Phi_{\mathbf{vv}}(j\omega T) = \mathrm{diag}\left\{\Phi_{v_1 v_1}(\Omega m) + j0, \Phi_{v_2 v_2}(\Omega m) + j0, \ldots, \Phi_{v_p v_p}(\Omega m) + j0\right\} \tag{34}$$

of a wide-sense stationary real-valued multivariate orthogonal random time-series with zero mean vector and the variance matrix:

$$\boldsymbol{\sigma}_{\mathbf{v}}^2 = \operatorname{diag}\left\{\sigma_{v_{11}}^2, \sigma_{v_{22}}^2, \ldots, \sigma_{v_{pp}}^2\right\} \tag{35}$$

where for $r = 1, 2, \ldots, p$ *we have*

$$\sigma_{v_{rr}}^2 = \frac{1}{2\pi T} \int_0^{2\pi} \Phi_{v_r v_r}(\omega T)\, \mathrm{d}(\omega T) \tag{36}$$

2. ϕ_n, $n = 1, 2, \ldots, N/2 - 1$ *are independent random variables uniformly distributed on* $[0, 2\pi)$*;*
3. A_n *converges to* 0 *as* $N \to \infty$ *in such a way that for* $r = 1, 2, \ldots, p$*:*

$$\frac{NTA_n^2}{4} = \Phi_{v_r v_r}(\Omega n) \tag{37}$$

where $n = 1, 2, \ldots, N/2 - 1$ *and* $\Omega n \in N_r^c$*;*

4. $A_0 = A_{\frac{N}{2}} = 0$ *or* $\phi_0 = \alpha = 0$*;*

then the extended MOMRS $\mathbf{u}(i)$ *with the consecutively circularly ordered frequencies converges in distribution as* $N \to \infty$ *to a Gaussian multivariate orthogonal multi-sine random time-series (GMOMRS)* $\mathbf{g}(i) = [g_1(i), g_2(i), \ldots, g_p(i)]^T$ *with zero mean vector and the variance matrix* $(1/p)\boldsymbol{\sigma}_{\mathbf{v}}^2$*:*

$$\mathbf{g}(i) \in As\mathcal{N}\left(\mathbf{0}, \frac{1}{p}\boldsymbol{\sigma}_{\mathbf{v}}^2\right) \tag{38}$$

Additionally

1. *The elements of the GMOMRS periodogram matrix are given by:*

$$\Phi_{g_r g_r}(\Omega m) = \begin{cases} \Phi_{v_r v_r}(\Omega m) & \text{if} \quad \Omega m \in N_r^c \setminus \{0, \pi\} \\ 0 & \text{if} \quad (\Omega m = 0) \wedge (\Omega m = \pi) \wedge (\Omega m \notin N_r^c) \end{cases} \tag{39}$$

where $r = 1, 2, \ldots, p$.

2. *The correlation function matrix of the GMOMRS converges to:*

$$\mathcal{E}\{\mathbf{g}(i)\mathbf{g}^T(i-\tau)\} = \mathbf{R}_{\mathbf{gg}}(\tau) = \frac{1}{2\pi pT} \int_0^{2\pi} \Phi_{\mathbf{vv}}(\omega T) \cos(\omega T \tau\, \mathrm{d}(\omega T) = \frac{1}{p}\mathbf{R}_{\mathbf{vv}}(\tau) \tag{40}$$

where $\tau = 0, 1, \ldots, \infty$.

Proof. The uniform distribution of independent, random phase shifts ϕ_n on $[0, 2\pi)$ for each frequency Ωn $(n = 1, 2, \ldots, N/2 - 1)$ implies that for any time instant i the random vector

$$\mathbf{l}_n(i) = \left[l_{1,n}(i), l_{2,n}(i), \ldots, l_{p,n}(i)\right]^T \tag{41}$$

where

$$l_{r,n}(i) = \begin{cases} \dfrac{\sqrt{NT}A_n}{2} \sin(\Omega n i + \phi_n) & \text{if} \quad \Omega n \in N_r^c \\ 0 & \text{if} \quad \Omega n \notin N_r^c \end{cases} \tag{42}$$

for $r = 1, 2, \ldots, p$, is characterised by

$$\mathcal{E}\{\mathbf{l}_n(i)\} = \mathbf{0} \tag{43}$$

Its variance matrix is

$$\mathcal{E}\{\mathbf{l}_n(i)\mathbf{l}_n^T(i)\} = \mathrm{diag}\left\{\mathcal{E}\{l_{1,n}^2(i)\}, \mathcal{E}\{l_{2,n}^2(i)\}, \ldots, \mathcal{E}\{l_{p,n}^2(i)\}\right\} \tag{44}$$

where

$$\mathcal{E}\{l_{r,n}^2(i)\} = \begin{cases} \dfrac{NTA_n^2}{8} & \text{if } \ \Omega n \in N_r^c \\ 0 & \text{if } \ \Omega n \notin N_r^c \end{cases} \tag{45}$$

for $r = 1, 2, \ldots, p$. From (41) it follows that

$$\mathbf{S}_N = \sum_{n=1}^{N/2-1} \mathbf{l}_n(i) = \frac{\sqrt{NT}}{2}\mathbf{u}^N(i) \tag{46}$$

and for each time instant i we have

$$\mathcal{E}\{\mathbf{u}(i)\} = 0 \tag{47}$$

The corresponding variance matrix is

$$\mathbf{\Sigma}_N^2 = \sum_{n=1}^{N/2-1} \mathcal{E}\{\mathbf{l}_n(i)\mathbf{l}_n^T(i)\} = \frac{NT}{4}\mathrm{diag}\left\{\sigma_{11,N}^2, \sigma_{22,N}^2, \ldots, \sigma_{pp,N}^2\right\} \tag{48}$$

where

$$\sigma_{rr,N}^2 = \frac{2}{2\pi T} \sum_{\Omega n \in N_r^c \setminus \{0,\pi\}} \Phi_{v_r v_r}(\Omega n)\Omega = \frac{2}{2\pi T p} \sum_{n=0}^{n_r-1} \Phi_{v_r v_r}\left(\Omega p n + (r-1)\Omega\right)\Omega p \tag{49}$$

As $N \to \infty$, the product $(r-1)\Omega$ tends to 0 and $(n_r - 1)\Omega p$ tends to π. This implies under Riemann's definition of the integral that

$$\lim_{N\to\infty} \sigma_{rr,N}^2 = \frac{1}{2\pi T p} \int_0^{2\pi} \Phi_{v_r v_r}(\omega T)\, \mathrm{d}(\omega T) = \frac{\sigma_{v_{rr}}^2}{p} \tag{50}$$

Let

$$\left\|\mathbf{l}_n(i)\right\|_2 = \sqrt{\sum_{r=1}^{p} l_{r,n}^2(i)} \tag{51}$$

denote the Euclidean norm of the vector $\mathbf{l}_n(i)$. It follows from the properties of the sine function that for each time instant i the sequence of random vectors $\mathbf{l}_n(i)$ ($n = 1, 2, \ldots, N/2 - 1$) is a uniformly bounded sequence (Karr, 1993), i.e. there exists a constant c such that

$$P\left\{\left\|\mathbf{l}_n(i)\right\|_2 \leq c\right\} = 1 \tag{52}$$

for $n = 1, 2, \ldots, N/2 - 1$. This implies that for every $\varepsilon > 0$, the extended Lindeberg condition

$$\lim_{N\to\infty} \frac{1}{N/2-1} \sum_{n=1}^{N/2-1} \mathcal{E}\left\{ \left\|\mathbf{l}_n(i)\right\|_2^2; \left\{ \left\|\mathbf{l}_n(i)\right\|_2 \geq \varepsilon\sqrt{\frac{N}{2}-1} \right\} \right\} = 0 \quad (53)$$

is satisfied by the sequence of $\mathbf{l}_n(i)$ $(n = 1, 2, \ldots, N/2 - 1)$ for each time instant i. It follows from an extension of the Lindeberg-Feller central limit theorem to the multivariate case (Serfling, 1980) that for each time instant i the random variable

$$\frac{2}{\sqrt{NT}} \mathbf{S}_N = \mathbf{u}^N(i) \quad (54)$$

converges in distribution as $N \to \infty$ to a Gaussian multivariate orthogonal random variable $\mathbf{g}(i)$ with zero mean vector and the variance matrix $(1/p)\sigma_{\mathbf{v}}^2$.

Property 1 follows directly from the approximation criterion (26). The proof of Property 2 follows from Riemann's definition of the integral applied for $N \to \infty$ to the correlation function matrix elements (10) of the power spectral density defined MOMRS. ■

The results of this lemma do not change when the zero-frequency phase shift ϕ_0 and Nyquist-frequency distribution parameter α are equal to $\pi/2$, and the corresponding sine component amplitudes are assumed to be chosen for $0 \in N_s^c$ and $\pi \in N_t^c$ as

$$NTA_0^2 = \Phi_{v_s v_s}(0) \quad (55)$$

$$NTA_{\frac{N}{2}}^2 = \Phi_{v_t v_t}(\pi) \quad (56)$$

because the amplitudes A_0 and $A_{\frac{N}{2}}$ tend to zero as $N \to \infty$.

It should be emphasised that for a given power spectral density matrix of a wide sense orthogonal random process the corresponding multivariate orthogonal multisine random time-series exhibits an interesting property: *asymptotically for $N \to \infty$ its periodogram matrix is a consistent estimator of the true power spectral density matrix.*

4. White Noise Approximation

When the power spectral density matrix of a multivariate white noise is approximated by the periodogram matrix of an MOMRS, the corresponding multisine time-series is an extended white MOMRS. For $p = 1, 2$ constant frequency bin spacings can be kept throughout the entire frequency range $[0, 2\pi)$ and whiteness holds for finite N-sample time-series, e.g. its autocorrelation function (or correlation matrices) behaves for a number of lags exactly as a pure white noise autocorrelation function (or correlation matrices) (Figwer and Niederliński, 1992; Niederliński and Figwer, 1995):

Example 1. When the power spectral density $\Phi_{vv}(\omega T) = \lambda^2$ of a scalar white noise is approximated by the periodogram of a scalar multisine random time-series, the corresponding extended MOMRS ($p = 1$) can be turned into an extended white multisine random time-series (WMRS) with the mean $\sqrt{\lambda^2/(TN)}$ and variance $\sigma^2 = (\lambda^2/T)(N-1)/N$. Its autocorrelation function behaves for lags $0, 1, \ldots, N-1$ as a pure white noise autocorrelation function:

$$R_{uu}(\tau) = \begin{cases} \dfrac{\lambda^2}{T} & \text{if } \tau = 0, N, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{57}$$

As $N \to \infty$, the variance of WMRS converges to λ^2/T, and its mean value tends to zero. ♦

Example 2. When the power spectral density matrix $\Phi_{\mathbf{vv}}(\omega T) = \lambda^2 \boldsymbol{I}$ of a bivariate orthogonal white noise ($p = 2$) is approximated by the periodogram matrix of an extended MOMRS with consecutively circularly ordered frequencies, a bivariate orthogonal white multisine random time-series (BOWMRS) is obtained. It is characterised by the mean value vector $\left[\sqrt{\lambda^2/(TN)}, 0\right]^T$ and the variance matrix

$$\sigma^2_{\mathbf{u}} = \frac{\lambda^2}{2T} \begin{bmatrix} \dfrac{N-2}{N} & 0 \\ 0 & 1 \end{bmatrix} \tag{58}$$

The elements of the correlation function matrix $\mathbf{R_{uu}}(\tau)$ for $\tau = 0, 1, \ldots, \infty$ are given by

$$R_{11}(\tau) = \begin{cases} \dfrac{\lambda^2}{2T} & \text{if } \tau = 0, N/2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{59}$$

$$R_{22}(\tau) = \begin{cases} \dfrac{\lambda^2}{2T} & \text{if } \tau = 0, N, \ldots \\ -\dfrac{\lambda^2}{2T} & \text{if } \tau = N/2, 3N/2, \ldots \\ 0 & \text{otherwise} \end{cases} \tag{60}$$

$$R_{12}(\tau) = R_{21}(\tau) = 0 \tag{61}$$

As $N \to \infty$, the variance matrix of the BOWMRS converges to $\lambda^2/(2T)\boldsymbol{I}$, and its mean value vector tends to a zero vector. ♦

It is worth noticing that the whiteness of the WMRS and BOWMRS holds for finite N-sample time-series. Unfortunately, this property cannot be extended to MOMRS having dimensions larger than 2. Correlation matrices of a white MOMRS (MOWMRS) with the number of elements $p > 2$ coincide only asymptotically for $N \to \infty$ with the correlation matrices of a p-variate white noise. Asymptotically the

MOWMRS is a Gaussian multivariate random time-series (Figwer and Niederliński, 1995).

Example 3. Assuming that $\Phi_{\mathbf{vv}}(\omega T)$ $(\omega T \in [0, 2\pi))$ is the power spectral density matrix of a real-valued multivariate white noise $(p > 2)$:

$$\Phi_{\mathbf{vv}}(\omega T) = \lambda^2 \mathbf{1} \tag{62}$$

the approximating extended MOMRS $\mathbf{u}(i)$ with consecutively circularly ordered frequencies converges in distribution as $N \to \infty$ to a Gaussian multivariate white multisine random time-series (GMOWMRS) $\mathbf{g}(i) = [g_1(i), g_2(i), \ldots, g_p(i)]^T$ with zero mean vector and the variance matrix $(\lambda^2/p)\mathbf{1}$:

$$\mathbf{g}(i) \in As\mathcal{N}\left(\mathbf{0}, \frac{\lambda^2}{p}\mathbf{1}\right) \tag{63}$$

The elements $R_{rs}(\tau)$ of the correlation function matrix $\mathbf{R_{gg}}(\tau)$ for $\tau = 0, 1, \ldots, \infty$ are given by

$$R_{rs}(\tau) = \begin{cases} R_{rr}(\tau) & \text{if} \quad r = s \\ 0 & \text{if} \quad r \neq s \end{cases} \tag{64}$$

where $r, s = 1, 2, \ldots, p$. The autocorrelation function $R_{rr}(\tau)$ of the r-th GMOWMRS element converges to

$$R_{rr}(\tau) = \begin{cases} \dfrac{\lambda^2}{p} & \text{if} \quad \tau = 0 \\ 0 & \text{otherwise} \end{cases} \tag{65}$$

♦

All white noises synthesised on the basis of multisine random time-series, even short ones, have very good correlation properties — their correlation matrices approximate very accurately the original ones. It is worth noticing the following exception: *for $N \to \infty$ the periodogram of WMRS and the periodogram matrices of BOWMRS and MOWMRS are consistent estimators of power spectral densities of the corresponding white noises* (see Lemma 1).

5. Gaussian Time-Series Simulation

It follows from the previous sections that statistical properties of multivariate orthogonal multisine random time-series synthesised based on the given power spectral density matrix of a random process to be simulated behave, asymptotically for $N \to \infty$, exactly as those for the corresponding true Gaussian random process. In computer simulation experiments there is no possibility to perform simulations for an infinite N. In order to simulate a Gaussian random process, a finite value of N must be chosen. This choice influences statistical properties of the synthesised multisine time-series. However, the original power spectral densities and autocorrelation functions are approximated very accurately by the corresponding properties of the synthesised multisine approximations, even for small values of N. The influence of a finite N can be seen while variances of the estimated parameters for power spectral density defined MOMRS are compared with the corresponding theoretically calculated Cramer-Rao bounds for true Gaussian random processes.

Example 4. The following bivariate orthogonal **AR** time-series:

$$\boldsymbol{A}(z^{-1})\mathbf{v}(i) = \mathbf{e}(i) \tag{66}$$

with

$$\boldsymbol{A}(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} -0.80 & 0.00 \\ 0.00 & -1.50 \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & 0.70 \end{bmatrix} z^{-2} \tag{67}$$

and a unity variance matrix of the white noise $\mathbf{e}(i)$ was simulated by using:

- its time-domain representation as a discrete-time filter excited by the Gaussian white noise $\mathbf{e}(i)$ generated on the base of a standard linear congruential random number generator (SGRNG);
- its frequency-domain representation as the power spectral density matrix

$$\Phi_{\mathbf{vv}}(j\omega T) = \begin{bmatrix} \dfrac{1.00 + j0}{1.64 - 1.60\cos\omega T} & 0 + j0 \\ 0 + j0 & \dfrac{1.00 + j0}{3.74 - 5.10\cos\omega T + 1.40\cos 2\omega T} \end{bmatrix} \tag{68}$$

  which was approximated by the periodogram matrix of a multivariate (bivariate) orthogonal multisine random time-series.

Each simulated N-sample **AR** time-series realisation ($N = 128$ and $N = 256$) was identified using the least-squares identification method (Box and Jenkins, 1976). The mean values and standard deviations (in parentheses) of the parameters estimated in 100 simulation experiments for the **AR** model with the structure of the matrix $\boldsymbol{A}(z^{-1})$ chosen as

$$\boldsymbol{A}(z^{-1}) = \begin{bmatrix} 1.00 & 0.00 \\ 0.00 & 1.00 \end{bmatrix} + \begin{bmatrix} a_{11}^{1} & 0.00 \\ 0.00 & a_{22}^{1} \end{bmatrix} z^{-1} + \begin{bmatrix} 0.00 & 0.00 \\ 0.00 & a_{22}^{2} \end{bmatrix} z^{-2} \tag{69}$$

and the corresponding Cramer-Rao bounds (CRB) are presented in Tab. 1.

The mean values of the estimated parameters do not differ from the true values but the standard deviations of the estimated parameters for the autoregressive multivariate orthogonal time-series simulated with multivariate orthogonal multisine random time-series are smaller than those obtained for the time-series simulated using the standard Gaussian white noise generator. These standard deviations are also much smaller than those which follow from the Cramer-Rao bound for the original Gaussian random process. ♦

Table 1. Mean values and standard deviations (in parentheses) of the orthogonal **AR** model parameter estimates obtained for 100 simulation experiments using the least-squares identification method.

Parameter (CRB)	Parameter Estimates		N
	SGRNG	MOMRS	
a_{11}^1 (0.053)	−0.791 (0.050)	−0.801 (0.006)	128
a_{22}^1 (0.063)	−1.476 (0.065)	−1.500 (0.009)	
a_{22}^2 (0.063)	0.680 (0.063)	0.700 (0.008)	
a_{11}^2 (0.037)	−0.801 (0.036)	−0.800 (0.002)	256
a_{22}^1 (0.044)	−1.496 (0.043)	−1.500 (0.004)	
a_{22}^2 (0.044)	0.695 (0.042)	0.700 (0.003)	

It follows from the spectral factorisation theorem (Pillai and Shim, 1993) that the results of the parameter estimation for a time-series simulated directly from the given power spectral density diagram and from the corresponding discrete-time filter excited by a multivariate orthogonal white multisine random time-series are comparable (Figwer, 1997; Figwer and Niederliński, 1995). This implies that the discussion of Cramer-Rao bounds for the results of parameter estimation of power spectral density defined MOMRS may be done by analysing only the results for the corresponding multivariate orthogonal white multisine random time-series.

Let l be the number of the simulated scalar white noise time-series samples taken from a synthesised WMRS with the period N $(l < N)$. It is well-known (Box and Jenkins, 1976) that, for a real-valued Gaussian white noise time-series of the length l, the estimates of its normalised autocorrelation function for all lags are asymptotically normally distributed with zero mean and the variance $1/l$. For the WMRS, the variance of the normalised unbiased autocorrelation estimator

$$\frac{\hat{R}_{uu}(\tau)}{\hat{R}_{uu}(0)} = \frac{l}{l-\tau} \frac{\sum_{i=0}^{l-1} u(i)u(i-\tau)}{\sum_{i=0}^{l} u^2(i)} \tag{70}$$

is lag dependent. The smallest value of this variance is for the lag $\tau = 1$. The variance $\mathcal{E}\left\{\left(\hat{R}_{uu}(1)/\hat{R}_{uu}(0)\right)^2\right\}$ may be approximated by the following formula:

$$\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\} \cong \frac{1}{l + \frac{(l-1)^2}{N-l+1}} \tag{71}$$

Similarly, the variance $\mathcal{E}\left\{\left(\hat{R}_{uu}(1)/\hat{R}_{uu}(0)\right)^2\right\}$ of the normalised autocorrelation function estimator (70) for all elements of MOWMRS is

$$\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\} \cong \frac{1}{l + \frac{(l-1)^2}{N/p-l+1}} \tag{72}$$

The analysis of the above expressions leads to the conclusion that, when using multivariate orthogonal multisine random time-series to simulate Gaussian random processes, the following two simulation schemes can be distinguished:

- Case $l \ll N/p$ in which the variances of the autocorrelation function estimator for the elements of the power spectral density defined multivariate orthogonal multisine random time-series are comparable with the corresponding values of the Cramer-Rao bounds for the true Gaussian random process.

- Case $l \approx N$ $(l < N)$ in which the variances of the autocorrelation function estimator for the elements of the power spectral density defined multivariate orthogonal multisine random time-series are always much smaller than the corresponding Cramer-Rao bounds for the true Gaussian random process. The results of autocorrelation estimation behave as for the true Gaussian random process with the number of samples

$$l' = \frac{1}{\mathcal{E}\left\{\left(\frac{\hat{R}_{uu}(1)}{\hat{R}_{uu}(0)}\right)^2\right\}} \tag{73}$$

  This means that to simulate an l'-sample time-series representation by using a classical Gaussian white noise random number generator you can simulate the corresponding l-sample $(l < l')$ multisine random time-series with the same statistical properties. This is an interesting property of the power spectral density defined multisine random time-series which may be called a simulation time-scale contraction. The simulation time-scale contraction allows us to reduce radically simulation efforts. This is especially important in real-world experiments in which test times are limited by the properties of the systems under tests (Figwer, 1996).

6. Conclusions

A synthesis and simulation method of wide-sense stationary multivariate orthogonal random processes characterised by their power spectral densities has been presented. It is based on approximating the power spectral densities by periodograms of orthogonal multisine time-series with deterministic amplitudes and random phase shifts, and transforming the frequency representations into the time-domain by using the inverse finite discrete Fourier transform. Multisine approximations of wide-sense stationary scalar and multivariate orthogonal random processes thus obtained are ergodic. Asymptotically, they turn into Gaussian time-series.

The proposed approach can be used when only a power spectral density diagram of the random process to be simulated is available. There is no necessity to find any parametric representation of the random process. This is especially important for random processes which have non-rational power spectral densities, because the accuracy of the parametric approximation is crucial in reconstructing the properties of original random processes (Pillai and Shim, 1993).

It was shown that power spectral density defined multivariate orthogonal multisine random time-series approximate very precisely the correlation properties of the original random processes. This method, when applied to the power spectral density of white noises, allows us to synthesise different types of interesting scalar and multivariate orthogonal, white or asymptotically white, ergodic random time-series.

A new technique of Gaussian random process simulation based on a time-scale contraction was proposed. It offers an opportunity to reduce radically the simulation time.

Theoretical arguments were presented for multivariate orthogonal multisine random time-series with an even length N of their period. For an odd N, the set of all relative frequencies does not include the Nyquist frequency π and there is no need to take into account terms involved by this frequency (the subscript $N/2$ for even N). Conveniently, it turned out that the results for the case of N odd are very similar to those for N even. The corresponding definitions and theorems can be obtained by eliminating terms implied by the sine component of the Nyquist frequency. Despite these simplifications, the choice of N even was more useful for the white noise synthesis. It allowed us to synthesise scalar and bivariate white multisine time-series for which whiteness holds for finite N, while for N odd it can be done only for the scalar case (Figwer and Niederliński, 1995).

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