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DECENTRALIZED VARIABLE STRUCTURE TRACKING FOR SYSTEMS WITH TIME-DOMAIN DOMINANCE

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In this paper, we consider the design of tracking controllers for linear MIMO systems described by an input-output model. The presence of known 'weak' interactions among SISO or MIMO subsystems may allow the designer to achieve objectives by using independent controllers of lower complexity than are necessary in general (control decentralization problem). Sufficient conditions for asymptotic tracking employing decentralized variable structure techniques are derived. The condition is shown to be closely related to (and in a sense, a time-domain counterpart of) dominance criteria used in frequency-domain techniques, as they have developed out of Rosenbrock's original diagonal dominance concept. The synthesis of a decentralized variable-structure controller for asymptotic tracking is illustrated for systems obeying some conditions on their nominal relative degrees.

Keywords: dominance conditions, decentralized control, robust control, variable structure control (VSC).

1. Introduction

The study of decentralized controllers for multivariable systems has attracted much attention in the last two decades, mainly because of its relevance to practical large-scale systems such as encountered in electric power systems, socioeconomic systems, chemical processes, space structures and robotic applications (Siljak, 1978). A frequent attitude of control system engineers in attacking large-scale systems is to try and find, by accurately analysing the system model, an underlying pattern of simpler SISO or MIMO subsystems connected by means of 'weak' relation links. Once such a structure has been recognized, an attempt can be made at controlling the subsystems by means of relatively simple controllers and only use information relative to each subsystem.

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1.1. Previous Work

The literature on decentralized control can be grossly divided in two main branches, dealing with methods in the frequency and in the time domains, respectively. Frequency domain methods have attracted the interest of researchers due to the fact that available models of large-scale systems are often of the input-output type, mostly in the form of approximated transfer function matrices. Rosenbrock's DNA and INA techniques (Rosenbrock, 1974), for the design of decentralized linear controllers for linear multivariable systems have proved to be among the most effective and practical tools for approaching large-scale systems that exhibit weak coupling among SISO subsystems.

Rosenbrock's diagonal dominance conditions are generally recognized to be rather difficult to meet in practical applications, while the design of a precompensator to help attain dominance might spoil the simplicity of the design procedure that makes the technique attractive. Less restrictive dominance conditions have been therefore sought for actively. Generalized diagonal dominance, investigated first by Araki and Nwokah (1975), allows arbitrary scaling of inputs and outputs to be applied to achieve dominance. Further extensions of Rosenbrock's work involve the use of generalized diagonal dominance by blocks, and applies to weakly interacting MIMO subsystems. Successive refinements of the idea led to a definition of block diagonal dominance (Bennet and Baras, 1962; Feingold and Varga, 1962), generalized block diagonal dominance (Limebeer, 1982), and quasi-block diagonal dominance (Nwokah, 1987; Ohta *et al.*, 1986). The latter, more general formulation, can be expressed as an M-matrix condition on a suitably defined matrix of norms applied to transfer function blocks. A coherent treatment of dominance concepts from both the standard and M-matrix approaches is developed in (Yeung and Bryant, 1992), where the more general concept of 'fundamental dominance' is also introduced. In that paper, the relationship between dominance, system approximation, and robust stability *à la* Doyle (Doyle, 1979; 1982) is also enlightened.

On the other hand, while state-space, centralized methods for multivariable tracking or servomechanism problems in LTI systems based on the internal model principle (see e.g. Wonham, 1979) attracted much attention, decentralized controllers for robust tracking in large-scale systems have also been an active research area. The pioneering work of Wang and Davison (1973) and Davison (1976) has been followed by important contributions, among which we mention (Chen *et al.*, 1991; Gavel and Siljak, 1989; Ioannou, 1986; Shi and Singh, 1993).

At about the same time, application of variable structure controllers (VSC) to multivariable systems has begun to be investigated (Utkin, 1977), and VS solutions to the MIMO servomechanism problem have been proposed (Young and Kwatny, 1982); also, see (DeCarlo *et al.*, 1988) for a tutorial introduction). A particularly prolific area of application has been the control of robot arms (Balestrino *et al.*, 1984; Slotine and Sastry, 1983; Young, 1978). More recently, efforts have been made to develop decentralized variable structure controllers (DVSC) in order to conjugate the outstanding performance provided by VSC even in the presence of nonlinear, uncertain plants with the requirements of limited controller complexity encountered in typical large-scale problems. Lefebvre *et al.* (1982), Matthews and DeCarlo (1987), Khurana *et al.* (1986) studied decentralized VS stabilizing controllers, while the tracking controller problem for a class of interconnected multivariable systems was given a solution by Matthews and DeCarlo (1988). Variable structure methods for tracking control of complex mechanical systems that cannot be considered algebraically interconnected (such as robot arms) have been discussed by Singh (1990). The authors (Balluchi and Bicchi, 1997) have derived necessary and sufficient conditions for robust perfect tracking under variable structure control.

1.2. Main Contributions and Organization

In this paper, we investigate the connections between input-output dominance concepts and VSC techniques for decentralized control of general multivariable systems. In order to retain the practice-oriented flavour of frequency-domain methods, the assumed model of the plant is an input/output relationship as represented e.g. by a transfer matrix $\mathbf{G}$.

The problem addressed is to find conditions on a plant $\mathbf{G}$ under which a VSC law that achieves asymptotic tracking on a block-diagonal approximation $\mathbf{G}_D$, is guaranteed to accomplish the same performance on the actual plant $\mathbf{G}$. A sufficient condition for the existence of such a controller is produced in Theorem 1. The condition can be regarded as a time-domain counterpart of Rosenbrock-like dominance conditions.

This theoretical result is utilized to synthesize practical decentralized tracking controllers for systems having unit row-relative degrees. Rather than on sliding-mode observer design, the proposed controller is based on a novel scheme, somewhat affine to reference-model tracking.

The problem is formulated precisely in Section 2. The new dominance conditions for decentralization are proposed in Section 3. Section 4 presents the synthesis of the tracking VS controller for systems with row-relative degree one and unaccessible states. Finally, in Section 5, simulation results are reported on the application of such a control scheme to systems with different dominant patterns.

2. Background and Problem Setup

Consider an m-input, m-output strictly proper MIMO system as comprised of $N \leq m$, m_i-input, m_i-output 'weakly' interacting square subsystems, with $m = m_1 + \cdots + m_N$. We assume at this stage that such an interconnection structure has been identified by the designer, and that inputs and outputs have been arranged and partitioned in contiguous groups so as to reflect the subsystem structure (algorithms to achieve this in a preliminary analysis phase have been discussed e.g. in (Balluchi *et al.*, 1993)). Accordingly, decompose the transfer matrix $\mathbf{G}(s)$ of the given system in the sum of a nominal part $\mathbf{G}_D(s)$ and an additive term $\mathbf{G}_C(s)$, namely

$$\mathbf{G}(s) = \mathbf{G}_D(s) + \mathbf{G}_C(s) \tag{1}$$

For decentralized control, the nominal part of the plant is block diagonal

$$\mathbf{G}_D(s) = \mathrm{diag}\big(\mathbf{G}_1(s), \dots, \mathbf{G}_N(s)\big)$$

It is assumed that the $m_i \times m_i$ transfer matrix of the i-th nominal subsystem $\mathbf{G}_i(s)$ is a strictly proper rational matrix with full rank over the field of complex numbers, implying that the number of effective inputs and the number of effective outputs of the i-subsystem is actually m_i. The system $\mathbf{G}_C$ is decomposed in blocks $\mathbf{G}_{ij}$ of dimension $m_i \times m_j$.

Based on such a decomposition, classical results on decentralized control show that, if a constant block-diagonal feedback $\mathbf{K}$ stabilizes the nominal system, i.e. if $\mathbf{I} + \mathbf{K}\mathbf{G}_D$ is Hurwitz, sufficient conditions for $\mathbf{K}$ to stabilize the real plant $\mathbf{G}$ can be given in terms of the matrix

$$\mathbf{Q} = (\mathbf{K}^{-1} + \mathbf{G}_D)^{-1}\mathbf{G}_C \tag{2}$$

with the inherited block partition

$$\mathbf{Q} = \{\mathbf{Q}_{ij}\}, \qquad \mathbf{Q}_{ij} = (\mathbf{K}_{ii}^{-1} + \mathbf{G}_i)^{-1}\mathbf{G}_{ij}$$

Denoting by $\mathcal{D}$ the Nyquist contour, two such sufficient conditions are

$$\rho\big(\{\mathbf{Q}_{ij}(s)\}\big) < 1, \quad \forall s \in \mathcal{D} \tag{3}$$

$$\rho_{PF}\big(\{\|\mathbf{Q}_{ij}(s)\|\}\big) < 1, \quad \forall s \in \mathcal{D} \tag{4}$$

where $\rho(\cdot)$ is the spectral radius of a matrix on the complex field, $\rho_{PF}(\cdot)$ is the Perron-Frobenius root of a real nonnegative matrix, and $\|\cdot\|$ is any induced norm on the space of complex matrices of given dimensions (for a proof of (3) and (4), see (Ohta *et al.*, 1986) and (Yeung and Bryant, 1992), respectively).

Motivated by these results, we investigate under what conditions a given VS controller that achieves asymptotic tracking of a given class of reference output trajectories $\mathbf{y}_r(t)$ with a specified error dynamics and bounded input disturbances $\boldsymbol{\nu}(t)$ for the nominal plant, maintains the same characteristics when connected to the real plant.

To make this idea more precise, consider a controllable realization $\mathcal{S}_i$ of the nominal subsystem $\mathbf{G}_i$,

$$\mathcal{S}_i : \begin{cases} \dot{\mathbf{x}}_i = \mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i(\mathbf{u}_i + \boldsymbol{\nu}_i), & \mathbf{x}_i(0) = \mathbf{x}_i^0 \\ \bar{\mathbf{y}}_i = \mathbf{C}_i\mathbf{x}_i \end{cases} \tag{5}$$

with initial conditions satisfying

$$\left\|\mathbf{x}_i^0\right\|_\infty \le \rho_i \in \mathbb{R}_+ \tag{6}$$

For further convenience, realization $\mathcal{S}_i$ is chosen in a column-wise controllable canonical form, namely

$$\begin{aligned} \mathbf{A}_i &= \mathrm{diag}\left(\mathbf{A}_i^{(1)}, \dots, \mathbf{A}_i^{(m_i)}\right), \qquad \mathbf{B}_i = \mathrm{diag}\left(\mathbf{b}_i^{(1)}, \dots, \mathbf{b}_i^{(m_i)}\right) \\ \mathbf{C}_i &= \left[\mathbf{C}_i^{(1)}, \dots, \mathbf{C}_i^{(m_i)}\right] \end{aligned} \tag{7}$$

where $(\mathbf{A}_i^{(j)}, \mathbf{b}_i^{(j)}, \mathbf{C}_i^{(j)})$ are minimal realizations (of order $n_i^{(j)}$) in controllable canonical-form of the j-th column of $\mathbf{G}_i$. We assume in this section that the states $\mathbf{x}_i \in \mathbb{R}^{n_i}$ are accessible to measurements (this hypothesis to be removed in Section 4). Input disturbances $\boldsymbol{\nu}_i$ represent noise on the actuators and possibly nonlinearities satisfying the so-called *matching conditions.*

Denoting by $\mathbf{L}_q^\infty$ the space of functions $\mathbf{f} : \mathbb{R} \to \mathbb{R}^q$ such that

$$\left\|\mathbf{f}(t)\right\|_\infty = \max_{k=1,q} \sup_{t\geq 0} \left|f_k(t)\right| < \infty$$

let the i-th disturbance vector $\boldsymbol{\nu}_i \in \mathbf{L}_{m_i}^\infty$, with

$$\left\|\boldsymbol{\nu}_i(t)\right\|_\infty \leq N_i \in \mathbb{R}_+ \tag{8}$$

and let the class of desired trajectories to be followed be described by the n_i-th order system $\mathcal{R}_i$,

$$\mathcal{R}_i : \begin{cases} \dot{\mathbf{r}}_i &= \mathbf{A}_{ri}\mathbf{r}_i + \mathbf{B}_{ri}\mathbf{v}_{ri}, \quad \mathbf{r}_i(0) = \mathbf{r}_i^0 \\ \mathbf{y}_{ri} &= \mathbf{C}_{ri}\mathbf{r}_i \end{cases} \tag{9}$$

where $(\mathbf{A}_{ri}, \mathbf{B}_{ri}, \mathbf{C}_{ri})$ are in column-wise controllable canonical form (hence $\mathbf{B}_{ri} = \mathbf{B}_i$), $\mathbf{A}_{ri}$ is stable, $\mathbf{C}_{ri} = \mathbf{C}_i$, and $\mathbf{v}_{ri} \in \mathbf{L}_{m_i}^\infty$ with

$$\left\|\mathbf{v}_{ri}(t)\right\|_\infty \leq V_i \in \mathbb{R}_+ \qquad \text{and} \qquad \left\|\mathbf{r}_i^0\right\|_\infty \leq \rho_{ri} \in \mathbb{R}_+ \tag{10}$$

Hence, restrictions on reference trajectories $\mathbf{y}_{ri}$ amount to boundedness and some mild regularity conditions in case $\mathcal{S}_i$ is minimum-phase. If $\mathcal{S}_i$ has some zero in the closed right half-plane (CRHP), reference trajectories are generated through a system with the same CRHP zeroes. Although more general tracking schemes can be devised (see e.g. (Kwatny and Kalnitsky, 1978)), the one considered above is sufficiently general for the purposes of this paper, while it lends itself to straightforward application of the theory of sliding modes (Utkin, 1977), which is briefly reviewed as follows.

The dynamics of tracking error between reference states $\mathbf{r}_i$ and states of the diagonal subsystem $\mathbf{x}_i$ (and hence, the dynamics of output tracking errors) can be chosen by enforcing a sliding motion on a linear manifold $\Sigma_i = \{\mathbf{x}_i \in \mathbb{R}^{n_i} \mid \boldsymbol{\sigma}_i = \mathbf{0}\}$, where $\boldsymbol{\sigma}_i \in \mathbb{R}^{m_i}$ is defined as

$$\boldsymbol{\sigma}_i = \boldsymbol{\Gamma}_i\left(\mathbf{x}_i - \mathbf{r}_i\right), \quad \boldsymbol{\Gamma}_i \in \mathbb{R}^{m_i \times n_i} \tag{11}$$

A convenient choice for the realization above is $\boldsymbol{\Gamma}_i = \operatorname{diag}(\boldsymbol{\Gamma}_i^{(1)}, \ldots, \boldsymbol{\Gamma}_i^{(m_i)})$, $\boldsymbol{\Gamma}_i^{(j)} \in \mathbb{R}^{1\times n_i^{(j)}}$ such that $\boldsymbol{\Gamma}_i^{(j)}\mathbf{b}_i^{(j)} = 1$ (hence $\boldsymbol{\Gamma}_i\mathbf{B}_i = \mathbf{I}_{m_i}$). Pole assignment or LQ techniques can be employed for choosing the remaining $n_i^{(j)} - 1$ free parameters in $\boldsymbol{\Gamma}_i^{(j)}$, as described e.g. by Dorling and Zinober (1986).

A well-known technique to study the behaviour of the system during the sliding motion is the method of the equivalent control. The equivalent control is the input signal $\mathbf{u}_{i_{eq}}$ that solves $\dot{\boldsymbol{\sigma}}_i = \mathbf{0}$. We have

$$\mathbf{u}_{i_{eq}} = -\boldsymbol{\Gamma}_i(\mathbf{A}_i\mathbf{x}_i + \mathbf{B}_i\boldsymbol{\nu}_i - \dot{\mathbf{r}}_i) \tag{12}$$

Therefore, by means of the equivalent control and noting that $(\mathbf{I}-\mathbf{B}_i\mathbf{\Gamma}_i)\mathbf{A}_{ri} = (\mathbf{I}-\mathbf{B}_i\mathbf{\Gamma}_i)\mathbf{A}_i$, the dynamics of the state error $\mathbf{x}_i - \mathbf{r}_i$ for the system restricted to the sliding surface Σ_i can be expressed as

$$\dot{\mathbf{x}}_i - \dot{\mathbf{r}}_i = (\mathbf{I}-\mathbf{B}_i\mathbf{\Gamma}_i)(\mathbf{A}_i\mathbf{x}_i - \mathbf{A}_{ri}\mathbf{r}_i - \mathbf{B}_{ri}\mathbf{v}_{ri}) = (\mathbf{I}-\mathbf{B}_i\mathbf{\Gamma}_i)\mathbf{A}_i(\mathbf{x}_i - \mathbf{r}_i) \quad (13)$$

where the column-wise controllable canonical form of the realizations of $\mathcal{S}_i$ and $\mathcal{R}_i$ is exploited. Note that only the coefficients of $\mathbf{\Gamma}_i$ actually appear in the sliding dynamics. Thus, the sliding motion on Σ_i yields the convergence of the states $\mathbf{x}_i$ to the states $\mathbf{r}_i$ with the dynamics imposed by the choice of $\mathbf{\Gamma}_i$.

Owing to a proper choice of $\mathbf{\Gamma}_i$, outputs $\bar{\mathbf{y}}_i$ during sliding asymptotically track reference ouputs $\mathbf{y}_{ri}$ under the equivalent control (12). However, since the disturbance is unknown, the equivalent control cannot be synthesized directly. Switching control laws are commonly designed as

$$\mathbf{u}_i = -\mathbf{\Gamma}_i(\mathbf{A}_i\mathbf{x}_i - \dot{\mathbf{r}}_i) - k_i\,\mathrm{sign}(\boldsymbol{\sigma}_i) \quad (14)$$

One says that a stable sliding regime exists on Σ_i if all system trajectories originating in a neighborhood of Σ_i point towards Σ_i, i.e. $\sigma_i^{(j)}\dot{\sigma}_i^{(j)} < 0$ for all components $\sigma_i^{(j)}$ of $\boldsymbol{\sigma}_i$. Such an existence condition is met globally on the state space if and only if

$$k_i > \left\|\boldsymbol{\nu}_i(t)\right\|_\infty \quad (15)$$

Furthermore, by choosing

$$k_i = N_i + \epsilon_i \quad (16)$$

where $\epsilon_i > 0$, it is guaranteed that the sliding manifold is reached in finite time, i.e. that $\boldsymbol{\sigma}_i = \mathbf{0}$ for all $t > \|\boldsymbol{\sigma}_i(0)\|/\epsilon_i$.

In this framework, we define the tracking performance of a VSC as

Definition 1. *A VSC law is said to achieve performance $\mathcal{P}_\Gamma$ on a system $\mathbf{G}$ if it ensures the stability of a sliding regime, during which outputs of $\mathbf{G}$ asymptotically track reference trajectories (9), (10), with error dynamics determined by $\mathbf{\Gamma}_i$, in spite of disturbances as in (8).*

The problem this paper is concerned with is therefore the following:

Problem 1. *Under what conditions on a system $\mathbf{G} = \mathbf{G}_D + \mathbf{G}_C$ will a decentralized VSC law (14) exist, which achieves $\mathcal{P}_\Gamma$ on $\mathbf{G}$?*

3. Time-Domain Dominance Conditions

The i-th block of outputs of the plant $\mathbf{G}$, $\mathbf{y}_i$, can be expressed in terms of the disturbed outputs of the nominal subsystem $\mathbf{G}_i$ in (5), as

$$\mathbf{y}_i = \bar{\mathbf{y}}_i + \boldsymbol{\varphi}_i \quad (17)$$

with

$$\varphi_i = \sum_{j=1,N} \varphi_{ij}$$

and

$$\begin{cases} \dot{\mathbf{x}}_{ij} &= \mathbf{A}_{ij}\mathbf{x}_{ij} + \mathbf{B}_{ij}(\mathbf{u}_j + \nu_j), \quad \mathbf{x}_{ij}(0) = \mathbf{x}_{ij}^0 \\ \varphi_{ij} &= \mathbf{C}_{ij}\mathbf{x}_{ij} \end{cases}$$

where N^2 MIMO systems $(\mathbf{A}_{ij}, \mathbf{B}_{ij}, \mathbf{C}_{ij})$ have been introduced, each providing a minimal realization of order n_{ij} of the transfer matrix $\mathbf{G}_{ij}$, with initial conditions satisfying

$$\left\|\mathbf{x}_{ij}^0\right\|_\infty \le \rho_{ij} \in \mathbb{R}_+ \tag{18}$$

Consider further N m_i-input, m_i-output systems $\mathcal{Z}_i$ of order n_i, with parameters and initial conditions equal to those of the nominal realizations in (5), and excited by an input signal $\psi_i(t)$,

$$\mathcal{Z}_i : \begin{cases} \dot{\mathbf{z}}_i &= \mathbf{A}_i\mathbf{z}_i + \mathbf{B}_i\psi_i, \quad \mathbf{z}_i(0) = \mathbf{x}_i^0 \\ \mathbf{w}_i &= \mathbf{C}_i\mathbf{z}_i \end{cases} \tag{19}$$

We are interested in conditions for ψ_i under which outputs $\mathbf{w}_i$ match the actual plant outputs $\mathbf{y}_i$. In order to investigate this point, we need to establish a preliminary result regarding the properness of rational matrices.

Given a proper rational function $G(s)$, let $\tilde{\delta}(G(s))$ denote its relative degree. If $\mathbf{M}(s)$ is a $p \times q$ proper rational matrix whose $(i,\, j)$ element is $M_{ij}(s)$, we define the relative degree of the i-th row of $\mathbf{M}(s)$, $\tilde{\delta}_{R_i}(\mathbf{M}(s))$, as the smallest relative degree in all entries of the i-th row of $\mathbf{M}(s)$

$$\tilde{\delta}_{R_i}(\mathbf{M}(s)) = \min_j \tilde{\delta}(M_{ij}(s))$$

A nonsingular $p \times p$ proper rational matrix $\mathbf{M}(s)$ is called *row reduced with respect to the relative degree* if

$$\tilde{\delta}(\det \mathbf{M}(s)) = \sum_{i=1,p} \tilde{\delta}_{R_i}(\mathbf{M}(s)) \tag{20}$$

Lemma 1. *Let $\mathbf{A}(s)$ and $\mathbf{B}(s)$ be $p \times p$ and $p \times q$ proper rational matrices, respectively, and, moreover, let $\mathbf{A}(s)$ be row reduced with respect to the relative degree. Then the rational matrix $\mathbf{A}^{-1}(s)\mathbf{B}(s)$ has μ poles at infinity if and only if*

$$\tilde{\delta}_{R_i}(\mathbf{A}(s)) \le \tilde{\delta}_{R_i}(\mathbf{B}(s)) + \mu \qquad \textit{for } i = 1, 2, \ldots, p$$

where an equality holds for at least one i. In particular, $\mathbf{A}^{-1}(s)\mathbf{B}(s)$ is proper (strictly proper) if and only if $\tilde{\delta}_{R_i}(\mathbf{A}(s)) \le \tilde{\delta}_{R_i}(\mathbf{B}(s))$ ($\tilde{\delta}_{R_i}(\mathbf{A}(s)) < \tilde{\delta}_{R_i}(\mathbf{B}(s))$) for $i = 1, 2, \ldots, p$.

Proof. The proof of this lemma, appearing in (Balluchi, 1996), is based on the theory of polynomial matrix reduction (Wolovich, 1974), and is not reported here because of space limitations. ■

We now turn back to the problem of finding an input disturbance ψ_i under which outputs $\mathbf{w}_i$ match $\mathbf{y}_i$ in (17).

Lemma 2. *Assume that, for all i and j, blocks $\mathbf{G}_i$ of $\mathbf{G}_D$ and $\mathbf{G}_{ij}$ of $\mathbf{G}_C$ satisfy the following:*

H1 *all CRHP transmission zeros of $\mathbf{G}_i$ cancel in all products $\mathbf{G}_i^{-1}\mathbf{G}_{ij}$;*

H2 *all CRHP poles of $\mathbf{G}_{ij}$ cancel in all products $\mathbf{G}_i^{-1}\mathbf{G}_{ij}$;*

H3 *$\mathbf{G}_i(s)$ is row reduced with respect to the relative degree;*

H4 $\tilde{\delta}_{R_k}(\mathbf{G}_{ij}(s)) \geq \tilde{\delta}_{R_k}(\mathbf{G}_i(s))$ *for* $k = 1, 2, \ldots, m_i$.

Under these conditions, there exist distributions $\psi_i(t) = \mathbf{u}_i(t) + \nu_i(t) + \zeta_i(t)$ such that $\mathbf{w}_i(t) = \mathbf{y}_i(t), \forall t > 0$. Distributions ζ_i may contain delta functions and derivatives of delta functions in the origin up to the order $\max_{k=1,m_i} \tilde{\delta}_{R_k}(\mathbf{G}_i(s)) - 1$. If the plant is initially relaxed and if $\mathbf{u} \in \mathbf{L}_m^\infty$ and $\nu \in \mathbf{L}_m^\infty$, then $\zeta \in \mathbf{L}_m^\infty$.

As a useful tool in the proof of this lemma, we recall the definition of the set of stable undelayed impulse response matrices $\mathcal{A}^{m\times n}$ (Vidyasagar, 1978), whose elements are matrices of distributions $\mathbf{f} : \mathbb{R} \to \mathbb{R}^{m\times n}$ of the form

$$\mathbf{f}(t) = \begin{cases} 0 & \text{if } t < 0 \\ \mathbf{F}\delta(t) + \mathbf{f}_a(t) & \text{if } t \geq 0 \end{cases}$$

where $\mathbf{F}$ is an $m \times n$ constant matrix, $\delta(t)$ is the unit delta distribution and $\mathbf{f}_a(t)$ is a matrix of measurable functions. The norm $\|\cdot\|_{\mathcal{A}}$ of a matrix of distributions $\mathbf{f}(t) \in \mathcal{A}^{m\times n}$ is defined by

$$\|\mathbf{f}(t)\|_{\mathcal{A}} = \max_{1\leq i\leq m} \sum_{j=1,n} \left(|F_{ij}| + \int_0^\infty |f_{a_{ij}}(t)|\,\mathrm{d}t \right)$$

Notice that this norm corresponds to the $\mathbf{L}_\infty$-induced norm of the convolution operator corresponding to distribution matrices: if $\mathbf{v}(t)$ is an n-vector signal in $\mathbf{L}_\infty$, we have

$$\|\mathbf{f}(t) * \mathbf{v}(t)\|_\infty \leq \|\mathbf{f}(t)\|_{\mathcal{A}} \|\mathbf{v}(t)\|_\infty$$

Proof of Lemma 2. Let $\mathbf{g}_i$ (respectively, $\mathbf{g}_{ij}$) denote the impulse response matrix of $\mathbf{G}_i$ ($\mathbf{G}_{ij}$). Equating $\mathbf{y}_i$ and $\mathbf{w}_i$, we have

$$\mathbf{g}_i * \zeta_i = \sum_{j=1,N} \mathbf{g}_{ij} * (\mathbf{u}_j + \nu_j) + \mathbf{C}_{ij} \exp(\mathbf{A}_{ij}t)\mathbf{x}_{ij}^0 \tag{21}$$

Let $\hat{\mathbf{g}}_i$ be defined such that

$$\hat{\mathbf{g}}_i * \mathbf{g}_i = \delta(t)\mathbf{I}_{m_i\times m_i}$$

and consider the system whose transfer function is the $m_i \times m_j$-matrix $\hat{\mathbf{G}}_i\mathbf{G}_{ij} = \mathbf{G}_i^{-1}\mathbf{G}_{ij}$. By Lemma 1, hypotheses **H3** and **H4** are necessary and sufficient conditions for $\hat{\mathbf{G}}_i\mathbf{G}_{ij}$ to be a proper rational matrix which, under conditions **H1** and **H2**, is stable. Hence $\hat{\mathbf{g}}_i * \mathbf{g}_{ij}$ belongs to $\mathcal{A}^{m_i \times m_j}$. From (21) we have

$$\zeta_i = \sum_{j=1,N} \hat{\mathbf{g}}_i * \mathbf{g}_{ij} * (\mathbf{u}_j + \boldsymbol{\nu}_j) + \zeta_i^0 \tag{22}$$

where ζ_i^0 stands for transient terms,

$$\zeta_i^0 = \hat{\mathbf{g}}_i * \left(\sum_{j=1,N} \mathbf{C}_{ij} \exp\left(\mathbf{A}_{ij}t\right)\mathbf{x}_{ij}^0 \right) \tag{23}$$

For arbitrary initial conditions of the plant, ζ_i^0 may contain delta functions and derivatives in the origin. Indeed, applying Lemma 1, the number μ_i of poles at infinity in the expression $\mathbf{G}_i^{-1}(s) \cdot \mathbf{C}_{ij}(s\mathbf{I} - \mathbf{A}_{ij})^{-1}\mathbf{x}_{ij}^0$ is such that

$$\mu_i \geq \tilde{\delta}_{R_k}\left(\mathbf{G}_i(s)\right) - \tilde{\delta}_{R_k}\left(\mathbf{C}_{ij}(s\mathbf{I} - \mathbf{A}_{ij})^{-1}\mathbf{x}_{ij}^0\right) \quad \text{for } k = 1, 2, \ldots, m_i$$

where an equality holds for at least one k. In particular, since for any k there exist $\mathbf{x}_{ij}^0$ such that

$$\tilde{\delta}_{R_k}\left(\mathbf{C}_{ij}(s\mathbf{I} - \mathbf{A}_{ij})^{-1}\mathbf{x}_{ij}^0\right) = \tilde{\delta}\left([\mathbf{C}_{ij}]_k(s\mathbf{I} - \mathbf{A}_{ij})^{-1}\mathbf{x}_{ij}^0\right) = 1$$

where $[\mathbf{C}_{ij}]_k$ denotes the k-th row of $\mathbf{C}_{ij}$, the maximum order of the derivatives of delta functions is

$$\mu_i = \max_{k=1,m_i} \tilde{\delta}_{R_k}\left(\mathbf{G}_i(s)\right) - 1$$

For relaxed initial conditions, the following upper bound holds:

$$\begin{aligned} \left\|\zeta_i(t)\right\|_\infty &\leq \sum_{j=1,N} \left\| (\hat{\mathbf{g}}_i * \mathbf{g}_{ij}) * (\mathbf{u}_j + \boldsymbol{\nu}_j) \right\|_\infty \\ &\leq \sum_{j=1,N} \left\|\hat{\mathbf{g}}_i * \mathbf{g}_{ij}\right\| \left(\left\|\mathbf{u}_j(t)\right\|_\infty + \left\|\boldsymbol{\nu}_j(t)\right\|_\infty \right) \end{aligned} \tag{24}$$

■

Under the hypotheses of Lemma 2, define $\mathbf{P} \in \mathbb{R}_+^{N \times N}$ as

$$\mathbf{P} = \{P_{ij}\}, \quad P_{ij} = \left\|\hat{\mathbf{g}}_i * \mathbf{g}_{ij}\right\|_{\mathcal{A}} \tag{25}$$

A sufficient condition solving Problem 1 stated in Section 2 is given in the following:

Theorem 1. (Sufficient condition for DVSC) *Given the MIMO system* $\mathbf{G} = \mathbf{G}_D + \mathbf{G}_C$ *satisfying the hypotheses of Lemma 2, consider a decentralized VSC law as in (14) with* $k_i = N_i + \epsilon_i$. *If*

$$\rho_{PF}\left(\mathbf{P}\right) < 1 \tag{26}$$

with $\mathbf{P}$ *as in (25), there exist values of* $\epsilon_i > 0$ *such as to guarantee performance* $\mathcal{P}_\Gamma$.

Proof. Applying the decentralized control law (14) to the actual plant, we have

$$\begin{cases} \dot{\mathbf{z}}_i &= \mathbf{A}_i\mathbf{z}_i + \mathbf{B}_i(\mathbf{u}_i + \nu_i + \zeta_i), \quad \mathbf{z}_i(0) = \mathbf{x}_i^0 \\ \mathbf{w}_i &= \mathbf{C}_i\mathbf{z}_i \end{cases}$$

with

$$\mathbf{u}_i = \mathbf{u}_i' - k_i \operatorname{sign}\big(\mathbf{\Gamma}_i(\mathbf{z}_i - \mathbf{r}_i)\big) \tag{27}$$

where

$$\mathbf{u}_i' = -\mathbf{\Gamma}_i(\mathbf{A}_i\mathbf{z}_i - \dot{\mathbf{r}}_i) \tag{28}$$

The existence of a stable sliding regime on Σ_i for $t \geq t_s$ is guaranteed if and only if

$$k_i > \big\|\nu_i(t+t_s) + \zeta_i(t+t_s)\big\|_\infty \tag{29}$$

The conditions on $\mathbf{G}$ and $\mathbf{G}_D$ under investigation are derived by studing the above inequality in terms of parameters k_i. Obviously, $N_i \geq \|\nu_i(t+t_s)\|$. From (22) and (24) we have

$$\big\|\zeta_i(t+t_s)\big\|_\infty \leq \sum_{j=1,N} P_{ij}\Big(\big\|\mathbf{u}_j(t+t_s)\big\|_\infty + \big\|\nu_j(t+t_s)\big\|_\infty\Big) + \big\|\zeta_i^0(t+t_s)\big\|_\infty$$

According to the control law defined in (27), an upper bound for $\|\mathbf{u}_j(t+t_s)\|$ is given by

$$\big\|\mathbf{u}_j(t+t_s)\big\|_\infty \leq \big\|\mathbf{u}_j'(t+t_s)\big\|_\infty + k_j$$

The first term on the right-hand side corresponds to a bound on the evolution of (28) after the onset time of the sliding mode. Since parameters in $\mathbf{\Gamma}_j$ have been chosen so as to have stable tracking dynamics, states will evolve on the hyperplane Σ_j asymptotically converging to the origin. A bound $U_j \geq \|\mathbf{u}_j'(t+t_s)\|$ can be therefore established in terms of the values of states at time t_s, which in turn are bounded due to finiteness of t_s.

From (23) we also have

$$\begin{aligned} \big\|\zeta_i^0(t+t_s)\big\|_\infty &= \left\|\hat{\mathbf{g}}_i * \left(\sum_{j=1,N} \mathbf{C}_{ij} \exp\left(\mathbf{A}_{ij}(t+t_s)\right)\mathbf{x}_{ij}^0\right)\right\|_\infty \\ &= \left\|\sum_{j=1,N} \tilde{\mathbf{C}}_{ij} \exp\left(\tilde{\mathbf{A}}_{ij}(t+t_s)\right)\tilde{\mathbf{B}}_{ij}\mathbf{x}_{ij}^0\right\|_\infty \end{aligned}$$

where $(\tilde{\mathbf{A}}_{ij}, \tilde{\mathbf{B}}_{ij}, \tilde{\mathbf{C}}_{ij})$ are minimal realizations of the causal part of $\mathbf{G}_i^{-1}(s)\mathbf{C}_{ij}(s\mathbf{I} - \mathbf{A}_{ij})^{-1}$. Hence, in the hypotheses of Lemma 2 and for any $\mathbf{x}_{ij}^0$ (18), an upper bound for $\|\zeta_i^0(t+t_s)\|$ is given by

$$Z_i^0 = \sum_{j=1,N} \big\|\tilde{\mathbf{C}}_{ij}\big\|_\infty \alpha(\tilde{\mathbf{A}}_{ij})\big\|\tilde{\mathbf{B}}_{ij}\big\|_\infty \rho_{ij} \tag{30}$$

Recapitulating, bounds on the peak norm of the vector of equivalent input disturbances $\zeta_i(\cdot)$ after time t_s are provided as

$$\left\|\zeta_i(t+t_s)\right\|_\infty \leq \sum_{j=1,N} P_{ij}(U_j + k_j + N_j) + Z_i^0 \tag{31}$$

Using vector notation

$$\mathbf{z} = \left[\left\|\zeta_1(t+t_s)\right\|_\infty, \dots, \left\|\zeta_N(t+t_s)\right\|_\infty\right]^T, \quad \mathbf{u} = [U_1, \dots, U_N]^T$$

$$\mathbf{k} = [k_1, \dots, k_N]^T, \quad \mathbf{n} = [N_1, \dots, N_N]^T, \quad \mathbf{z}^0 = \left[Z_1^0, \dots, Z_N^0\right]^T$$

inequalities (31) are rewritten as

$$\mathbf{z} \leq \mathbf{P}(\mathbf{k} + \mathbf{u} + \mathbf{n}) + \mathbf{z}^0 \tag{32}$$

(inequality signs in vectorial relations are to be interpreted elementwise). Accordingly, condition (29) is verified provided that

$$\mathbf{k} > \mathbf{Pk} + (\mathbf{I} + \mathbf{P})\mathbf{n} + \mathbf{Pu} + \mathbf{z}^0 \geq \mathbf{n} + \mathbf{z} \tag{33}$$

Introducing $\boldsymbol{\epsilon} = [\epsilon_1, \dots, \epsilon_N]^T$, the VSC law (14) with $\mathbf{k} = \mathbf{n} + \boldsymbol{\epsilon}$ guarantees the existence of a sliding regime yielding performance $\mathcal{P}_\Gamma$ on $\mathbf{G}$, provided that

$$\boldsymbol{\epsilon} > \mathbf{P}\boldsymbol{\epsilon} + \mathbf{P}(\mathbf{u} + 2\mathbf{n}) + \mathbf{z}^0 \tag{34}$$

From the theory of positive matrices (Gantmacher, 1977), a nonnegative solution $\boldsymbol{\epsilon}$ to this equation exists for nonnegative $\mathbf{P}$, $\mathbf{n}$, $\mathbf{u}$, and $\mathbf{z}^0$, if and only if the Perron-Frobenius root of $\mathbf{P}$ is less than 1. ■

Remark 1. The proof of Theorem 1 directly offers a formula for the DVS controller parameter $\mathbf{k}$, i.e.

$$\mathbf{k} = \mathbf{n} + (\mathbf{I} - \mathbf{P})^{-1}(2\mathbf{Pn} + \mathbf{Pu} + \mathbf{z}^0 + \boldsymbol{\beta}), \quad \boldsymbol{\beta} > 0 \tag{35}$$

The set of all controller amplitude vectors $\mathbf{k}$ accomplishing decentralization is therefore the cone $\mathbf{C} \subset \mathbf{R}_+^N$ with vertex in $\mathbf{n} + (\mathbf{I} - \mathbf{P})^{-1}(2\mathbf{Pn} + \mathbf{Pu} + \mathbf{z}^0)$ and positively spanned by the columns of $(\mathbf{I} - \mathbf{P})^{-1}$.

Remark 2. Notice that condition (26) is related to the quasi-block diagonal dominance condition (4), in the limit case of interest dominance is sought for high gains $\mathbf{K}_{ii}$ that enforce arbitrary small tracking errors on minimum-phase nominal systems. In such a case, in fact, from (2) one gets

$$\lim_{\|\mathbf{K}\| \to \infty} \mathbf{Q}(s) = \hat{\mathbf{Q}}(s) = \mathbf{G}_D^{-1}(s)\mathbf{G}_C(s)$$

so conditions (4) and (26) can be rewritten as

$$\rho_{PF}\left(\left\{\left\|\hat{\mathbf{Q}}_{ij}(s)\right\|\right\}\right) < 1, \quad \forall s \in \mathcal{D} \tag{36}$$

$$\rho_{PF}\left(\left\{\left\|\hat{\mathbf{q}}_{ij}(t)\right\|_{\mathcal{A}}\right\}\right) < 1 \tag{37}$$

respectively, where $\hat{\mathbf{q}}_{ij}(t)$ denotes the impulse response matrix of $\hat{\mathbf{Q}}_{ij}(s)$.

Remark 3. Conditions equivalent to (26) can be obtained from the theory of non-negative and M-matrices as:

- there exists an induced norm $\|\cdot\|$ on $\mathbb{R}^{N\times N}$ such that $\|\mathbf{P}\| < 1$;
- $\mathbf{W} = \mathbf{I} - \mathbf{P}$ is an M-matrix,

Furthermore, easy-to-check sufficient conditions for (26) to be met are derived from Gershgorin's theorem as

$$\|\mathbf{P}\|_\infty < 1, \quad \|\mathbf{P}\|_1 < 1 \tag{38}$$

i.e. in terms of conventional row or column dominance. Note also that, according to the theory of generalized diagonal dominance (Araki and Nwokah, 1975), conditions in Theorem 1 guarantee the existence of an input-output scaling matrix $\mathbf{S}$ with positive elements such that $\mathbf{S}^{-1}\ \mathbf{PS}$ satisfies one of (38).

4. Synthesis of a Robust Decentralized VSC

The results of the previous section indicate general conditions for deciding whether a decentralized control can be attempted on a given plant model. In this section, we are interested in demonstrating how a practical synthesis of a VS controller can also be derived from the presented techniques. The first obstacle to the straightforward application of a controller of type (27) to a given plant is that being the controller based on a particular realization (19) of the nominal part of the plant, it is necessary to set up observers for systems (19) with suitable dynamics to reject input disturbances ζ_i. Nonlinear, variable structure observers have been proposed in the literature that can be applied in principle to this problem (Slotine *et al.*, 1987; Walkott and Zak, 1988). An alternative approach to this problem is developed in this section. Another concern in the practical design of a VS controller is to discuss the attractivity of the sliding manifolds in the large (Section 3 was only concerned with existence conditions for the sliding regime, i.e., with local stability of the sliding manifolds). Also this concern will be addressed in what follows.

Consider a MIMO system with nominal description given by the block-diagonal matrix $\mathbf{G}_D$ and structured interconnections $\mathbf{G}_C$. Assume that the matrix $\mathbf{G}_D = \text{diag}(\mathbf{G}_i)$ satisfies a stricter version of hypotheses **H1** of Lemma 2:

H1′ all $\mathbf{G}_i$ are minimum phase,

as well as hypotheses **H3** and the further hypothesis

H5 each row of all $\mathbf{G}_i(s)$ has relative degree one.

Note that hypotheses **H3** and **H5** are necessary and sufficient conditions for the product $\mathbf{C}_i\mathbf{B}_i$ to be nonsingular.

Assume also that the interconnections $\mathbf{G}_C$ satisfies with respect to nominal matrix $\mathbf{G}_D$ the sufficient condition for decentralized control (26) given in Theorem 1. Let the desired output trajectories to be followed be described by the outputs of systems $\mathcal{R}_i$ in (9). The proposed structure of the i-th controller is based on an auxiliary system $\mathcal{M}_i$ consisting of a column-wise controllable canonical-form realization of $\mathbf{G}_i$,

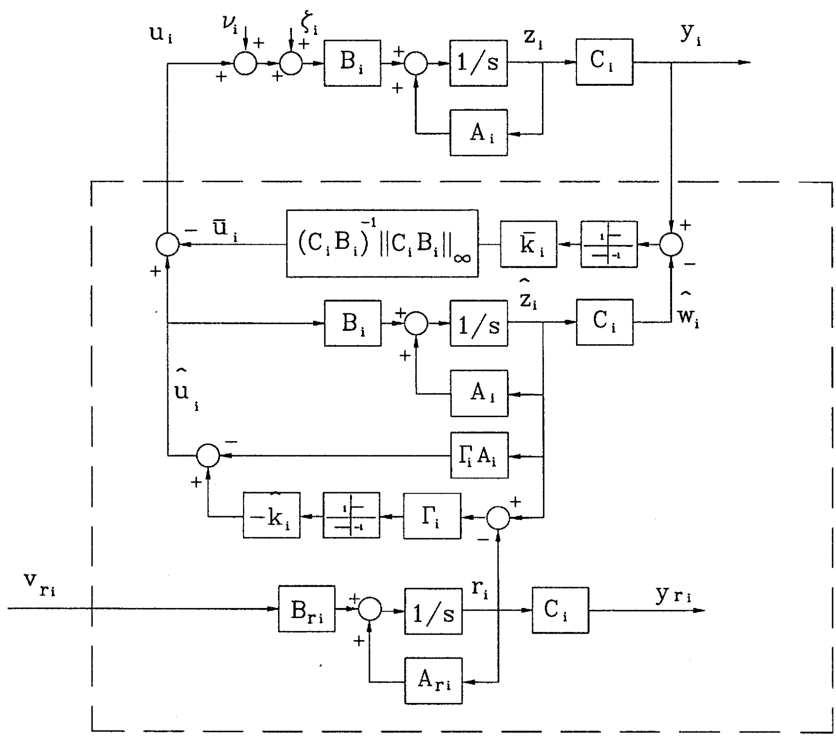

Fig. 1. The i-th channel of the proposed control scheme. The dashed box contains the controller. The outer part of the scheme represents the i-th row of the actual plant, with the input disturbance ζ_i replacing the effects on the output $\mathbf{y}_i$ of unmodelled dynamics $\mathbf{G}_{ii}$ and off-diagonal blocks $\mathbf{G}_{ij}$.

and is reported in detail in the dashed box of Fig. 1. Inputs, states, and outputs of the auxiliary system are $\hat{\mathbf{u}}_i$, $\hat{\mathbf{z}}_i$, and $\hat{\mathbf{w}}_i$, respectively, and its initial conditions are assumed to be zero,

$$\mathcal{M}_i : \begin{cases} \dot{\hat{\mathbf{z}}}_i &= \mathbf{A}_i\hat{\mathbf{z}}_i + \mathbf{B}_i\hat{\mathbf{u}}_i, \quad \hat{\mathbf{z}}_i(0) = \mathbf{0} \\ \hat{\mathbf{w}}_i &= \mathbf{C}_i\hat{\mathbf{z}}_i \end{cases} \tag{39}$$

The control inputs to the plant $\mathbf{u}_i$ consist of the sum of two signals, $\hat{\mathbf{u}}_i$ and $\bar{\mathbf{u}}_i$, synthesized by two switching controllers (see Fig. 1). Within a specified finite time t_0, control $\hat{\mathbf{u}}_i$ enforces a sliding regime that produces asymptotic tracking of the auxiliary system's outputs $\hat{\mathbf{w}}_i$ on the reference signals $\mathbf{y}_{ri}$, while control $\bar{\mathbf{u}}_i$ enforces a different sliding condition, which yields $\mathbf{y}_i \equiv \hat{\mathbf{w}}_i$, for all $t \geq t_0$.

The design of the control input $\hat{\mathbf{u}}_i$ is obtained by applying the VSC techniques of Section 2. Similarly to (11), introduce a linear manifold $\Sigma_i = \{\hat{\mathbf{z}}_i \in \mathbb{R}^{n_i} \mid \boldsymbol{\sigma}_i = \mathbf{0}\}$, where $\boldsymbol{\sigma}_i = \boldsymbol{\Gamma}_i(\hat{\mathbf{z}}_i - \mathbf{r}_i)$, and $\boldsymbol{\Gamma}_i \in \mathbb{R}^{m_i \times n_i}$ is such that $\boldsymbol{\Gamma}_i\mathbf{B}_i = \mathbf{I}_{m_i}$. The control law

$$\hat{\mathbf{u}}_i = -\boldsymbol{\Gamma}_i(\mathbf{A}_i\hat{\mathbf{z}}_i - \dot{\mathbf{r}}_i) - \hat{k}_i\,\mathrm{sign}(\boldsymbol{\sigma}_i) \tag{40}$$

with

$$\hat{k}_i = \left\|\mathbf{\Gamma}_i\right\|_\infty \frac{\left\|\hat{\mathbf{z}}_i(0) - \mathbf{r}_i(0)\right\|_\infty}{t_0} = \left\|\mathbf{\Gamma}_i\right\|_\infty \frac{\rho_{ri}}{t_0} \tag{41}$$

where ρ_{ri} is as in (10), guarantees that a sliding regime on Σ_i is maintained for all $t \geq t_0$. During such a sliding motion auxiliary states $\hat{\mathbf{z}}_i$ asymptotically track reference states $\mathbf{r}_i$, i.e. outputs $\hat{\mathbf{w}}_i$ converge to reference outputs $\mathbf{y}_{ri}$, with an error dynamics fixed by $\mathbf{\Gamma}_i$.

To design the second control input $\bar{\mathbf{u}}_i$, consider the system $\mathcal{Z}_i$ in (19), whose outputs $\mathbf{w}_i$ coincide with the i-th output channel $\mathbf{y}_i$ of the plant in the hypotheses of Lemma 2. The goal of $\bar{\mathbf{u}}_i$ is to counteract noise $\boldsymbol{\nu}_i$ and interconnection effects $\boldsymbol{\zeta}_i$, so as to have $\mathbf{w}_i(t)$ effectively tracking $\hat{\mathbf{w}}_i(t)$, hence $\mathbf{y}_{ri}(t)$. Denoting the state error between the plant $\mathcal{Z}_i$ and the auxiliary system $\mathcal{M}_i$ by $\mathbf{e}_i = \mathbf{z}_i - \hat{\mathbf{z}}_i$, we have the error dynamics

$$\dot{\mathbf{e}}_i = \mathbf{A}_i\mathbf{e}_i + \mathbf{B}_i(\bar{\mathbf{u}}_i + \boldsymbol{\nu}_i + \boldsymbol{\zeta}_i), \quad \mathbf{e}_i(0) = \mathbf{z}_i(0) = \mathbf{x}_i^0 \tag{42}$$

Since under assumptions **H3** and **H5** matrix $\mathbf{C}_i\mathbf{B}_i$ is invertible, let us consider the sliding manifold $S_i = \{\mathbf{e}_i \in \mathbb{R}^{n_i} \mid \varsigma_i = \mathbf{0}\}$ with

$$\varsigma_i = \mathbf{C}_i\mathbf{e}_i$$

The closed loop dynamics obtained by enforcing a sliding motion on the surface S_i have poles coincident with the transmission zeros of $\mathbf{G}_i(s)$. This can be easily verified by noting that

$$\det\left\{s\mathbf{I} - \left(\mathbf{A}_i - \mathbf{B}_i(\mathbf{C}_i\mathbf{B}_i)^{-1}\mathbf{C}_i\mathbf{A}_i\right)\right\} = s^{m_i} \det\left\{\mathbf{C}_i\mathbf{B}_i\right\} \det\left\{\begin{bmatrix} s\mathbf{I} - \mathbf{A}_i & -\mathbf{B}_i \\ \mathbf{C}_i & \mathbf{0} \end{bmatrix}\right\}$$

Hence, under assumptions **H1**$'$ these dynamics are stable. Futhermore, consider the control law

$$\bar{\mathbf{u}}_i(t) = -(\mathbf{C}_i\mathbf{B}_i)^{-1}\left\|\mathbf{C}_i\mathbf{B}_i\right\|_\infty \bar{k}_i \operatorname{sign}(\varsigma_i) \tag{43}$$

with

$$\bar{k}_i \geq \left\|(\mathbf{C}_i\mathbf{B}_i)\right\|_\infty^{-1} \left\|\mathbf{C}_i\mathbf{A}_i\mathbf{e}_i(t)\right\|_\infty + \left\|\boldsymbol{\nu}_i(t)\right\|_\infty + \left\|\boldsymbol{\zeta}_i(t)\right\|_\infty + \bar{\epsilon}_i \tag{44}$$

and

$$\bar{\epsilon}_i = \frac{\left\|\mathbf{C}_i\right\|_\infty}{\left\|\mathbf{C}_i\mathbf{B}_i\right\|_\infty} \frac{\left\|\mathbf{e}_i(0)\right\|_\infty}{t_0} = \frac{\left\|\mathbf{C}_i\right\|_\infty}{\left\|\mathbf{C}_i\mathbf{B}_i\right\|_\infty} \frac{\rho_i}{t_0} \tag{45}$$

with ρ_i as in (6). Assuming (44) to hold, all sliding surfaces S_i are reached within time t_0 as well, and in the ensuing sliding regime, the plant outputs equal the auxiliary system's, i.e. $\mathbf{y}_i(t) = \hat{\mathbf{w}}_i$ for all $t \geq t_0$.

Due to interconnections among nominal subsystems $\mathbf{G}_i$, the evolutions of $\mathbf{e}_i(t)$ and $\boldsymbol{\zeta}_i(t)$ on the right-hand side of inequality (44) depend on all $\hat{k}_j$, chosen as in (41), and on all $\bar{k}_j$ for $j = 1, \dots, N$. Then the problem of the synthesis of a decentralized VSC for the given plant amounts to finding parameters

$$\bar{\mathbf{k}} = \left[\bar{k}_1, \dots, \bar{k}_N\right]^T$$

which solve the set of the N inequalities (44) for $i = 1, \dots, N$.

The discussion of the right-hand side terms of (44) is subdivided in two successive time intervals, namely the 'reaching phase', $t \in [0, t_0]$, and the 'sliding regimes', $t \in (t_0, +\infty)$. Note that conditions (41) and (44) ensure that sliding motions along manifolds Σ_i and S_i, respectively, are established before time t_0. However, the instants at which the different manifolds are reached are not specified and depend on the initial conditions. To study inequality (44), an upper bound must be provided for the terms $\zeta_i(t)$ and $\mathbf{C}_i\mathbf{A}_i\mathbf{e}_i(t)$ both for $t \in [0, t_0]$ and for $t \in (t_0, +\infty)$.

The detailed computations for these two cases are reported in the Appendix. Results for the case $t \in [0, t_0]$ are summarized by the inequality

$$\bar{\mathbf{k}} \geq (\mathbf{P}+\mathbf{R}(\mathbf{I}+\mathbf{P}))\bar{\mathbf{k}}+(\mathbf{I}+\mathbf{R})(\mathbf{P}(\hat{\mathbf{u}}+\hat{\mathbf{k}})+\mathbf{z}^0)+(\mathbf{I}+\mathbf{R})(\mathbf{I}+\mathbf{P})\mathbf{n}+\mathbf{e}^0+\bar{\boldsymbol{\epsilon}} \tag{46}$$

where $\mathbf{R}$ can be made arbitrarily small (at the cost of larger control inputs) by a suitable choice of t_0. On the other hand, for $t \in (t_0, \infty)$, we have

$$\bar{\mathbf{k}} \geq \mathbf{P}\bar{\mathbf{k}} + \mathbf{P}(\hat{\mathbf{u}}' + \hat{\mathbf{k}}) + \mathbf{z}^0 + (\mathbf{I} + \mathbf{P})\mathbf{n} + \mathbf{e} \tag{47}$$

From the results of the previous section, therefore, since the condition (26) holds by assumption, positive solutions $\bar{\mathbf{k}}$ satisfying the above inequalities can be found.

In conclusion, a set of gains $\bar{\mathbf{k}}$ ensuring convergence to the sliding manifolds S_i and sliding motions along them is given by the intersection of the solutions of (46) and (47). To find a set of possible solutions note that any $\bar{\mathbf{k}}$ satisfying

$$\bar{\mathbf{k}} \geq \mathbf{P}\bar{\mathbf{k}} + \mathbf{P}(\hat{\mathbf{u}}' + \hat{\mathbf{k}}) + \mathbf{z}^0 + (\mathbf{I} + \mathbf{P})\mathbf{n} + \mathbf{e} + \mathbf{R}(\mathbf{I} + \mathbf{P})\bar{\mathbf{k}} \tag{48}$$

also satisfies inequality (46). Then any $\bar{\mathbf{k}}$ satisfying

$$\begin{aligned}\bar{\mathbf{k}} \;\geq\; & (\mathbf{P} + \mathbf{R}(\mathbf{I} + \mathbf{P}))\bar{\mathbf{k}} + (\mathbf{I} + \mathbf{R})\big(\mathbf{P}(\hat{\mathbf{u}} + \hat{\mathbf{k}}) + \mathbf{z}^0\big) \\ & + (\mathbf{I} + \mathbf{R})(\mathbf{I} + \mathbf{P})\mathbf{n} + \mathbf{e}^0 + \bar{\boldsymbol{\epsilon}} + \mathbf{P}\hat{\mathbf{u}}' + \mathbf{e}\end{aligned}$$

satisfies both (46) and (47). Hence, it follows that the time-domain dominance condition (26) is a sufficient condition for the synthesis of a decentralized VSC scheme such as that described in this section.

5. Simulation Results

In this section, we present simulation results to illustrate how various degrees of decentralization can be imposed on a controller, without compromising performance, at the expenses of higher control energy. A 4×4 system with two possible decompositions (four and two blocks resp.) is considered for that purpose.

It is to be noted that direct application of the DVSC control described in the preceding sections to plants would lead to the well-known phenomenon of 'chattering control', which is almost ubiquitous in sliding-mode control schemes. Chattering may be a serious disadvantage of variable structure controllers, because of the high activity imposed on the actuators and of possible excitation of unmodelled dynamics. Elimination of chattering has been widely studied in the VSC literature (Slotine and Sastry, 1983). One effective technique for chattering suppression is the so-called 'boundary layer' control, roughly consisting in replacing the switching part of the controls with a steep saturation function. Such a replacement basically affects the asymptotic stability of the resulting design, but still guarantees uniform ultimate boundedness of trajectories (bounds being arbitrarily reduced by increasing the saturation function gain), which is a satisfactory goal under any practical regard. In the simulations reported in this section, we apply a standard boundary layer technique to eliminate chattering from inputs to the plant. Degradation of expected tracking performance of our proposed control technique is negligible.

5.1. 4 × 4 System with 1 × 1 Nominal Blocks

Consider the plant described by the transfer function matrix

$$
\mathbf{G}(s) = \left[\begin{matrix}
\frac{(s+.4)}{(s-.05)(s+.2)} + \frac{.02(s+.3)}{(s+.5)(s+.7)} & \frac{.35(s+.5)}{(s+.2)(s+.4)} + \frac{.02(s+.1)}{(s+.9)(s+1.1)} \\
\frac{.35(s+.1)}{(s+.3)(s+.5)} + \frac{.02(s+.1)}{(s+.9)(s+1.1)} & \frac{(s+.2)}{(s-.025)(s+.7)} + \frac{.02(s+.6)}{(s+.4)(s+1)} \\
\frac{.02(s+.4)}{(s+.6)(s+.8)} & \frac{.02(s+.2)}{(s+.3)(s+.5)} \\
\frac{.02(s+.3)}{(s+.4)(s+1.2)} & \frac{.02(s+.4)}{(s+.6)(s+.8)}
\end{matrix}\right.
$$

$$
\cdots \left.\begin{matrix}
\frac{.02(s+.4)}{(s+.6)(s+.8)} & \frac{.02(s+.2)}{(s+.3)(s+.5)} \\
\frac{.02(s+.8)}{(s+.4)(s+1.2)} & \frac{.02(s+.4)}{(s+.6)(s+.8)} \\
\frac{(s+.1)}{(s-.025)(s+.2)} + \frac{.02(s+.2)}{(s+.5)(s+.7)} & \frac{.35(s+.2)}{(s+.4)(s+1.2)} + \frac{.02(s+.1)}{(s+.3)(s+1.1)} \\
\frac{.35(s+.4)}{(s+.3)(s+.5)} + \frac{.02(s+.1)}{(s+.3)(s+1.1)} & \frac{(s+.5)}{(s-.05)(s+.7)} + \frac{.02(s+.3)}{(s+.4)(s+.8)}
\end{matrix}\right] \tag{49}
$$

Assume input disturbances to be the sinusoidal signals

$$
\begin{aligned}
\nu_1 &= 5\sin(.07t), & \nu_2 &= 5\sin(.02t) \\
\nu_3 &= 5\sin(.10t), & \nu_4 &= 5\sin(.05t)
\end{aligned} \tag{50}
$$

Consider first a four-block decomposition $\mathbf{G} = \mathbf{G}_D + \mathbf{G}_C$, with

$$
\mathbf{G}_D = \operatorname{diag}\left(\frac{(s+.4)}{(s-.05)(s+.2)}, \frac{(s+.2)}{(s-.025)(s+.7)}, \frac{(s+.1)}{(s-.025)(s+.2)}, \frac{(s+.5)}{(s-.05)(s+.7)}\right)
$$

The corresponding matrix $\mathbf{P}$ is

$$\mathbf{P} = \begin{bmatrix} 0.0437 & 0.7953 & 0.0433 & 0.0437 \\ 0.8000 & 0.0427 & 0.0430 & 0.0416 \\ 0.0422 & 0.0418 & 0.0424 & 0.8064 \\ 0.0413 & 0.0413 & 0.8060 & 0.0416 \end{bmatrix}$$

Since $\rho_{PF}(\mathbf{P}) = 0.92916$, a decentralized VSC of type (40) and (43) can be applied. Let the output trajectories to be tracked be generated for each channel by filtering the sinusoidal inputs

$$\begin{aligned} v_{r1} &= 50\sin(.10t - 0.92), \qquad & v_{r2} &= 20\sin(.15t + 1.67) \\ v_{r3} &= 30\sin(.18t - 1.43), \qquad & v_{r4} &= 60\sin(.08t + 0.56) \end{aligned} \tag{51}$$

through a second-order filter with poles at -0.5 and -1 and initial conditions bounded by $\rho_{ri} = 10^{-3}$. Choose sliding manifolds Σ_i so as to obtain a tracking dynamics with pole at -2, i.e. $\mathbf{\Gamma}_i = [2,\ 1]$.

Assuming a reaching time t_0 equal to 6×10^{-5}, by means of (41), we set $\hat{\mathbf{k}} = [33.3,\ 33.3,\ 33.3,\ 33.3]^T$.

Let the initial conditions of systems $\mathbf{G}_D$ and $\mathbf{G}_C$ be bounded by $\rho_1 = 2.5 \times 10^{-3}, \rho_2 = \rho_3 = \rho_4 = 5 \times 10^{-3}$, and $\rho_{ij} = 10^{-3}$ for $i, j = 1, 4$.

By (A10) one gets $\mathbf{R} = \mathrm{diag}[1.50,\ 2.85,\ 0.45,\ 0.90] \times 10^{-5}$, and, since $\rho_{PF}(\mathbf{P} + \mathbf{R}(\mathbf{I} + \mathbf{P})) = 0.92919$, positive solutions $\bar{\mathbf{k}}$ for eqn. (46) exist. By (30) and (50) $\mathbf{z}^0 = [10^{-5},\ 1.50,\ 2 \times 10^{-3},\ 2.15 \times 10^{-3}]^T$ and $\mathbf{n} = [5,\ 5,\ 5,\ 5]^T$. Applying (A2), $\hat{\rho}_i(t_0) = 4 \times 10^{-3}$, for $i = 1, \ldots, 4$; and, according to (A7) and (A9), $\hat{\mathbf{u}} = [50,\ 20,\ 30,\ 60]^T$ and $\mathbf{e}^0 = [1.25,\ 4.75,\ 0.75,\ 1.50]^T \times 10^{-3}$.

Further, by (45), the reaching of the sliding manifolds S_i within time t_0 is guaranteed if $\bar{\epsilon} = [41.6,\ 83.3,\ 83.3,\ 83.3]^T$. The solutions $\bar{\mathbf{k}}$ to (46) taken with the equality sign evaluate to

$$\bar{\mathbf{k}} = \left[2038,\ 2077,\ 2225,\ 2209\right]^T$$

It is easy to verify that such a value for $\bar{\mathbf{k}}$ also satisfies (47). Indeed, by (B2), $\hat{\mathbf{u}}' = [107, 40.7, 64.1, 124]^T$, by (A8) $\bar{\rho}_1(t_0) = 0.250$, $\bar{\rho}_2(t_0) = 0.255$, $\bar{\rho}_3(t_0) = 0.273$, $\bar{\rho}_4(t_0) = 0.273$, and, by (B3), $\mathbf{e} = [0.087,\ 0.145,\ 0.022,\ 0.061]^T$. The outputs of the plant and of the auxiliary systems are compared with the desired trajectories in Fig. 2. Input signals for the four channels are reported in Fig. 3.

5.2. 4 × 4 System with 2 × 2 Nominal Blocks

Finally, consider 2×2 decomposition of the matrix (49) obtained by choosing

$$\mathbf{G}_D = \mathrm{diag}\left(\begin{bmatrix} \frac{(s+.4)}{(s-.05)(s+.2)} & \frac{.35(s+.5)}{(s+.2)(s+.4)} \\ \frac{.35(s+.1)}{(s+.3)(s+.5)} & \frac{(s+.2)}{(s-.025)(s+.7)} \end{bmatrix}, \begin{bmatrix} \frac{(s+.1)}{(s-.025)(s+.2)} & \frac{.35(s+.2)}{(s+.4)(s+1.2)} \\ \frac{.35(s+.4)}{(s+.3)(s+.5)} & \frac{(s+.5)}{(s-.05)(s+.7)} \end{bmatrix}\right)$$

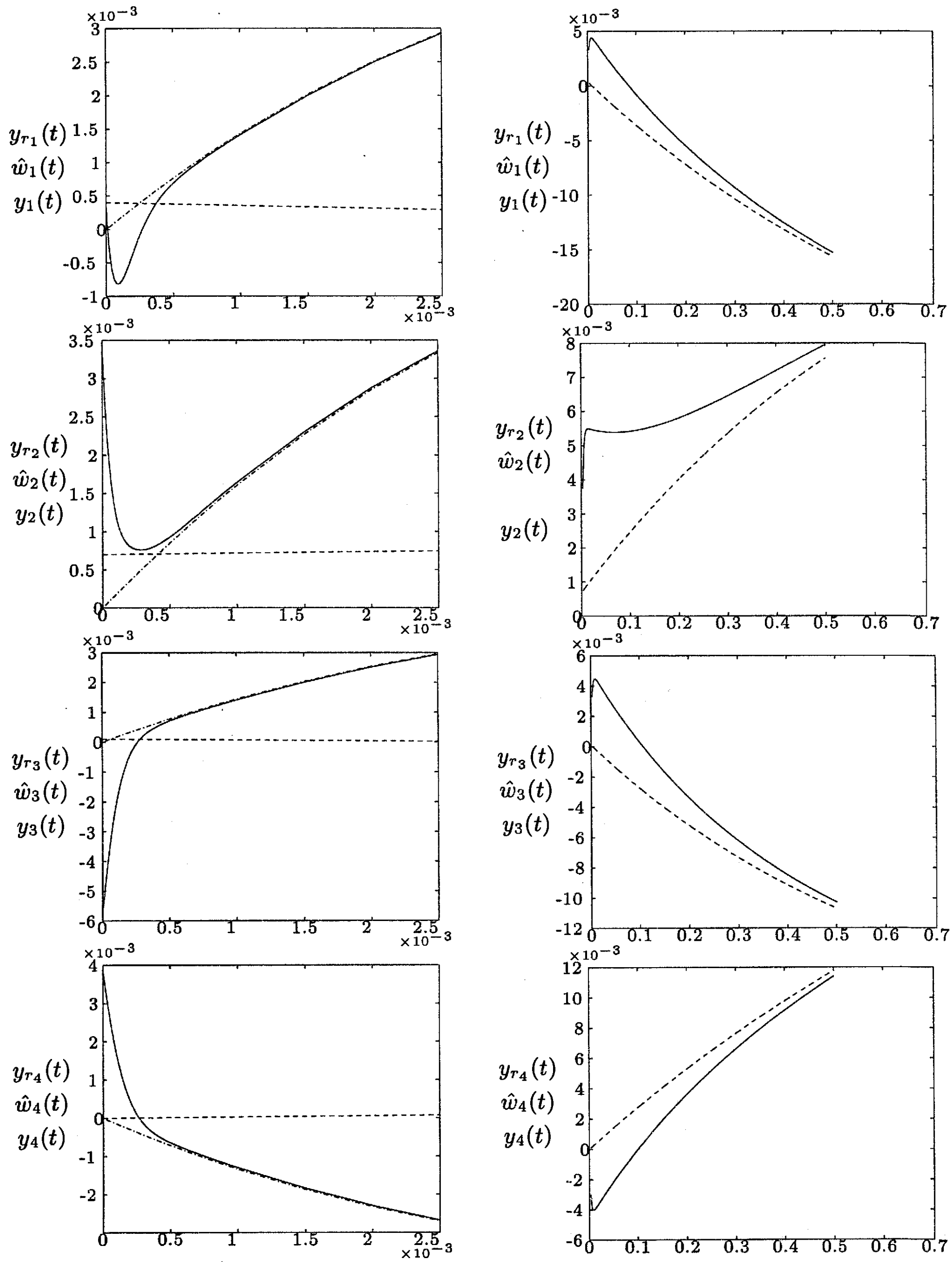

Fig. 2. Simulations for the control based on a 4-by-4 decomposition described in Section 5.1. Desired trajectories (dashed), auxiliary system outputs (dash-dot) and plant outputs (solid) during the reaching phase, $t \in [0, t_0]$ (left), and during sliding motion, $t \in (t_0, \infty)$ (right).

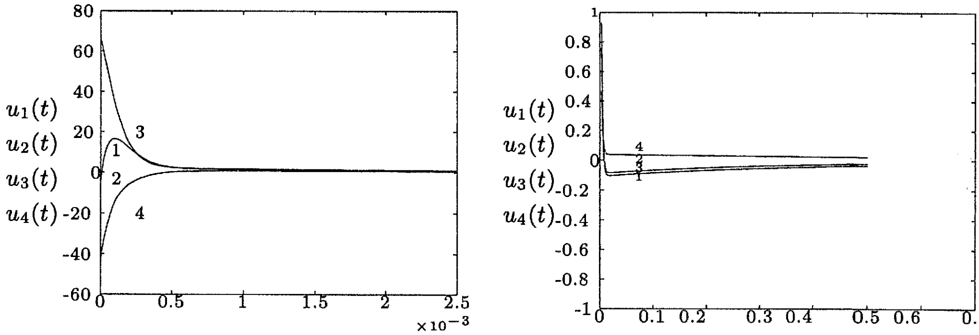

Fig. 3. Control signals ($u_1(t)$ solid, $u_2(t)$ dashed, $u_3(t)$ dash-dot and $u_4(t)$ dotted) during the reaching phase (on the left) and during sliding motion (on the right) for the case of Section 5.1.

Reference trajectories are generated by filtering the signals (51) through a 4-th order filter with poles at $-.5, -.75, -1$ and -1.5 and initial conditions bounded by $\rho_{ri} = 10^{-4}$. In this case, the coupling among the nominal subsystems in $\mathbf{G}_D$ is much weaker than in the previous one. In fact,

$$\mathbf{P} = \begin{bmatrix} 0.0778 & 0.0704 \\ 0.0654 & 0.0643 \end{bmatrix}$$

and $\rho_{PF}(\mathbf{P}) = 0.1393$ is lower. Each row vector of the 2×2 block-diagonal matrix $\mathbf{\Gamma}_i$ is chosen so as to provide tracking dynamics during the motion on the sliding manifolds Σ_i with poles at $-2, -2.2$ and -3. Assuming the constraint on the reaching time as in the previous case, namely $t_0 = 6\times 10^{-5}$, amplitudes $\hat{\mathbf{k}}$ have been set as $\hat{\mathbf{k}} = [64,\ 64]^T$. Matrix $\mathbf{R}$ evaluates to $\mathbf{R} = \operatorname{diag}[4.94,\ 5.38\,]\times 10^{-5}$ and $\rho_{PF}(\mathbf{P}+\mathbf{R}(\mathbf{I}+\mathbf{P})) = 0.1393$.

Let the initial conditions of systems $\mathbf{G}_D$ and $\mathbf{G}_C$ be bounded by $\rho_1 = \rho_2 = 10^{-3}$ and $\rho_{11} = \rho_{12} = \rho_{22} = 10^{-3}$, $\rho_{21} = 7\times 10^{-4}$. By the same procedure one gets $\mathbf{z}^0 = [0.69,\ 3.22]^T$, $\mathbf{n} = [5,\ 5]^T$, and since $\hat{\rho}_1(t_0) = \hat{\rho}_2(t_0) = 4.6\times 10^{-3}$, according to (A7) and (A9), $\hat{\mathbf{u}} = [274,\ 283]^T$ and $\mathbf{e}^0 = [6.6,\ 7.2]^T \times 10^{-3}$. Setting $\bar{\epsilon} = [44.5,\ 64.1]^T$, in order to ensure reaching the manifolds S_i in the specified time t_0, the solution to (46) with the equality sign is

$$\bar{\mathbf{k}} = [120,\ 134]^T$$

Furthermore, since $\hat{\mathbf{u}}' = [538,\ 559]^T$, $\bar{\rho}_1(t_0) = 2.28\times 10^{-2}$, $\bar{\rho}_2(t_0) = 2.36\times 10^{-2}$ and $\mathbf{e} = [1.38,\ 3.11]^T$, one can verify that (47) is also satisfied.

Note that the weaker interactions among interconnected subsystems allow the use of much lower control efforts. The outputs of the plant and those of the auxiliary systems are compared with the desired trajectories in Fig. 4 while the control inputs are reported in Fig. 5. The behaviour of the controlled output system is comparable with that obtained with the stricter, 1×1 decentralization scheme.

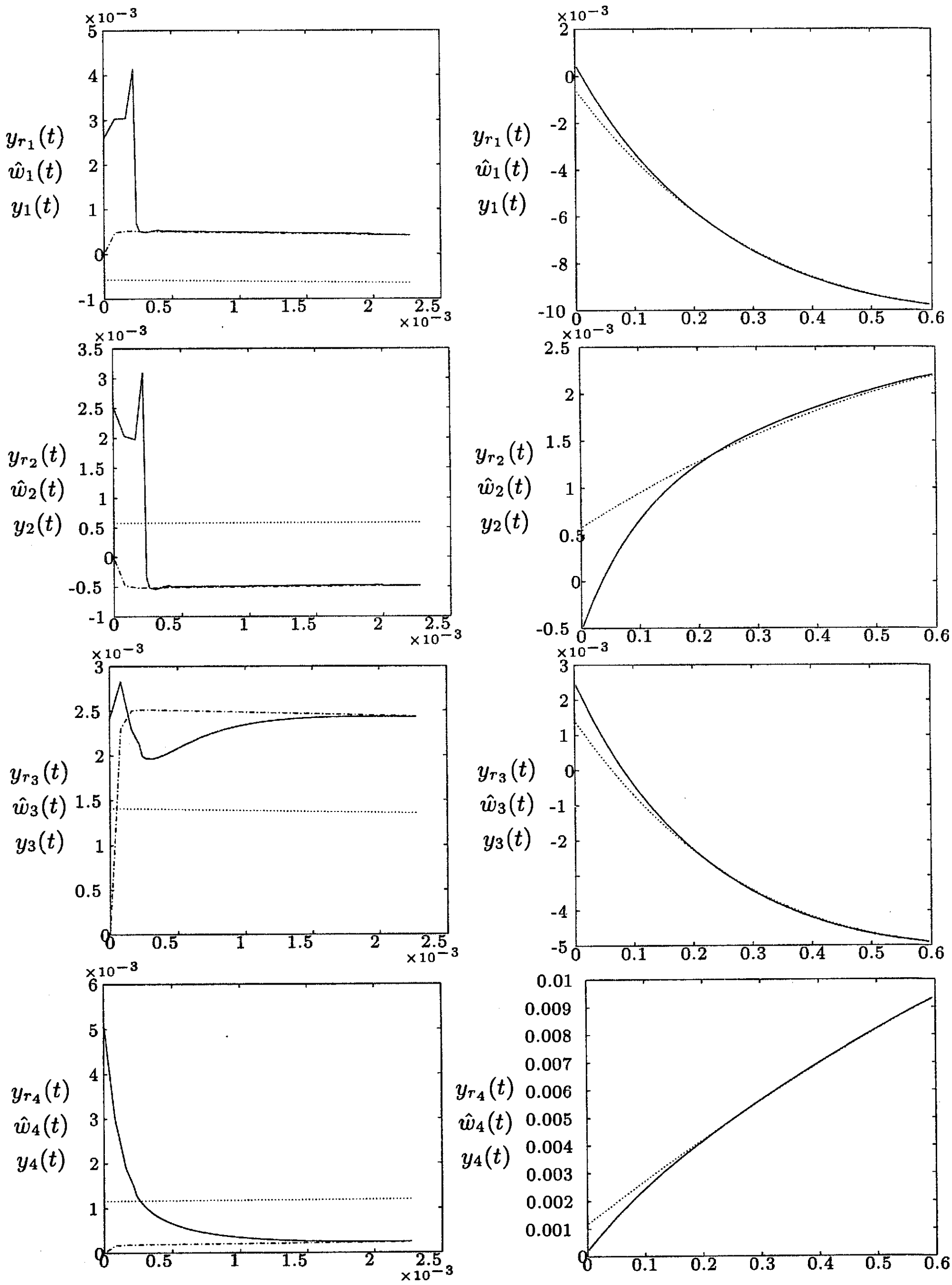

Fig. 4. Simulations for the control based on a 2-by-2 decomposition described in Section 5.2. Desired trajectories (dashed), auxiliary system outputs (dash-dot) and plant outputs (solid) during the reaching phase, $t \in [0, t_0]$ (left), and during sliding motion, $t \in (t_0, \infty)$ (right).

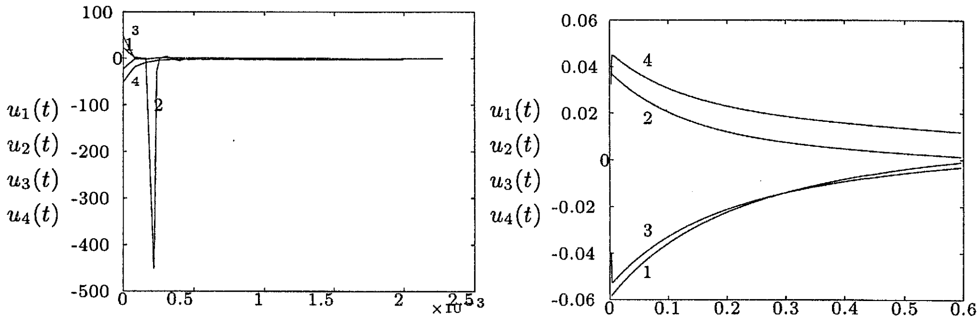

Fig. 5. Control signals ($u_1(t)$ solid, $u_2(t)$ dashed, $u_3(t)$ dash-dot and $u_4(t)$ dotted) during the reaching phase (on the left) and during sliding motion (on the right) for the control described in Section 5.2.

6. Conclusions

We considered under what conditions a variable structure controller designed for asymptotic output tracking on a set of nominal, decoupled MIMO subsystems, retains its performance when applied to a real plant with modelling errors and interactions among subsystems. A sufficient condition to obtain such a property has been derived, which was shown to be a time-domain analogous to well-known frequency-domain dominance conditions employed in classical decentralized stabilization theory *à la* Rosenbrock. As a final remark, we stress that, although we made use of the theory of sliding modes and of tools of variable structure control design, the time-domain dominance condition of Theorem 1 turns out to be independent of this choice, as well as of the specifications of tracking performance. Further investigations will be devoted to understand to what extent the results found in this paper can be extended to different styles of controller design.

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Appendices

A. Case $t \in [0, t_0]$

By (24), under controls (40),(43) $\zeta_i(t)$ is bounded for $t \in [0, t_0]$ as follows:

$$\sup_{\tau \in [0,t_0]} \|\zeta_i(\tau)\|_\infty \leq \sum_{j=1,N} P_{ij}(\hat{U}_j + \hat{k}_j + \bar{k}_j + N_j) + Z_i^0 \tag{A1}$$

where P_{ij} is defined in (25), $\hat{U}_j$ stand for upper bounds on terms $\|\mathbf{\Gamma}_j(\mathbf{A}_j\hat{\mathbf{z}}_j(\tau) - \dot{\mathbf{r}}_j(\tau))\|$ for $\tau \in [0, t_0]$, N_j is as in (8) and Z_i^0 is given by (30).

Provided that $\hat{k}_j$ is chosen according to (41), at some time $\hat{t}_j \in [0, t_0]$ manifold Σ_j is reached and a sliding motion satisfying $\sigma_j = \mathbf{\Gamma}_i(\hat{\mathbf{z}}_j - \mathbf{r}_j) = 0$, is established for $t \geq \hat{t}_j$. At $t = \hat{t}_j$ the distance $\|\hat{\mathbf{z}}_j(\hat{t}_j) - \mathbf{r}_j(\hat{t}_j)\|$ between the states of the auxiliary and reference system is bounded by

$$\hat{\rho}_j(\hat{t}_j) = \alpha_{(\hat{t}_j)}\big((\mathbf{I}-\mathbf{B}_j\mathbf{\Gamma}_j)\mathbf{A}_j\big)\rho_{rj} + \big\|\exp\big((\mathbf{I}-\mathbf{B}_j\mathbf{\Gamma}_j)\mathbf{A}_j t\big)\mathbf{B}_j h(\hat{t}_j - t)\big\|_{\mathcal{A}}\hat{k}_j \quad \text{(A2)}$$

where $\alpha_{(\tau)}(\mathbf{M}) = \sup_{0\leq t\leq \tau}\|\exp(\mathbf{M}t)\|_\infty$, ρ_{rj} is as in (10), and $h(t)$ is the Heaviside function. Efficient techniques for providing such bounds of matrix exponentials can be found e.g. in (Kagstrom, 1977; Van Loan, 1977). Consider the k-th channel of the j-th block of the nominal system (see (7)). Introduce the transformed state variables $\hat{\eta}_j^{(k)} = \hat{\mathbf{T}}_j^{(k)}(\hat{\mathbf{z}}_j^{(k)} - \mathbf{r}_j^{(k)})$, with $\hat{\mathbf{T}}_j^{(k)} \in \mathbb{R}^{n_j^{(k)} \times n_j^{(k)}}$ given by

$$\hat{\mathbf{T}}_j^{(k)} = \begin{bmatrix} \mathbf{I}_{n_j^{(k)}-1} & \mathbf{0} \\ \multicolumn{2}{c}{\mathbf{\Gamma}_j^{(k)}} \end{bmatrix} \quad \text{(A3)}$$

Partitioning the transformed state $\hat{\eta}_j^{(k)}$ as $\hat{\eta}_j^{(k)} = \begin{bmatrix} \hat{\eta}_j^{(1,k)} \\ \hat{\eta}_j^{(2,k)} \end{bmatrix}$ with $\hat{\eta}_j^{(1,k)} \in \mathbb{R}^{n_j^{(k)}-1}$, $\hat{\eta}_j^{(2,k)} \in \mathbb{R}$, the sliding regime condition $\sigma_j = \mathbf{\Gamma}_i(\hat{\mathbf{z}}_j - \mathbf{r}_j) = 0$, is rewritten as

$$\hat{\eta}_j^{(2,k)} = 0 \quad \text{for } k = 1, \ldots, m_j$$

while the sliding mode evolution in the $(n_j - m_j)$ reduced state space is described by

$$\hat{\eta}_j^{(1,k)}(t) = \exp\Big(\hat{\mathbf{A}}_j(t - \hat{t}_j)\Big)\hat{\mathbf{T}}_j\Big(\mathbf{z}_j(\hat{t}_j) - \mathbf{r}_j(\hat{t}_j)\Big) \quad \text{for } k = 1, \ldots, m_j \quad \text{(A4)}$$

where $\hat{\mathbf{A}}_j = \text{diag}(\hat{\mathbf{A}}_j^{(1)}, \ldots, \hat{\mathbf{A}}_j^{(m_j)})$ with $\hat{\mathbf{A}}_j^{(k)}$ the upper-left $(n_j - 1) \times (n_j - 1)$-block in $\hat{\mathbf{T}}_j^{(k)}\mathbf{A}_j^{(k)}(\hat{\mathbf{T}}_j^{(k)})^{-1}$, and $\hat{\mathbf{T}}_j = \text{diag}(\hat{\mathbf{T}}_j^{(1)}, \ldots, \hat{\mathbf{T}}_j^{(m_j)})$.

By means of (9),

$$\begin{aligned} \mathbf{\Gamma}_j\big(\mathbf{A}_j\hat{\mathbf{z}}_j(t) - \dot{\mathbf{r}}_j(t)\big) &= \mathbf{\Gamma}_j\mathbf{A}_j\hat{\mathbf{T}}_j^{-1}\hat{\eta}_j + \mathbf{\Gamma}_j(\mathbf{A}_j - \mathbf{A}_{rj})\exp(\mathbf{A}_{rj}t)\mathbf{r}_i^0 \\ &+ \big(\mathbf{\Gamma}_j(\mathbf{A}_j - \mathbf{A}_{rj})\exp(\mathbf{A}_{rj}t)\mathbf{B}_{rj} - \delta(t)\mathbf{I}\big) * \mathbf{v}_{rj} \quad \text{(A5)} \end{aligned}$$

with $\hat{\eta}_j = [\hat{\eta}_j^{(1)}, \ldots, \hat{\eta}_j^{(m_j)}]^T$. Since $\hat{\mathbf{A}}_j$ is Hurwitz (this results from the choice of stable tracking dynamics in $\mathbf{\Gamma}_j$), from (A5) and (A4)

$$\begin{aligned} \sup_{\tau\in[0,t_0]} \|\mathbf{\Gamma}_j\Big(\mathbf{A}_j\hat{\mathbf{z}}_j(\tau) - \dot{\mathbf{r}}_j(\tau)\Big)\|_\infty \leq\ & \|\mathbf{\Gamma}_j\mathbf{A}_j\hat{\mathbf{T}}_j^{-1}\|_\infty \alpha_{(\hat{t}_j)}(\hat{\mathbf{A}}_j)\|\hat{\mathbf{T}}_j\|_\infty \hat{\rho}_j(\hat{t}_j) \\ & + \big\|\mathbf{\Gamma}_j(\mathbf{A}_j - \mathbf{A}_{rj})\big\|_\infty \alpha_{(\hat{t}_j)}(\mathbf{A}_{rj})\rho_{rj} \\ & + \big\|\big(\mathbf{\Gamma}_j(\mathbf{A}_j - \mathbf{A}_{rj})\ \exp\ (\mathbf{A}_{rj}t)\mathbf{B}_{rj} - \delta(t)\mathbf{I}\big) \\ & \times h(\hat{t}_j - t)\big\|_{\mathcal{A}} V_j \quad \text{(A6)} \end{aligned}$$

Further, since $\|\hat{\mathbf{z}}_j(\hat{t}_j)-\mathbf{r}_j(\hat{t}_j)\| \leq \|\hat{\mathbf{z}}_j(t_0)-\mathbf{r}_j(t_0)\| \leq \hat{\rho}_j(t_0)$ with $\hat{\rho}_j(t_0)$ given by (A2), and for any $\tau_1 > \tau_2$ it holds $\alpha_{(\tau_1)}(\mathbf{M}) \geq \alpha_{(\tau_2)}(\mathbf{M})$, upper bounds $\hat{U}_j$ in (A1) can be obtained from (A6) as follows:

$$\begin{aligned}\hat{U}_j \;=\; & \left\|\mathbf{\Gamma}_j\mathbf{A}_j\hat{\mathbf{T}}_j^{-1}\right\|_\infty\alpha_{(t_0)}(\hat{\mathbf{A}}_j)\left\|\hat{\mathbf{T}}_j\right\|_\infty\hat{\rho}_j(t_0) + \left\|\mathbf{\Gamma}_j(\mathbf{A}_j-\mathbf{A}_{rj})\right\|_\infty\alpha_{(t_0)}(\mathbf{A}_{rj})\rho_{rj} \\ & + \left\|\left(\mathbf{\Gamma}_j(\mathbf{A}_j-\mathbf{A}_{rj})\exp\left(\mathbf{A}_{rj}t\right)\mathbf{B}_{rj} - \delta(t)\mathbf{I}\right)h(t_0-t)\right\|_{\mathcal{A}}V_j \qquad \text{(A7)}\end{aligned}$$

with $\hat{\mathbf{T}}_j$ as in (A3) and V_j as in (10).

By similar arguments, the evolution of $\mathbf{e}_i(t)$ for $t \in [0, t_0]$, according to (42) under control (43), is bounded as follows:

$$\begin{aligned}\sup_{\tau\in[0,t_0]}\left\|\mathbf{e}_i(\tau)\right\|_\infty \leq \bar{\rho}_i(t_0) \;&=\; \alpha_{(t_0)}(\mathbf{A}_i t_0)\rho_i + \left\|\exp(\mathbf{A}_i\tau)\mathbf{B}_i h(t_0-\tau)\right\|_{\mathcal{A}} \\ &=\; \times\left(\bar{k}_i + N_i + \left\|\zeta_i(\tau)h(t_0-\tau)\right\|_\infty\right) \qquad \text{(A8)}\end{aligned}$$

with ρ_i as in (6). Hence, introducing

$$E_i^0 \;=\; \left\|\mathbf{C}_i\mathbf{B}_i\right\|_\infty^{-1}\left\|\mathbf{C}_i\mathbf{A}_i\right\|_\infty \alpha_{(t_0)}(\mathbf{A}_i)\rho_i \qquad \text{(A9)}$$

$$R_i \;=\; \left\|\mathbf{C}_i\mathbf{B}_i\right\|_\infty^{-1}\left\|\mathbf{C}_i\mathbf{A}_i\right\|_\infty\left\|\exp(\mathbf{A}_i t)\mathbf{B}_i h(t_0-t)\right\|_{\mathcal{A}} \qquad \text{(A10)}$$

from (A8) and (A1), we get

$$\begin{aligned}&\sup_{\tau\in[0,t_0]}\left\|\mathbf{C}_i\mathbf{B}_i\right\|_\infty^{-1}\left\|\mathbf{C}_i\mathbf{A}_i\mathbf{e}_i(\tau)\right\|_\infty \\ &\qquad \leq E_i^0 + R_i\Big(\bar{k}_i + N_i + \sum_{j=1,N} P_{ij}(\hat{U}_j + \hat{k}_j + \bar{k}_j + N_j) + Z_i^0\Big) \qquad \text{(A11)}\end{aligned}$$

Note that, since t_0 is a design parameter, terms E_i^0 and R_i can be made arbitrarily small at the expenses of the control effort.

Let $\mathbf{R}$ be $\mathrm{diag}(R_i)$ and let $\hat{\mathbf{k}}$, $\mathbf{z}^0$ $\mathbf{e}^0$, $\hat{\mathbf{u}}$ and $\bar{\epsilon}$ stand for the N-dimensional vectors collecting terms $\hat{k}_i$ (as in (41)), Z_i^0 (as in (30)), E_i^0 (as in (A9)), $\hat{U}_i$ (as in (A7)) and $\bar{\epsilon}_i$ (as in (45)), respectively. By (A11) and (A1), attractivity conditions (44) for $t \in [0, t_0]$ are met by any $\bar{\mathbf{k}}$ solving

$$\bar{\mathbf{k}} \geq \left(\mathbf{P} + \mathbf{R}(\mathbf{I}+\mathbf{P})\right)\bar{\mathbf{k}} + (\mathbf{I}+\mathbf{R})\left(\mathbf{P}(\hat{\mathbf{u}}+\hat{\mathbf{k}}) + \mathbf{z}^0\right) + (\mathbf{I}+\mathbf{R})(\mathbf{I}+\mathbf{P})\mathbf{n} + \mathbf{e}^0 + \bar{\epsilon}$$

Hence, since by hypothesis matrix $\mathbf{P}$ satisfies the dominance condition (26) and matrix $\mathbf{R}$ can be made arbitrarily small by a suitable choice of t_0, it is always possible to obtain $\rho_{PF}(\mathbf{P} + \mathbf{R}(\mathbf{P}+\mathbf{I})) < 1$ so that the inequality can be solvable for any positive $\bar{\mathbf{k}}$.

B. Case $t \in (t_0, \infty)$

Also in this case, an upper bound for $\zeta_i(t)$ for $t > t_0$ can be given as

$$\sup_{\tau \in (t_0,\infty)} \left\|\zeta_i(\tau)\right\|_\infty = \left\|\zeta_i(t+t_0)\right\|_\infty \le \sum_{j=1,N} P_{ij}(\hat{U}'_j + \hat{k}_j + \bar{k}_j + N_j) + Z_i^0 \tag{B1}$$

where N_j is as in (8), Z_i^0 is as in (30), and upper bounds $\hat{U}'_j$ on terms $\|\mathbf{\Gamma}_j(\mathbf{A}_j\hat{\mathbf{z}}_j(t+t_0) - \dot{\mathbf{r}}_j(t+t_0))\|$ are obtained as in (A7) with $\alpha(\cdot)$ evaluated on the infinite horizon, i.e.

$$\begin{aligned} \hat{U}'_j &= \left\|\mathbf{\Gamma}_j\mathbf{A}_j\hat{\mathbf{T}}_j^{-1}\right\|_\infty \alpha(\hat{\mathbf{A}}_j)\left\|\hat{\mathbf{T}}_j\right\|_\infty \hat{\rho}_j(t_0) + \left\|\mathbf{\Gamma}_j(\mathbf{A}_j - \mathbf{A}_{rj})\right\|_\infty \alpha(\mathbf{A}_{rj})\rho_{rj} \\ &\quad + \left\|\mathbf{\Gamma}_j(\mathbf{A}_j - \mathbf{A}_{rj}) \exp(\mathbf{A}_{rj}t)\mathbf{B}_{rj} - \delta(t)\mathbf{I}\right\|_{\mathcal{A}} V_j \end{aligned} \tag{B2}$$

Furthermore, introducing the state variables

$$\bar{\eta}_i^{(k)} = \begin{bmatrix} \bar{\eta}_i^{(1,k)} \\ \bar{\eta}_i^{(2,k)} \end{bmatrix} = \bar{\mathbf{T}}_i^{(k)}\mathbf{e}_i$$

with $\bar{\eta}_i^{(1,k)} \in \mathbb{R}^{n_i^{(k)}-1}$, $\bar{\eta}_i^{(2,k)} \in \mathbb{R}$, where matrix $\bar{\mathbf{T}}_i^{(k)}$ is defined similarly to (A3):

$$\bar{\mathbf{T}}_i^{(k)} = \begin{bmatrix} \mathbf{I}_{n_j-1} \; \mathbf{0} \\ \mathbf{C}_i^{(k)} \end{bmatrix}$$

the motion along the sliding surface S_i can be described by

$$\bar{\eta}_i^{(1)}(t - t_0) = \exp(\bar{\mathbf{A}}_i(t - t_0))\bar{\eta}_i^{(1)}(t_0)$$

with $\bar{\mathbf{T}}_i\mathbf{A}_i\bar{\mathbf{T}}_i^{-1} = \begin{bmatrix} \bar{\mathbf{A}}_i & \bar{\mathbf{A}}_i^{12} \\ \bar{\mathbf{A}}_i^{21} & \bar{\mathbf{A}}_i^{22} \end{bmatrix}$ and $\bar{\mathbf{T}}_i = \mathrm{diag}(\bar{\mathbf{T}}_i^{(1)}, \ldots, \bar{\mathbf{T}}_i^{(m_i)})$.

Under hypothesis **H1′**, this sliding motion is stable. Therefore

$$\begin{aligned} &\sup_{\tau \in (t_0,\infty)} \left\|\mathbf{C}_i\mathbf{B}_i\right\|_\infty^{-1} \left\|\mathbf{C}_i\mathbf{A}_i\mathbf{e}_i(\tau)\right\|_\infty \\ &\qquad \le E_i = \left\|\mathbf{C}_i\mathbf{B}_i\right\|_\infty^{-1} \left\|\mathbf{C}_i\mathbf{A}_i\bar{\mathbf{T}}^{-1}\right\|_\infty \alpha(\bar{\mathbf{A}}_i^{11})\left\|\bar{\mathbf{T}}\right\|_\infty \bar{\rho}_i(t_0) \end{aligned} \tag{B3}$$

with $\bar{\rho}_i(t_0)$ as in (A8). Denoting by $\hat{\mathbf{u}}'$ and $\mathbf{e}$ the N-dimensional vectors of entries $\hat{U}'_i$ (as in (B2)) and E_i (as in (B3)), respectively, conditions (44) are met for $t > t_0$ by any $\bar{\mathbf{k}}$ satisfying

$$\bar{\mathbf{k}} \ge \mathbf{P}\bar{\mathbf{k}} + \mathbf{P}(\hat{\mathbf{u}}' + \hat{\mathbf{k}}) + \mathbf{z}^0 + (\mathbf{I} + \mathbf{P})\mathbf{n} + \mathbf{e}$$

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