DETECTION AND REGULATION PROBLEMS FOR DISCRETE-TIME DISTRIBUTED SYSTEMS WITH DELAYS

ABDELWAHED NAMIR*, FOUAD LAHMIDI*
OMAR ALIF*

A distributed discrete-time hereditary system is considered. An unknown input is supposed to be a perturbation. First, we investigate the possibility of reconstructing this input using the information provided by an output equation. Then we treat the problem of keeping the observation as close as possible to some desired values (with the system still perturbed by the unknown input). To illustrate the results, some examples are presented.

Keywords: detection, discrete and delay systems, regulation, unknown inputs.

1. Introduction

In recent years, many works have been devoted to the study of discrete-time distributed systems, we can cite e.g. the works (Kern and Przyłuski, 1988; Kubrusly, 1989; Phat and Dieu, 1992). More recently, the case of discrete-time distributed systems with delays in the state, the control or the observation has also been considered (see the works of Karrakchou and Rachik, 1995; Karrakchou *et al.*, 1999; Namir *et al.*, 1998; etc.). In these studies, several concepts related to discrete-time systems have been investigated such as controllability, observability, stability, observers, compensators, etc.

The aim of this work is to investigate the 'detection' and 'regulation' problems for discrete-time distributed systems with delays in the state, the input and the output. When considering a mathematical model for a system of physical, chemical or economic type, it is often necessary to take into account some unknown parameters that affect the system. Depending on the nature of the considered system, these parameters can be of different origins: errors in the approximation of the original system, some external perturbations, excitations of an unknown source, etc.

Many works have been devoted to the study of systems with an unknown input. The problems considered depend on the nature of the unknown action. We can note e.g. the works (Afifi and El Jai, 1994; 1995) where the input has been considered as the excitation of an unknown source. The aim of those works was to investigate

* Département de Mathématiques et d'Informatique, Faculté des Sciences Ben M'Sik, Casablanca, Morocco, e-mail: lfouad@mailcity.com.

the possibility of reconstructing the input using the information given by an output equation (the detection problem).

A more classical problem is the so-called *regulation problem.* In this case the unknown input is considered as an undesirable perturbation (noise) whose effects on the system must be 'regulated' instantly by a suitable control.

In this work, we study the detection and regulation problems in the case of systems whose states evolution is described by a set of discrete-time delayed equations in Hilbert spaces. More precisely, we consider the systems which can be written after transformation as

$$\begin{cases} \xi_{i+1} = \sum_{j=0}^{p} A_j \xi_{i-j} + \sum_{j=0}^{q} D_j f_{i-j} + \sum_{j=0}^{r} B_j u_{i-j}, \quad i \geq 0 \\ f_k = \phi_k \ \text{given for } \ -q \leq k \leq -1 \ \text{ and } \ \xi_k \in \mathcal{X}, -p \leq k \leq 0 \end{cases} \tag{1}$$

with the output equation

$$y_i = \sum_{j=0}^{m} C_j \xi_{i-j}, \quad i \geq 1, \quad m \leq p \tag{2}$$

where $\xi_i \in \mathcal{X}$, $f_i \in F$, $u_i \in U$, $y_i \in Y$. The spaces $\mathcal{X}$, F, U and Y are supposed to be Hilbert spaces with Y of a finite dimension. The operators A_j, D_j, B_j, and C_j are linear and bounded in the appropriate spaces. In the sequel, the sequence $(f_i)_{i\geq 0}$ will denote the unknown input.

In the first part of this work, we investigate the possibility of reconstructing the input $(f_i)_{i\geq 0}$ using the information given by the observation (the detection problem). The problem considered can be formulated as follows:

$$(\mathcal{P}1) \begin{cases} \text{Given eqn. (1), with } (f_i)_{i\geq 0} \text{ unknown, and the observation (2),} \\ \text{is it possible to reconstruct the sequence } (f_i)_{i\geq 0}? \end{cases}$$

It can be considered as a mathematical model of many practical phenomena, especially the problem of environmental pollution. The danger of pollution increases when the pollution source is unknown. In this case the source can be considered as an unknown action (input) which is observed via an output equation and must be detected.

In the second part, our objective is to keep the output y_i as close as possible to a desired value y_i^d. To achieve this, we apply, at every instant i, a suitable control u_i to the system (the regulation problem). More precisely, the problem considered in this section is as follows:

$$(\mathcal{P}2) \begin{cases} \text{At every instant } i, \text{ find a control } u_i \text{ which will ensure that the} \\ \text{observation } y_i \text{ is maintained as close as possible to the desired} \\ \text{output } y_i^d. \end{cases}$$

The work contains four sections. In the second section, we treat the detection problem. We begin by considering the case of systems with delays only in the input and output (without delays in the state). Then we give a generalization of all the presented results to the case of systems with delays in the state, input and output. The regulation problem is investigated in Section 3. We develop two methods to calculate the appropriate control that will ensure that the output is maintained as close as possible to the desired values. The work also contains examples that illustrate the developed results.

2. Detection Problem

In this section, we assume that the system (1) is not controlled ($(u_i)_{i\geq -q} \equiv 0$)

$$\begin{cases} \xi_{i+1} = \sum_{j=0}^{p} A_j \xi_{i-j} + \sum_{j=0}^{q} D_j f_{i-j}, \quad 0 \leq i \leq N-1 \\ f_k = \phi_k \text{ given, } -q \leq k \leq -1 \text{ and } \xi_k \in \mathcal{X}, \ -p \leq k \leq 0 \end{cases} \tag{3}$$

where $N = 1, 2, \ldots$. First, we investigate the detection problem in the case of systems without delays in the state and then we generalize the results.

2.1. The Case of Systems with Delays in the Input and Output (without Delays in the State)

Consider the system

$$\begin{cases} \xi_{i+1} = A\xi_i + \sum_{j=0}^{q} D_j f_{i-j}, \quad 0 \leq i \leq N-1 \\ f_k = \phi_k \text{ given for } -q \leq k \leq -1 \text{ and } \xi_0 \in \mathcal{X} \end{cases} \tag{4}$$

where $A \in \mathcal{L}(\mathcal{X})$ and the operators D_j, $j = 0, \ldots, q$ satisfy the same hypothesis as in the Introduction. The solution to (4) is given by

$$\xi_i = A^i \xi_0 + \sum_{k=0}^{q} \sum_{j=0}^{i-1} A^{i-j-1} D_k f_{j-k}, \quad 0 \leq i \leq N$$

In the following, we assume $A^{-i} = (A^*)^{-i} = 0$, for $i = 1, 2, \ldots$ Hence, for all $l \geq 0$, we can write

$$\xi_{i-l} = A^{i-l} \xi_0 + \sum_{k=0}^{q} \sum_{j=0}^{i-l-1} A^{i-j-l-1} D_k f_{j-k}, \quad 1 \leq i \leq N$$

We have

$$\begin{aligned}
y_i &= \sum_{l=0}^{m} C_l \xi_{i-l} = \sum_{l=0}^{m} C_l A^{i-l}\xi_0 + \sum_{l=0}^{m}\sum_{k=0}^{q}\sum_{j=0}^{i-l-1} C_l A^{i-j-l-1} D_k f_{j-k} \\
&= \sum_{l=0}^{m} C_l A^{i-l}\xi_0 + \sum_{l=0}^{m}\sum_{k=0}^{q}\sum_{j=-k}^{i-l-k-1} C_l A^{i-j-l-k-1} D_k f_j \\
&= \sum_{l=0}^{m} C_l A^{i-l}\xi_0 + \sum_{l=0}^{m}\sum_{k=0}^{q}\sum_{j=-k}^{i-1} C_l A^{i-j-l-k-1} D_k f_j \\
&= \sum_{l=0}^{m} C_l A^{i-l}\xi_0 + \sum_{l=0}^{m}\sum_{j=0}^{i-1} C_l A^{i-j-l-1} D_0 f_j \\
&\quad + \sum_{l=0}^{m}\sum_{k=1}^{q}\sum_{j=-k}^{-1} C_l A^{i-j-l-k-1} D_k \phi_j + \sum_{l=0}^{m}\sum_{k=1}^{q}\sum_{j=0}^{i-1} C_l A^{i-j-l-k-1} D_k f_j \\
&= a_i + \sum_{l=0}^{m}\sum_{k=0}^{q}\sum_{j=0}^{i-1} C_l A^{i-j-l-k-1} D_k f_j, \quad 1 \leq i \leq N
\end{aligned}$$

where $a_i = \sum_{l=0}^{m} C_l A^{i-l}\xi_0 + \sum_{l=0}^{m}\sum_{k=1}^{q}\sum_{j=-k}^{-1} C_l A^{i-j-l-k-1} D_k \phi_j, \ 1 \leq i \leq N.$

Without loss of generality, we assume that $a_i = 0, \ 1 \leq i \leq N$. (If $a_i \neq 0$ for some $i \in \{1, \ldots, N\}$, we can consider the observation $y_i - a_i$). Introduce the following operator:

$$Q : (f_i)_{0 \leq i \leq N-1} \in F^N \longmapsto (y_i)_{1 \leq i \leq N} \in Y^N$$

where $(y_i)_{1 \leq i \leq N}$ is the observation corresponding to the input $(f_i)_{0 \leq i \leq N-1}$. The operator Q can be written in the matrix form as

$$Q = \begin{pmatrix}
C_0 D_0 & 0 & \cdots & \cdots & 0 \\
\sum_{l=0}^{m}\sum_{k=0}^{q} C_l A^{1-l-k} D_k & C_0 D_0 & \ddots & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \ddots & \vdots \\
\vdots & & & & 0 \\
\sum_{l=0}^{m}\sum_{k=0}^{q} C_l A^{N-l-k-1} D_k & \cdots & \cdots & \sum_{l=0}^{m}\sum_{k=0}^{q} C_l A^{1-l-k} D_k & C_0 D_0
\end{pmatrix}$$

It is easy to check that Q is linear and bounded. Its adjoint is given by

$$Q^* : (y_i)_{1 \leq i \leq N} \in Y^N \longmapsto (f_i)_{0 \leq i \leq N-1} \in F^N$$

where

$$f_i = \sum_{l=0}^{m} \sum_{k=0}^{q} \sum_{j=i+1}^{N} D_k^*(A^*)^{j-i-k-l-1} C_l^* y_j, \quad 0 \le i \le N-1$$

or in the matrix form

$$Q^* = \begin{pmatrix} D_0^* C_0^* & \cdots & \cdots & \cdots & \cdots & \sum_{l=0}^{m}\sum_{k=0}^{q} D_k^*(A^*)^{N-l-k-1} C_l^* \\ 0 & \ddots & & & & \vdots \\ \vdots & \ddots & & \ddots & & \vdots \\ \vdots & & \ddots & & \ddots & \vdots \\ \vdots & & & \ddots & & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & D_0^* C_0^* \end{pmatrix}$$

The main result in this subsection is given by

Proposition 1. *Consider the system (4) with the output equation (2). If the operator Q is injective, then every input $(f_i)_{0 \le i \le N-1}$ can be reconstructed.*

Proof. We have $Q^*Q(f_i)_{0 \le i \le N-1} = Q^*(y_i)_{1 \le i \le N}$. The operator Q^*Q is positive definite, since Q is injective, and hence it is invertible. Therefore the sequence $(f_i)_{0 \le i \le N-1}$ is given by

$$(f_i)_{0 \le i \le N-1} = (Q^*Q)^{-1} Q^*(y_i)_{1 \le i \le N}$$

■

To check if the operator Q is injective, one can use the following characterization:

Proposition 2. *The following statements are equivalent:*

(a) the operator Q is injective,

(b) the mapping $C_0 D_0$ is injective.

Proof.

(a) ⇒ (b) Assume that there exists $f \in F$ such that $C_0 D_0 f = 0$. Set $f_i = 0$ for $i = 0, \dots, N-2$ and $f_{N-1} = f$. We obtain $Q(f_i)_{0 \le i \le N-1} = 0$ and hence $f = 0$.

(b) ⇒ (a) Let $(f_i)_{0 \le i \le N-1} \in F^N$ such that $Q(f_i)_{0 \le i \le N-1} = (y_i)_{1 \le i \le N} = 0$. We have

- $0 = y_1 = C_0 D_0 f_0$ and hence $f_0 = 0$.
- Assume that $f_j = 0$ for $j = 0, \dots, i-1$, where $i \in \{1, \dots, N-1\}$. Then

$$0 = y_{i+1} = \sum_{j=0}^{i-1} \sum_{l=0}^{m} \sum_{k=0}^{q} C_l A^{i-j-l-k} D_k f_j + \sum_{l=0}^{m} \sum_{k=0}^{q} C_l A^{-l-k} D_k f_i = C_0 D_0 fi$$

Thus $f_i = 0$ and therefore $(f_i)_{0 \le i \le N-1} \equiv 0$. ■

Assume that Q is injective. To reconstruct an input $(f_i)_{0\leq i\leq N-1}$, it is not necessary to invert the operator Q^*Q as it is seemingly suggested in the proof of Proposition 1. Instead, we can proceed as follows. Write $Q_0 = C_0 D_0$. The operator Q_0 is linear and bounded and its adjoint is given by $Q_0^* = D_0^* C_0^*$. In addition, Q_0 is injective and so is $Q_0^* Q_0$. Furthermore,

- We have $Q_0 f_0 = y_1$ and so $f_0 = (Q_0^* Q_0)^{-1} Q_0^* y_1$.
- Assume that the f_j, $j = 0, \dots, i-1$, where $i \in \{1, \dots, N-1\}$, have been calculated. We have

$$y_{i+1} = \sum_{j=0}^{i-1} \sum_{l=0}^{m} \sum_{k=0}^{q} C_l A^{i-j-l-k} D_k f_j + Q_0 f_i$$

Hence $Q_0^* Q_0 f_i = Q_0^* z_{i+1}$, where

$$z_{i+1} = y_{i+1} - \sum_{j=0}^{i-1} \sum_{l=0}^{m} \sum_{k=0}^{q} C_l A^{i-j-l-k} D_k f_j$$

Therefore $f_i = (Q_0^* Q_0)^{-1} Q_0^* z_{i+1}$.

Example 1. As an illustration of the above result, consider the following hyperbolic system:

$$\begin{cases} \dfrac{\partial^2}{\partial t^2} z = \Delta z + \mathcal{X}_{\Omega_1}(x) f_1(t) + \mathcal{X}_{\Omega_2}(x) f_2(t) + \mathcal{X}_{\Omega_3}(x) f_1(t-h), \quad (x,t) \in]0,1[^2 \\ z(x,0) = \dfrac{\partial}{\partial t} z(x,0) = 0, \quad x \in]0,1[\\ z(0,t) = z(1,t) = 0, \quad t \in]0,1[\\ f_1(\theta) = 0, \quad \theta \in [-h, 0[, \quad h = 0.25 \end{cases} \tag{5}$$

where $\Omega_1 =]0, 1/3[$, $\Omega_2 =]2/3, 1[$, $\Omega_3 =]1/3, 2/3[$ and $\mathcal{X}_{\Omega_i}$ denotes the characteristic function of the interval Ω_i.

We assume that the observation is given by

$$y(t) = \begin{pmatrix} \langle z(\cdot,t), \mathcal{X}_{\Omega_4} \rangle \\ \left\langle \dfrac{\partial}{\partial t} z(\cdot,t), \mathcal{X}_{\Omega_5} \right\rangle \end{pmatrix} \tag{6}$$

where $\Omega_4 =]0, 1/4[$, $\Omega_5 =]3/4, 1[$ and $\langle \cdot, \cdot \rangle$ denotes the inner product in the Hilbert space $L^2(]0,1[)$.

Let $\mathcal{X} = H_0^1(0,1) \times L^2(0,1)$, $\xi = (z(\cdot,t), \partial z(\cdot,t)/\partial t) \in \mathcal{X}$ and define the operators

$$E_0 : \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mapsto \begin{pmatrix} 0 \\ a\mathcal{X}_{\Omega_1} + b\mathcal{X}_{\Omega_2} \end{pmatrix} \in \mathcal{X}, \quad E_1 : \begin{pmatrix} a \\ b \end{pmatrix} \in \mathbb{R}^2 \mapsto \begin{pmatrix} 0 \\ a\mathcal{X}_{\Omega_1} \end{pmatrix} \in \mathcal{X}$$

The system (5)–(6) can then be written in a state-equation form as follows:

$$\begin{cases} \dot{\xi}(t) = \Phi\xi(t) + E_0 f(t) + E_1 f(t-h) \\ \xi(0) = 0, \quad f(\theta) = 0, \quad \theta \in [-h, 0[\end{cases} \tag{7}$$

$$y(t) = C\xi(t) \tag{8}$$

where

$$f(t) = \begin{pmatrix} f_1(t) \\ f_2(t) \end{pmatrix} \in \mathbb{R}^2, \quad \Phi = \begin{pmatrix} 0 & I \\ \Delta & 0 \end{pmatrix}, \quad C = \begin{pmatrix} \langle \cdot, \mathcal{X}_{\Omega_4} \rangle \\ \langle \cdot, \mathcal{X}_{\Omega_5} \rangle \end{pmatrix}$$

and I is the identity mapping in $L^2(]0,1[)$. The operator Φ is the infinitesimal generator of the strongly-continuous semigroup $S(t)_{t\geq 0}$ defined by

$$S(t)\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} \sum_{n=1}^{\infty} \left[\cos(n\pi t)\,\langle x_1, b_n\rangle + \frac{1}{n}\sin(n\pi t)\,\langle x_2, b_n\rangle\right] b_n \\ \sum_{n=1}^{\infty} \left[-\sin(n\pi t)\,\langle x_1, b_n\rangle + \cos(n\pi t)\,\langle x_2, b_n\rangle\right] b_n \end{pmatrix}$$

where $b_n = \sqrt{2}\sin(n\pi t)$, $n \geq 1$.

Let $\delta = 1/N$, where $N = 1, 2, \ldots$, $t_i = i\delta$, $i \in \mathbb{Z}$, $f(t) = f_i$, $t \in]t_i, t_{i+1}[$, $j = E(h/\delta)$, $A_0 = S(\delta)$, $\xi_i = z(t_i)$ and $y_i = y(t_i)$. The system (7)–(8) can then be described by the following discrete system:

$$\begin{cases} \xi_{i+1} = A_0\xi_i + D_0 f_i + D_j f_{i-j} + D_{j+1} f_{i-j-1}, \quad 0 \leq i \leq N-1 \\ \xi_0 = 0, \quad f_k = 0, \quad k < 0 \end{cases} \tag{9}$$

$$y_i = C\xi_i, \quad 1 \leq i \leq N \tag{10}$$

where D_0, D_j and $D_{j+1} \in \mathcal{L}(\mathbb{R}^2, \mathcal{X})$ are given by

$$D_0 = \int_0^{\delta} S(t)E_0\,\mathrm{d}t, \quad D_j = \int_h^{(j+1)\delta} S(t-h)E_1\,\mathrm{d}t, \quad D_{j+1} = \int_{(j+1)\delta}^{h+\delta} S(t-h)E_1\,\mathrm{d}t$$

If the system (5) is perturbed by the input $f(t) = (\exp(-t), \sin(\pi t/2))$, then one can check by a direct computation that the observation is given by $y(t) = (y_1(t), y_2(t))$, where

$$y_1(t) = \begin{cases} y_{1,1}(t) & \text{if } 0 \leq t \leq h \\ y_{1,2}(t) & \text{if } h \leq t \leq 1 \end{cases}, \quad y_2(t) = \begin{cases} y_{2,1}(t) & \text{if } 0 \leq t \leq h \\ y_{2,2}(t) & \text{if } h \leq t \leq 1 \end{cases}$$

$$\begin{aligned}
y_{1,1}(t) &= \sum_{n\geq 1}\frac{2}{(n\pi)^3}\left[\frac{1}{1+n^2\pi^2}\left[\sin(n\pi t) - n\pi\cos(n\pi t) + n\pi\exp(-t)\right]\left(1-\cos\left(\frac{n\pi}{3}\right)\right)\right.\\
&\left. + \frac{2}{\pi(1-4n^2)}\left(\sin(n\pi t) - 2n\sin\left(\frac{\pi t}{2}\right)\right)\left(\cos\left(\frac{2n\pi}{3}\right) - \cos(n\pi)\right)\right]\\
&\times\left(1-\cos\left(\frac{n\pi}{4}\right)\right)
\end{aligned}$$

$$\begin{aligned}
y_{1,2}(t) &= y_{1,1}(t) + \sum_{n\geq 1}2\big(\sin(n\pi(t-h)) - n\pi\cos(n\pi(t-h)) + n\pi\exp(h-t)\big)\\
&\times\left(\cos\left(\frac{n\pi}{3}\right) - \cos\left(\frac{2n\pi}{3}\right)\right)\frac{(1-\cos(n\pi/4))}{(n\pi)^3(1+n^2\pi^2)}
\end{aligned}$$

$$\begin{aligned}
y_{2,1}(t) &= \sum_{n\geq 1}\frac{2}{(n\pi)^2}\left[\frac{1}{1+n^2\pi^2}\big(\cos(n\pi t) + n\pi\sin(n\pi t) - \exp(-t)\big)\left(1-\cos\left(\frac{n\pi}{3}\right)\right)\right.\\
&\left. + \frac{2}{\pi(1-4n^2)}\left(\cos(n\pi t) - \cos\left(\frac{\pi t}{2}\right)\right)\left(\cos\left(\frac{2n\pi}{3}\right) - \cos(n\pi)\right)\right]\\
&\times\left(\cos(3n\pi 4 - \cos\left(\frac{n\pi}{4}\right)\right)
\end{aligned}$$

$$\begin{aligned}
y_{2,2}(t) &= y_{2,1}(t) + \sum_{n\geq 1}2\big(\cos(n\pi(t-h)) + n\pi\sin(n\pi(t-h)) - \exp(h-t)\big)\\
&\times\left(\cos\left(\frac{n\pi}{3}\right) - \cos\left(\frac{2n\pi}{3}\right)\right)\frac{(\cos(3n\pi/4) - \cos(n\pi/4))}{(n\pi)^2(1+n^2\pi^2)}
\end{aligned}$$

If we take as the output the sequence $(y(t_i))_{1\leq i\leq N-1}$, then we will obtain the numerical results given in Table 1. In the second column, we give the exact values of $f_1(\cdot)$ and $f_2(\cdot)$ for different points in $[0,1]$. In the others columns, we present some approximations of the values of $f_1(\cdot)$ and $f_2(\cdot)$ corresponding to different values of N. Figures 1 and 2 represent the results in graphical form. ♦

Example 2. Consider the diffusion system described by the following parabolic equation:

$$\begin{cases}
\frac{\partial}{\partial t}z - \Delta z = \mathcal{X}_{\Omega_0}f(t) + \mathcal{X}_{\Omega_1}f(t-h), & (x,t)\in]0,1[\times]0,1[\\
z(x,0) = 0, \quad x\in]0,1[& \\
z(0,t) = z(1,t) = 0, \quad t\in]0,1[& \\
f(\theta) = 0, \quad \theta\in[-h,0[, \quad h = 0.25 &
\end{cases} \tag{11}$$

Table 1. Approximation of $f_1(\cdot)$ and $f_2(\cdot)$.

		$N=100$	$N=200$	$N=400$	$N=600$	$N=800$	$N=1000$
$f_1(0)$	1	0.966719	0.998335	0.999167	0.999445	0.999583	0.999667
$f_2(0)$	0	0.007874	0.003929	0.001964	0.001309	0.000981	0.000785
$f_1(0.1)$	0.904837	0.901983	0.903410	0.904123	0.904361	0.904481	0.904552
$f_2(0.1)$	0.156434	0.164293	0.160338	0.158380	0.157730	0.157406	0.157211
$f_1(0.2)$	0.818731	0.816306	0.817518	0.818124	0.818326	0.818426	0.818485
$f_2(0.2)$	0.309017	0.316649	0.312791	0.310894	0.310266	0.309953	0.309765
$f_1(0.3)$	0.740818	0.738784	0.739800	0.740309	0.740477	0.740568	0.740013
$f_2(0.3)$	0.453990	0.461194	0.457539	0.455752	0.455162	0.454868	0.454692
$f_1(0.4)$	0.670320	0.668635	0.669478	0.669898	0.670039	0.670112	0.670157
$f_2(0.4)$	0.587785	0.594347	0.591013	0.589386	0.588850	0.588583	0.588423
$f_1(0.5)$	0.606531	0.605116	0.605836	0.606186	0.606301	0.606363	0.606409
$f_2(0.5)$	0.707107	0.712886	0.709939	0.708509	0.708038	0.707804	0.707664
$f_1(0.6)$	0.548812	0.547623	0.548246	0.548533	0.548632	0.548674	0.548729
$f_2(0.6)$	0.809017	0.813882	0.811386	0.810186	0.809793	0.809560	0.809481
$f_1(0.7)$	0.496585	0.495610	0.496139	0.496371	0.496451	0.496495	0.496554
$f_2(0.7)$	0.891006	0.894819	0.892850	0.891913	0.891607	0.891456	0.891365
$f_1(0.8)$	0.449329	0.448573	0.448995	0.449172	0.449228	0.449256	0.449306
$f_2(0.8)$	0.951056	0.953732	0.952331	0.951678	0.951468	0.951364	0.951302
$f_1(0.9)$	0.406570	0.406064	0.406351	0.406467	0.406520	0.406527	0.406616
$f_2(0.9)$	0.987688	0.989185	0.988369	0.988011	0.987900	0.987846	0.987814

where $\Omega_0 =]0, 1/2[$, $\Omega_1 =]1/2, 1[$. The observation is given by

$$y(t) = \sum_{n=1}^{\infty} \frac{1}{n} \langle z(\cdot,t), b_n \rangle \, b_n \tag{12}$$

The system (11) can be written in a state-equation form as follows:

$$\begin{cases} \dot{z}(t) = \Delta z(t) + E_0 f(t) + E_1 f(t-h) \\ z(0) = 0, \quad f(\theta) = 0, \quad \theta \in [-h, 0[\end{cases} \tag{13}$$

where $z(t) \in \mathcal{X} = L^2(]0,1[)$, E_0 and $E_1 \in \mathcal{L}(\mathbb{R}, \mathcal{X})$ are given by

$$E_i : a \in \mathbb{R} \longmapsto a\mathcal{X}_{\Omega_i} \in \mathcal{X}, \quad i \in \{0, 1\}$$

The laplacian Δ is the infinitesimal generator of the strongly-continuous semigroup $S(t)_{t \geq 0}$ defined by

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \langle x, b_n \rangle \, b_n$$

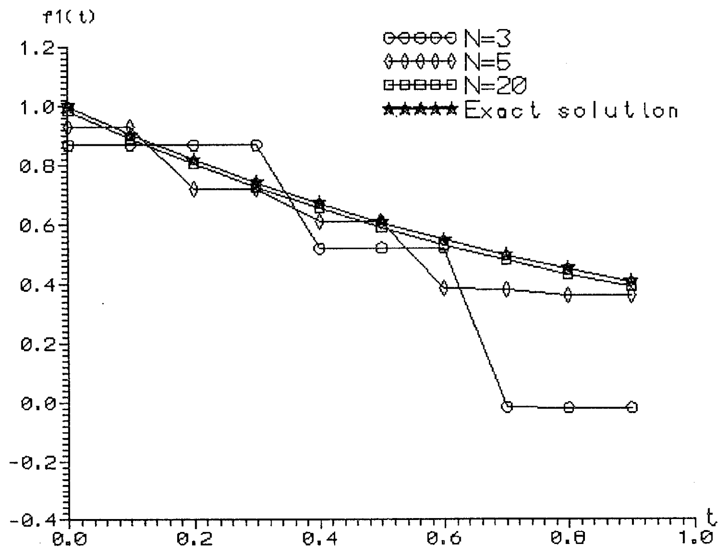

Fig. 1. Approximation of $f_1(\cdot)$.

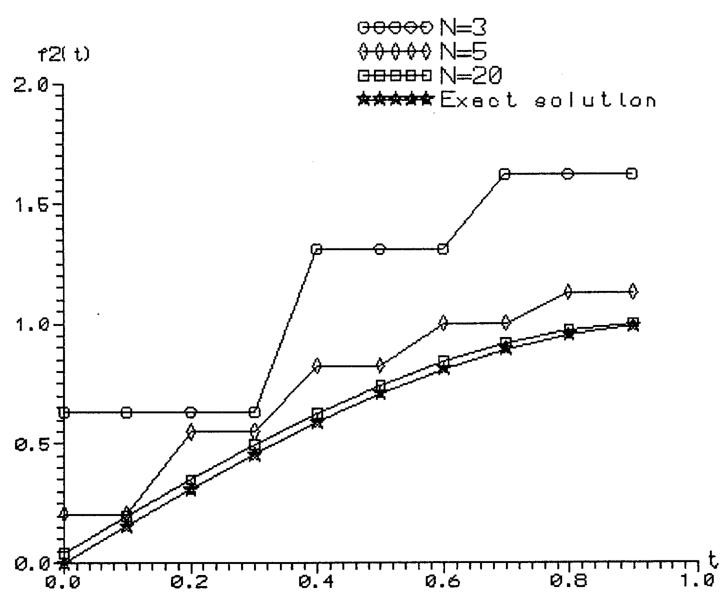

Fig. 2. Approximation of $f_2(\cdot)$.

Using the same technique as in the previous example, we can rewrite the system (11)–(12) in the following discrete form:

$$\begin{cases} \xi_{i+1} = A_0\xi_i + D_0 f_i + D_j f_{i-j} + D_{j+1} f_{i-j-1}, & 0 \le i \le N-1 \\ \xi_0 = 0, \quad f_k = 0, \quad k < 0 \end{cases} \tag{14}$$

$$y_i = \sum_{n=1}^{\infty} \frac{1}{n} \langle \xi_i, b_n \rangle \tag{15}$$

where D_0, D_j and $D_{j+1} \in \mathcal{L}(\mathbb{R}, \mathcal{X})$ are given by

$$D_0 = \int_0^{\delta} S(t)E_0 \,\mathrm{d}t, \quad D_j = \int_h^{(j+1)\delta} S(t-h)E_1 \,\mathrm{d}t, \quad D_{j+1} = \int_{(j+1)\delta}^{h+\delta} S(t-h)E_1 \,\mathrm{d}t$$

As a numerical example, consider

$$y(t) = \begin{cases} \tilde{y}(t) & \text{if } 0 \le t \le h \\ \hat{y}(t) & \text{if } h < t \le 1 \end{cases}$$

where

$$\tilde{y}(t) = \frac{\sqrt{2}}{\pi^3}\left(\pi^2 t e^{-\pi^2 t} + \sum_{n=2}^{\infty} \frac{1}{n^2(n^2-1)} e^{-n^2\pi^2 t}\left(e^{\pi^2(n^2-1)t} - 1\right)\left(1 - \cos\left(\frac{n\pi}{2}\right)\right)\right)$$

and

$$\begin{aligned} \hat{y}(t) &= \frac{\sqrt{2}}{\pi^3}\Big[\pi^2\left(te^{-\pi^2 t} + (t-h)e^{-\pi^2(t-h)}\right) \\ &\quad + \sum_{n=2}^{\infty} \frac{1}{n^2(n^2-1)}\Big(e^{-n^2\pi^2 t}\left(e^{\pi^2(n^2-1)t} - 1\right)\left(1 - \cos\left(\frac{n\pi}{2}\right)\right) \\ &\quad + e^{-n^2\pi^2(t-h)}\left(e^{\pi^2(n^2-1)(t-h)} - 1\right)\left(\cos\left(\frac{n\pi}{2}\right) - \cos(n\pi)\right)\Big)\Big] \end{aligned}$$

The results are gathered in Table 2. In the second column, we give the exact values of $f(\cdot)$ for different points in $[0, 1]$. In the other columns, we present some approximations of the values of $f(\cdot)$ corresponding to different values of N. The same results are also visualized graphically in Fig. 3. ♦

Table 2. Approximation of $f(\cdot)$.

		$N=100$	$N=200$	$N=400$	$N=600$	$N=800$	$N=1000$
$f(0)$	1	0.949890	0.974938	0.987498	0.991680	0.993768	0.995020
$f(0.1)$	0.372708	0.354572	0.363575	0.368125	0.369649	0.370413	0.370871
$f(0.2)$	0.138911	0.132166	0.135511	0.137204	0.137771	0.138056	0.138227
$f(0.3)$	0.051773	0.049636	0.050607	0.051164	0.051360	0.051461	0.051523
$f(0.4)$	0.019296	0.018394	0.018833	0.190616	0.019139	0.019178	0.019202
$f(0.5)$	0.007192	0.064378	0.006898	0.007067	0.007115	0.007136	0.007149
$f(0.6)$	0.002680	0.002398	0.002574	0.002637	0.002654	0.002661	0.002666
$f(0.7)$	0.000999	0.000931	0.000969	0.000985	0.000990	0.000993	0.000994
$f(0.8)$	0.000372	0.000597	0.000427	0.000384	0.000377	0.000374	0.000373
$f(0.9)$	0.000139	0.000205	0.000155	0.000142	0.000140	0.000139	0.000139

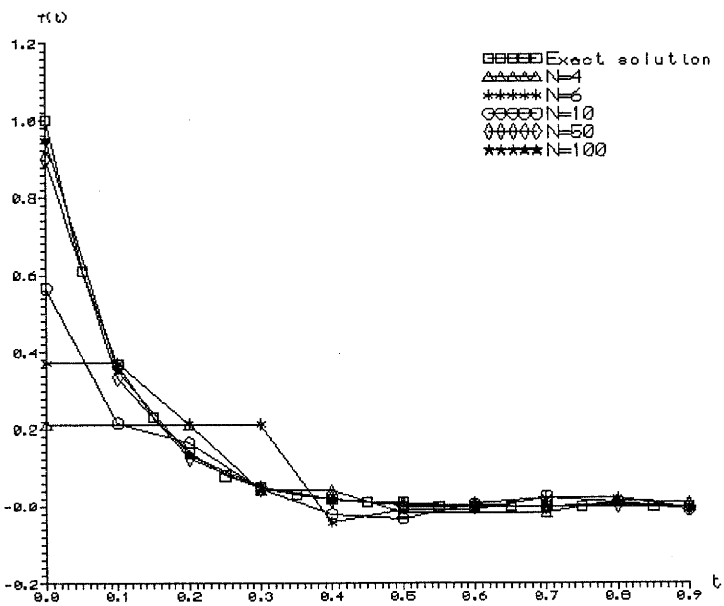

Fig. 3. Approximation of $f(\cdot)$.

The assumption that Q is injective is not always verified, especially if the space F is of an infinite dimension. Define then the operator

$$\overline{Q} : \overline{(f_i)}_{0\leq i\leq N-1} \in F^N / \ker Q \longmapsto \overline{Q}\overline{(f_i)}_{0\leq i\leq N-1} = Q(f_i)_{0\leq i\leq N-1} \in Y^N$$

where $\overline{(f_i)}_{0\leq i\leq N-1} = (f_i)_{0\leq i\leq N-1} + \ker Q$. The operator $\overline{Q}$ is linear, bounded and injective. Therefore, to each $(y_i)_{1\leq i\leq N} \in A_N := \operatorname{Im}(Q)$ will correspond a unique element $\overline{(f_i)}_{0\le i\le N-1} \in F^N/\ker Q$ such that

$$\overline{Q}\,\overline{(f_i)}_{0\le i\le N-1} = Q(f_i)_{0\le i\le N-1} = (y_i)_{1\le i\le N}$$

To reconstruct $\overline{(f_i)}_{0\le i\le N-1}$, it is sufficient to find a sequence $(f_i^*)_{0\le i\le N-1} \in F^N$ such that $Q(f_i^*)_{0\le i\le N-1} = (y_i)_{1\le i\le N}$, because we then have

$$\overline{(f_i)}_{0\le i\le N-1} = (f_i^*)_{0\le i\le N-1} + \ker Q$$

Therefore the idea is to search for an input $(f_i^*)_{0\le i\le N-1}$ of minimal norm which satisfies $Q(f_i^*)_{0\le i\le N-1} = (y_i)_{1\le i\le N}$. The existence and uniqueness result is given by

Proposition 3. *If $(y_i)_{1\le i\le N} \in A_N$, then there exists a unique sequence $(f_i^*)_{0\le i\le N-1} \in F^N$ of minimal norm such that $Q(f_i^*)_{0\le i\le N-1} = (y_i)_{1\le i\le N}$.*

Proof. The set

$$\mathcal{F}_{\text{ad}} = \left\{(f_i)_{0\le i\le N-1} \in F^N / Q(f_i)_{0\le i\le N-1} = (y_i)_{1\le i\le N}\right\}$$

is non-empty by the hypotheses and, since Q is bounded, it is closed. In addition, one can easily check that $\mathcal{F}_{\text{ad}}$ is convex. Hence, by the projection principle, there exists a unique element $(f_i^*)_{0\le i\le N-1} \in \mathcal{F}_{\text{ad}}$ such that

$$\|(f_i^*)_{0\le i\le N-1}\| = \inf_{(v_i)_{0\le i\le N-1}\in\mathcal{F}_{\text{ad}}} \|(v_i)_{0\le i\le N-1}\|$$

■

In order to present a method to calculate the sequence $(f_i^*)_{0\le i\le N-1}$, we introduce the operator

$$\Lambda : (y_i)_{1\le i\le N} \in Y^N \longmapsto QQ^*(y_i)_{1\le i\le N} = (z_i)_{1\le i\le N} \in Y^N$$

with

$$z_i = \sum_{j=0}^{i-1} \sum_{\alpha=j+1}^{N} \sum_{l,\beta=0}^{m} \sum_{k,\gamma=0}^{q} C_l A^{i-j-k-l-1} D_k D_\gamma^* (A^*)^{\alpha-j-\gamma-\beta-1} C_\beta^* y_\alpha$$

Lemma 1. *Consider the set A_N and the operator Λ. We have*

$$A_N = \operatorname{Im}(\Lambda)$$

Proof. It is readily verified that $\ker(Q^*) = \ker(QQ^*)$. Moreover, since Y is a finite-dimensional space, we have

$$A_N = \operatorname{Im}(Q) = \left(\ker(Q^*)\right)^{\perp} = \left(\ker(QQ^*)\right)^{\perp} = \operatorname{Im}(QQ^*) = \operatorname{Im}(\Lambda)$$

■

Proposition 4. *Let* $(y_i)_{1\le i\le N} \in A_N$. *The sequence* $(f_i^*)_{0\le i\le N-1}$ *is given by*

$$(f_i^*)_{0\le i\le N-1} = Q^*(\eta_i)_{1\le i\le N}$$

where $(\eta_i)_{1\le i\le N} \in Y^N$ *is such that*

$$\Lambda(\eta_i)_{1\le i\le N} = (y_i)_{1\le i\le N}$$

Proof. Since $(y_i)_{1\le i\le N} \in A_N$, by Lemma 1 there exists $(\eta_i)_{1\le i\le N} \in Y^N$ such that $\Lambda(\eta_i)_{1\le i\le N} = (y_i)_{1\le i\le N}$. Set $(f_i^*)_{0\le i\le N-1} = Q^*(\eta_i)_{1\le i\le N}$. Then we have

- $Q(f_i^*)_{0\le i\le N-1} = QQ^*(\eta_i)_{1\le i\le N} = \Lambda(\eta_i)_{1\le i\le N} = (y_i)_{1\le i\le N}$.
- Let $(v_i)_{0\le i\le N-1}$ be a sequence in F^N such that $(y_i)_{1\le i\le N} = Q(v_i)_{0\le i\le N-1}$. We have

$$\begin{aligned} 0 &= \langle Q((f_i^*)_{0\le i\le N-1} - (v_i)_{0\le i\le N-1}), (\eta_i)_{1\le i\le N}\rangle \\ &= \langle (f_i^*)_{0\le i\le N-1} - (v_i)_{0\le i\le N-1}, Q^*((\eta_i)_{1\le i\le N})\rangle \\ &= \langle (f_i^*)_{0\le i\le N-1} - (v_i)_{0\le i\le N-1}, (f_i^*)_{0\le i\le N-1}\rangle \end{aligned}$$

Hence

$$\begin{aligned} \|(f_i^*)_{0\le i\le N-1}\|^2 &= \langle (f_i^*)_{0\le i\le N-1}, (v_i)_{0\le i\le N-1}\rangle \\ &\le \|(f_i^*)_{0\le i\le N-1}\| \, \|(v_i)_{0\le i\le N-1}\| \end{aligned}$$

and so $\|(f_i^*)_{0\le i\le N-1}\| \le \|(v_i)_{0\le i\le N-1}\|$. ■

2.2. The General Case (System with Delays in the State, Input and Output)

Consider again the system (3) with the output equation (2). If we set

$$x_i = (\xi_i, \ldots, \xi_{i-p}, f_{i-1}, \ldots, f_{i-q})^T \in X = \mathcal{X}^{p+1} \times F^q, \quad 0 \le i \le N$$

then (3) reduces to the difference system

$$\begin{cases} x_{i+1} = Ax_i + Df_i, & 0 \le i \le N-1 \\ x_0 \in X \end{cases} \tag{16}$$

where

$$A = \begin{pmatrix} A_0 & A_1 & \cdots & \cdots & A_p & D_1 & \cdots & \cdots & \cdots & D_q \\ I_{\mathcal{X}} & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & & \vdots & \vdots & & & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\ 0 & \cdots & 0 & I_{\mathcal{X}} & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & & & & \vdots & I_F & \ddots & & & \vdots \\ \vdots & & & & \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & I_F & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_0 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ I_F \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

with $I_{\mathcal{X}}$ and I_F being the identity mappings on $\mathcal{X}$ and F, respectively. Moreover, if we define

$$C = (C_0\, C_1\, \cdots\, C_m\, \underbrace{0\, 0 \cdots 0\, 0}_{p+q-m \text{ times}}) \in \mathcal{L}(X, Y)$$

then the output equation (3) can be written as

$$y_i = Cx_i, \quad 1 \leq i \leq N \tag{17}$$

By this transformation, it is possible to tackle the detection problem for the system (2)–(3) using the difference equations (16)–(17). Remark that the system (16)–(17) is a particular case of (2)–(4) (with $q = m = 0$). Therefore, all the results presented in the precedent section are still applicable. In the sequel, we give a concise formulation of these results and the proofs will be omitted.

Without loss of generality, we assume that $x_0 = (\xi_0, \ldots, \xi_{-p}, \phi_{-1}, \ldots, \phi_{-q})^T = 0$. The operators Q and Λ are in this case of the form

$$\begin{cases} \widetilde{Q} : (f_i)_{0 \leq i \leq N-1} \in F^N \longmapsto (y_i)_{1 \leq i \leq N} \in Y^N \\ y_i = \displaystyle\sum_{j=0}^{i-1} CA^{i-j-1} D f_j \end{cases}$$

$$\widetilde{Q} = \begin{pmatrix} CD & 0 & \cdots & 0 \\ CAD & CD & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ CA^{N-1}D & \cdots & CAD & CD \end{pmatrix}$$

and

$$\begin{cases} \widetilde{\Lambda}(y_i)_{1\leq i\leq N} \in Y^N \longmapsto \widetilde{Q}\widetilde{Q}^*((y_i)_{1\leq i\leq N}) = (z_i)_{1\leq i\leq N} \in Y^N \\ z_i = \sum_{j=0}^{i-1} \sum_{k=j+1}^{N} CA^{i-j-1}DD^*(A^*)^{k-j-1}C^*y_k \end{cases}$$

Let $\widetilde{A}_N = \Big\{(y_i)_{1\leq i\leq N} \in Y^N / \exists (f_i)_{0\leq i\leq N-1} \in F^N : \widetilde{Q}(f_i)_{0\leq i\leq N-1} = (y_i)_{1\leq i\leq N}\Big\}$.

Proposition 5. *(a) If $(y_i)_{1\leq i\leq N} \in \widetilde{A}_N$, then there exists a unique sequence $(f_i^*)_{0\leq i\leq N-1} \in F^N$ of minimal norm such that $\widetilde{Q}(f_i^*)_{0\leq i\leq N-1} = (y_i)_{1\leq i\leq N}$. It is given by*

$$\begin{cases} (f_i^*)_{0\leq i\leq N-1} = Q(\eta_i)_{1\leq i\leq N} \\ \widetilde{\Lambda}(\eta_i)_{1\leq i\leq N} = (y_i)_{1\leq i\leq N} \end{cases}$$

(b) Moreover, we have the following results:

- *The operator $\widetilde{Q}$ is injective if and only if $CD = C_0D_0$ is one-to-one.*
- *If $\widetilde{Q}$ is injective, then every input can be reconstructed. It is given by*

$$(f_i)_{0\leq i\leq N-1} = (\widetilde{Q}^*\widetilde{Q})^{-1}\widetilde{Q}^*(y_i)_{1\leq i\leq N}$$

Remark 1. In the case where the operator $\widetilde{Q}$ is one-to-one, we can use the method presented in the precedent section to reconstruct the unknown input.

3. Regulation Problem

As can be seen in the previous section, it is always possible to reduce a hereditary system to difference equations by rewriting the delayed system in an appropriate product space. This transformation permits us to consider in this section, without loss of generality, only discrete systems of difference equations. For example, the system (1)–(2) can be reduced as follows. Set

$$x_i = (\xi_i, \ldots, \xi_{i-p}, f_{i-1}, \ldots, f_{i-q}, u_{i-1}, \ldots, u_{i-r})^T \in X = \mathcal{X}^{p+1} \times F^q \times U^r, \quad i \geq 0$$

It is easily verified that the sequence $(x_i)_{i\geq 0}$ satisfies

$$\begin{cases} x_{i+1} = Ax_i + Df_i + Bu_i, \quad i \geq 0 \\ x_0 \in X \end{cases} \tag{18}$$

and

$$y_i = Cx_i, \quad i \geq 1 \tag{19}$$

where the operators $A \in \mathcal{L}(X)$, $D \in \mathcal{L}(F, X)$, $B \in \mathcal{L}(U, X)$ and $C \in \mathcal{L}(X, Y)$ are respectively given by

$$A = \begin{pmatrix}
A_0 & A_1 & \cdots & \cdots & A_p & D_1 & \cdots & \cdots & \cdots & D_q & B_1 & \cdots & \cdots & \cdots & B_r \\
I_X & 0 & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \ddots & \ddots & & \vdots & \vdots & & & & \vdots & \vdots & & & & \vdots \\
\vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & \vdots & \vdots & & & & \vdots \\
0 & \cdots & 0 & I_X & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & \vdots & I_F & \ddots & & & \vdots & \vdots & & & & \vdots \\
\vdots & & & & \vdots & 0 & \ddots & \ddots & & \vdots & \vdots & & & & \vdots \\
\vdots & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots & & & & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & I_F & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 \\
\vdots & & & & \vdots & \vdots & & & & \vdots & I_U & \ddots & & & \vdots \\
\vdots & & & & \vdots & \vdots & & & & \vdots & 0 & \ddots & \ddots & & \vdots \\
\vdots & & & & \vdots & \vdots & & & & \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & \cdots & \cdots & 0 & 0 & \cdots & 0 & I_U & 0
\end{pmatrix}$$

$$D = (D_0 \underbrace{0 \cdots 0}_{p \text{ times}} I_F \underbrace{0 \cdots 0}_{q+r-1 \text{ times}})^T, \quad B = (B_0 \underbrace{0 \cdots 0}_{p+q \text{ times}} I_U \underbrace{0 \cdots 0}_{r-1 \text{ times}})^T$$

and

$$C = (C_0\, C_1 \cdots C_m \underbrace{0 \cdots 0}_{p+q+r-m \text{ times}})$$

If the system (18)–(19) is not perturbed ($(f_i)_i \equiv 0$) and not controlled ($(u_i)_i \equiv 0$), then the output is given by $y_i = CA^i x_0$, $i \geq 1$. We recall that the problem considered in this section is as follows:

$(\mathcal{P}2)$ $\begin{cases} \text{Find, at every instant } i, \text{ a control } u_i \text{ which will ensure that the} \\ \text{observation } y_i \text{ is maintained as close as possible to } CA^i x_0. \end{cases}$

Set $e_i = y_i - CA^i x_0$. A method to calculate the control $(u_i)_{i \geq 0}$ is given by

Proposition 6. *Assume*

(a) $\dim U = \dim Y < \infty$,

(b) CB *and* CD *are injective.*

Then there exists a control sequence $(u_i)_{i \geq 0}$ *such that* $e_i = CDf_{i-1}$, $\forall i \geq 1$. *In addition, if* $f_i \longrightarrow 0$ *as* $i \longrightarrow \infty$, *then* $e_i \longrightarrow 0$ *as* $i \longrightarrow \infty$.

Proof. Define

$$\begin{cases} u_0 = 0 \\ u_i = -(CB)^{-1}C\left[\sum_{j=0}^{i-1} D\left(D^*C^*CD\right)^{-1} D^*C^*e_{j+1} + \sum_{j=0}^{i-1} A^{i-1}Bu_j\right], \quad i \geq 1 \end{cases} \tag{20}$$

We have

- $y_1 = Cx_1 = CAx_0 + CDf_0$, hence $e_1 = CDf_0$.
- Assume that $e_j = CDf_{j-1}$ for $j = 1, \ldots, i$, where $i \geq 1$. Then, by (b),

$$f_{j-1} = \left(D^*C^*CD\right)^{-1} D^*C^*e_j, \quad j = 1, \ldots, i$$

On the other hand,

$$\begin{aligned} x_{i+1} &= A^{i+1}x_0 + \sum_{j=0}^{i} A^{i-j}Df_j + \sum_{j=0}^{i} A^{i-j}Bu_j \\ &= A^{i+1}x_0 + Df_i + \sum_{j=0}^{i-1} A^{i-j}Df_j + \sum_{j=0}^{i-1} A^{i-j}Bu_j + Bu_i \end{aligned}$$

Hence

$$\begin{aligned} y_{i+1} &= CA^{i+1}x_0 + CDf_i + C\left[\sum_{j=0}^{i-1} A^{i-j}D(D^*C^*CD)^{-1}D^*C^*e_{j+1} + \sum_{j=0}^{i-1} A^{i-j}Bu_j\right] \\ &\quad - CB(CB)^{-1}C\left[\sum_{j=0}^{i-1} A^{i-j}D(D^*C^*CD)^{-1}D^*C^*e_{j+1} + \sum_{j=0}^{i-1} A^{i-j}Bu_j\right] \end{aligned}$$

and therefore $e_{i+1} = CDf_i$. ∎

Example 3. Consider the system (9)–(10) of Example 1. In the product space $X = \mathcal{X} \times (\mathbb{R}^2)^{j+1}$ this system reduces to

$$\begin{cases} x_{i+1} = Ax_i + Df_i, \quad i \geq 0 \\ x_0 \in X \end{cases} \tag{21}$$

$$y_i = Cx_i, \quad i \geq 1 \tag{22}$$

where

$$A = \begin{pmatrix} A_0 & 0 & \cdots & 0 & D_j & D_{j+1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & I & \ddots & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_0 \\ I \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

and

$$C = (C_0 \; \underbrace{0\,0 \cdots 0\,0}_{j+1 \text{ times}}) \in \mathcal{L}(X, Y)$$

Suppose that $U = \mathbb{R}^2$ and the operator $B \in \mathcal{L}(U, X)$ is given by

$$B : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto \begin{pmatrix} \begin{pmatrix} \alpha b_1 \\ \beta b_2 \end{pmatrix} \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

Then one can easily check that the operators CB and CD are one-to-one. Let $f_i = (\exp(-i/10), 0)$ and $x_0 = 0$. In this case $e_i = (e_{1,i}, e_{2,i}) = y_i$, $i \geq 1$. If we apply the control $(u_i)_{i \geq 0}$ given by (20) to the system (21), we obtain the numerical results of Table 3. They are also represented graphically in Figs. 4 and 5.

Table 3. Approximation of e_1 and e_2.

i	10	20	30	40	50
$e_{1,i}$	4.40242×10^{-4}	1.61956×10^{-4}	5.95803×10^{-5}	2.19184×10^{-5}	8.06330×10^{-6}
$e_{2,i}$	1.22236×10^{-8}	4.49290×10^{-9}	1.65164×10^{-9}	5.52972×10^{-10}	0

The hypotheses of Proposition 6 are strong. However, Problem ($\mathcal{P}2$) can be expressed as an optimization one which gives a method to calculate the control $(u_i)_{i \geq 0}$ with weaker hypotheses. For every $i \geq 0$, set

$$J_i(u_i) = \|e_{i+1} - CDf_i\|^2 + \|u_i\|^2 \tag{23}$$

($\mathcal{P}2$) can be considered as equivalent to the problem of finding, at every instant i, a control $u_i^* \in U$ that minimizes the criterion J_i given by (23). ♦

Proposition 7. *Assume that the operator CD is injective. Then the control $(u_i^*)_{i \geq 0}$ is given by the following recurrence relations:*

$$\begin{cases} u_0^* = 0 \\ f_0 = (D^*C^*CD)^{-1} D^*C^* e_1 \end{cases}$$

$$\begin{cases} u_i^* = -(I + B^*C^*CB)^{-1} B^*C^* \left(\sum_{j=0}^{i-1} CA^{i-j}Df_j + \sum_{j=0}^{i-1} CA^{i-j}Bu_j^* \right) \\ f_i = (D^*C^*CD)^{-1} D^*C^* \left(e_{i+1} - \sum_{j=0}^{i-1} CA^{i-j}Df_j - \sum_{j=0}^{i} CA^{i-j}Bu_j^* \right), \quad i \geq 1 \end{cases}$$

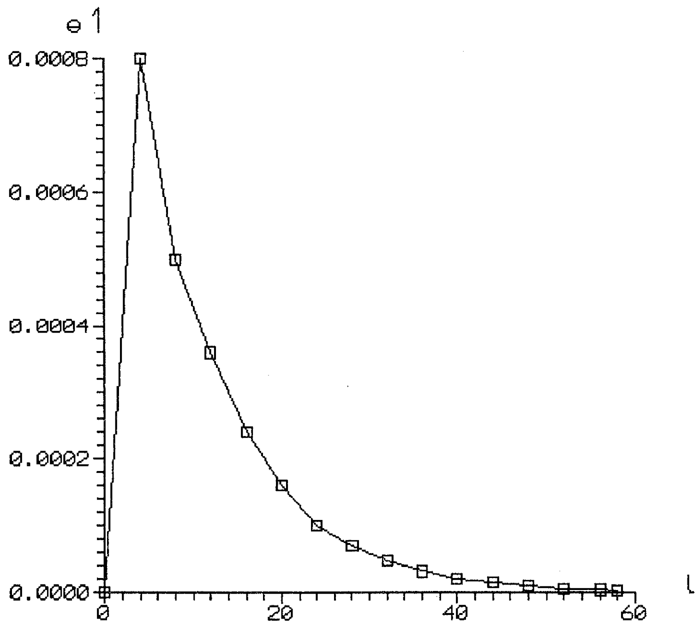

Fig. 4. Approximation of e_1.

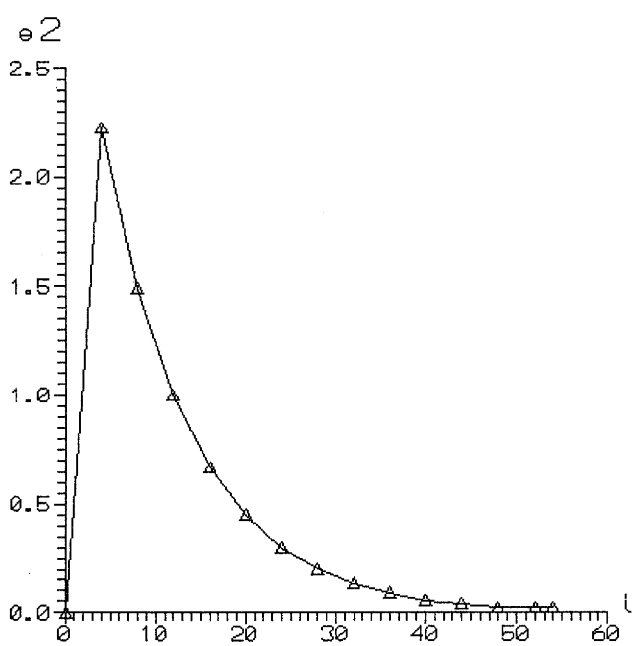

Fig. 5. Approximation of e_2.

Proof. We have

$$y_1 = CAx_0 + CDf_0 + CBu_0 \tag{24}$$

Hence $J_0(u_0) = \|u_0\|^2 + \|CBu_0\|^2$ and therefore $u_0^* = 0$. By (24), $CDf_0 = e_1$ and hence $f_0 = (D^*C^*CD)^{-1} D^*C^*e_1$.

Assume that u_j^* and f_j, $j = 0, \dots, i-1$ have been calculated. We have

$$x_{i+1} = A^{i+1}x_0 + Df_i + \sum_{j=0}^{i-1} A^{i-j}Df_j + \sum_{j=0}^{i-1} A^{i-j}Bu_j^* + Bu_i$$

Hence

$$e_{i+1} - CDf_i = \sum_{j=0}^{i-1} CA^{i-j}Df_j + \sum_{j=0}^{i-1} CA^{i-j}Bu_j^* + CBu_i \tag{25}$$

Therefore

$$\begin{aligned} J_i(u_i) &= \left\| \sum_{j=0}^{i-1} CA^{i-j}Df_j + \sum_{j=0}^{i-1} CA^{i-j}Bu_j^* \right\|^2 + \langle (I + B^*C^*CB)\, u_i, u_i \rangle \\ &\quad + 2 \left\langle B^*C^* \left(\sum_{j=0}^{i-1} CA^{i-j}Df_j + \sum_{j=0}^{i-1} CA^{i-j}Bu_j^* \right), u_i \right\rangle \end{aligned}$$

The minimum u_i^* of J_i satisfies

$$(I + B^*C^*CB)\, u_i^* = -B^*C^* \left(\sum_{j=0}^{i-1} CA^{i-j}Df_j + \sum_{j=0}^{i-1} CA^{i-j}Bu_j^* \right)$$

Hence

$$u_i^* = -\,(I + B^*C^*CB)^{-1} B^*C^* \left(\sum_{j=0}^{i-1} CA^{i-j}Df_j + \sum_{j=0}^{i-1} CA^{i-j}Bu_j^* \right)$$

and, by (25), we have

$$f_i = (D^*C^*CD)^{-1} D^*C^* \left(e_{i+1} - \sum_{j=0}^{i-1} CA^{i-j}Df_j - \sum_{j=0}^{i} CA^{i-j}Bu_j^* \right)$$

■

Example 4. The system (14)–(15) of Example 2 can be rewritten in the product space $X = L^2(]0,1[) \times \mathbb{R}^{j+1}$ as follows:

$$\begin{cases} x_{i+1} = Ax_i + Df_i, & i \geq 0 \\ x_0 \in X \end{cases} \tag{26}$$

$$y_i = Cx_i, \quad i \geq 1 \tag{27}$$

where

$$A = \begin{pmatrix} A_0 & 0 & \cdots & 0 & D_j & D_{j+1} \\ 0 & 0 & \cdots & \cdots & \cdots & 0 \\ \vdots & I & \ddots & & & \vdots \\ \vdots & 0 & \ddots & \ddots & & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & I & 0 \end{pmatrix}, \quad D = \begin{pmatrix} D_0 \\ I \\ 0 \\ \vdots \\ \vdots \\ 0 \end{pmatrix}$$

and

$$C = (C_0 \ \underbrace{0\,0 \cdots 0\,0}_{j+1 \text{ times}})$$

Let $U = \mathbb{R}^2$ and assume that the input operator $B \in \mathcal{L}(U, X)$ is given by

$$B : \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \longmapsto \begin{pmatrix} \alpha b_1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$$

It is easy to check that the operator CD is one-to-one. As a numerical application we assume that $f_i = e^{-i/10}$ and $x_0 = 0$. In Table 4 we give different values of the error e_i obtained when the system is excited by the control $(u_i^*)_i$ given in Proposition 7. Figure 6 shows the same results graphically. ♦

Table 4. Approximation of e_i.

i	10	20	40	80	120	140
e_i	0.086648	0.032527	0.004403	0.000806	0.000001	0.000000

4. Conclusion

In this work, we have investigated the problems of detection and regulation for discrete-time delayed systems in Hilbert spaces. In all the presented results we have assumed that the involved operators are linear. We think that it will be interesting to study the detection and regulation problems for nonlinear systems. Such systems require different approaches from those used in this work. We think that the fixed-point technique can be a useful tool to resolve the detection and regulation problems for nonlinear systems. This possibility is now under investigation.

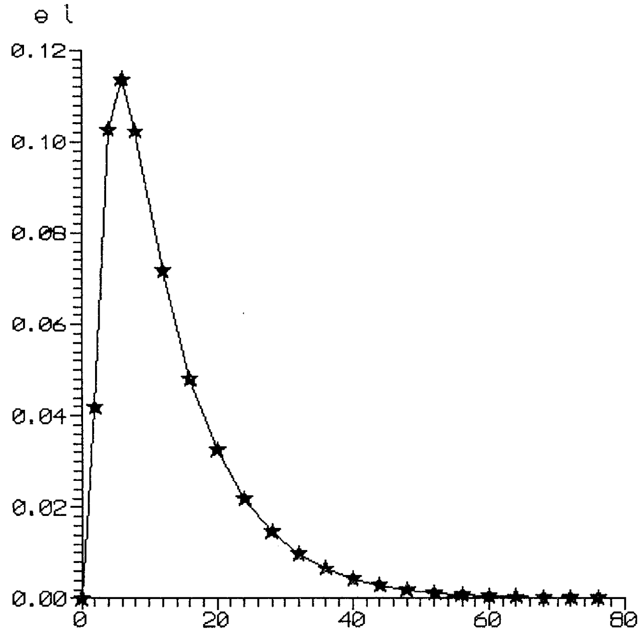

Fig. 6. Approximation of e_i.

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