Int. J. Appl. Math. Comput. Sci., 2001, Vol.11, No.4, 835–858

ROBUST IDENTIFICATION OF PARASITIC FEEDBACK DISTURBANCES FOR LINEAR LUMPED PARAMETER SYSTEMS

VYACHESLAV MAKSIMOV*, LUCIANO PANDOLFI**

We study the problem of identification of an input to a linear finite-dimensional system. We assume that the input has a feedback form, which is related to a problem often encountered in fault detection. The method we use is to embed the identification problem in a class of inverse problems of dynamics for controlled systems. Two algorithms for identification of a feedback matrix based on the method of feedback control with a model are constructed. These algorithms are stable with respect to noise-corrupted observations and computational errors.

Keywords: input identification, feedback control, fault detection

1. Introduction

Our basic assumption throughout the paper is that we know the matrices C and A of a given linear finite-dimensional system

$$\dot{x} = Ax + u, \qquad y = Cx, \tag{1}$$

where $x \in \mathbb{R}^q$ and A is a $q \times q$ matrix, $y \in \mathbb{R}^p$, and C is a $p \times q$ matrix. However, the initial condition $x_0 = x(0)$ is unknown. The q-vector u describes a disturbance which is unknown and which we want to estimate based on the measurements taken during the evolution of the system. Moreover, we assume that the disturbance u has a feedback form, i.e.

$$u = Fx, \tag{2}$$

due to unmodelled components of systems, e.g. due to parasitic couplings or viscous dampings. What is more, unknown inputs of the form (2) are encountered in the problem of *fault detection:* it may happen that the nominal value of u is 0 while $u = Fx \neq 0$ is due to a failure of an internal component of the system, e.g. an interconnection which transforms to zero some component of the matrix A.

* Institute of Mathematics and Mechanics, Ural Branch, Academy of Sciences of Russia, S. Kovalevskaya Str. 16, 620219, Ekaterinburg, Russia, e-mail: maksimov@imm.uran.ru

** Politecnico di Torino, Dipartimento di Matematica, Corso Duca degli Abruzzi 24, I–10129 Torino, Italy, e-mail: lucipan@polito.it

We formulate the following problem: Let the values of $y(t)$ be read on $[0,T]$ at discrete time instants $\tau_i = iT/n$. The measurements are affected by errors so that at time τ_i we obtain a vector

$$\xi_i = Cx(\tau_i) + z_i, \qquad \|z_i\| < h, \tag{3}$$

where h is a prescribed tolerance. The aim of this paper is to propose a method for the reconstruction of the input $u(\cdot)$ within a certain tolerance μ, i.e. we want to construct a function $v(\cdot)$ on $[0,T]$ such that

$$\|u_*(t) - v(t)\|_{L^2(0,T)} \leq \mu,$$

on the basis of the information (2). Here $u_*(\cdot)$ is a function which produces the same output as $u(\cdot)$. We shall see that $u_*(\cdot)$ turns out to have a feedback form so that $v(\cdot)$ is (within a certain tolerance) a feedback too.

Summarizing, the goal of this paper is to solve the identification problem for unknown parameters of system (1), (2). The literature on this subject, for both continuous and discrete-time systems, is abundant, see, e.g. (Unbehauen, 1990) and the references therein.

We shall present two algorithms for identification of the matrix F. The reconstruction procedure that we use in the first case is inspired by the methods used in (Osipov and Kryazhimskii, 1995) for input reconstruction when the input belongs to a known convex bounded closed set of $L^2(0,T;\mathbb{R}^n)$. We note that our feedback input $u(t) = Fx(t)$ indeed belongs to a bounded subset of $L^2(0,T;\mathbb{R}^n)$, but this subset is not known, since it depends on the initial datum $x(0)$ and, what is most important, on the unknown feedback F as well. Hence this set will not enter directly into the reconstruction process until the proof of Theorem 9.

The second algorithm is based on constructions from (Blizorukova and Maksimov, 1997; Kryazhimskii, 1999; Kryazhimskii and Osipov, 1987; Kryazhimskii *et al.*, 1997) and, in essence, it also uses the principle of feedback control with a model. It assumes that F belongs to a known set (which is not restrictive from the point of view of fault detection) and that $C = I$.

2. The First Method

The presentation of the first identification method is split into several parts, each in a subsection, whose results are of independent interest.

2.1. A Simplified Version of the Problem

As has already been stated, we assume that we know the $q \times q$ matrix A of the linear system

$$\dot{x} = Ax + u, \qquad u(t) = Fx(t), \qquad y = Cx, \tag{4}$$

while the feedback matrix F is unknown. Let us now observe that the matrices A and F in (4) play similar roles. Hence, instead of (4), we can consider the simplified problem

$$\dot{x} = \tilde{u}, \qquad y = Cx, \tag{5}$$

$$\tilde{u}(t) = (A + F)x(t). \tag{6}$$

It is clear that, once $\tilde{u}(\cdot)$ is identified from (5), the quantity $\tilde{x}(\cdot)$ is known and we can obtain $F\tilde{x}(\cdot)$ as $\tilde{u}(\cdot) - A\tilde{x}(\cdot)$. Thus it is not restrictive to assume that $A = 0$, and we adopt this assumption in what follows.

We used $\tilde{u}(\cdot)$ to denote the special input $(A + F)x(\cdot) = Fx(\cdot)$. For clarity, we shall use the redundant symbol $\tilde{x}(\cdot) = \tilde{x}(\cdot; x_0)$ to denote the solution of the differential equation (5), and $\tilde{y} = C\tilde{x}$.

We observe that the feedback control (6) is smooth and bounded. So, our first step will be the study of a more general problem: we shall investigate the problem of approximation of an input $\tilde{u}(\cdot)$ (equivalently, the approximation of $\mathrm{d}\tilde{x}(\cdot)/\mathrm{d}t$) which acts on system (5), on the assumption that it is bounded, differentiable and with bounded derivative (the boundedness of the derivative is used in Section 2.3 in order to obtain quantitative estimates) without any reference to its feedback form. This will be outlined in Sections 2.2 and 2.3. In Sections 2.4 and 3, we shall make explicit use of the assumption that $\tilde{u}(\cdot)$ has a feedback form, and we shall present two (non-recursive!) procedures for the approximation of the matrix F (or, more general, $A + F$).

2.2. The Reconstruction Procedure

In this section we shall study the simplified problem described by (5) with bounded input $\tilde{u}(\cdot)$. We associate the following auxiliary model to system (5):

$$\dot{w} = v, \qquad w(0) = w_0, \qquad r = Cw. \tag{7}$$

Moreover, we choose any w_0 such that

$$\|Cx_0 - Cw_0\| \le h.$$

The idea of the reconstruction process is as follows: We fix a number n and the observation instants $\tau_i = iT/n$. We assume that the input $\tilde{u}(\cdot)$ was estimated on $[0, \tau_i)$. In order to estimate the input on the next interval $[\tau_i, \tau_{i+1})$, we feed a test input $v(\cdot)$ to the auxiliary system (7) and we compare its output with the measured output of the given system (5). Among all the possible inputs $v(\cdot)$ on $[\tau_i, \tau_{i+1})$, we choose the one that reduces a certain functional of the error as much as possible, as described below.

We introduce the functional

$$\epsilon(t) = \|r(t) - \tilde{y}(t)\|^2 + \alpha \int_0^t [\|v(s)\|^2 - \|u_*(s)\|^2] \,\mathrm{d}s,$$

where $w(t) = w(t; \xi_0, v)$, and $u_*(\cdot)$ denotes the input of minimal L^2-norm, which gives the output $\tilde{y}(\cdot)$. Clearly, $u_*(t)$ is, for every t, the projection of $\tilde{u}(t)$ on $[\ker C]^{\perp}$.

Hence $u_*(\cdot)$ is bounded on $[0,T]$. The values of $\epsilon(t)$ cannot be computed since they depend on the unknown values of $u_*(\cdot)$ and $\tilde{x}(\cdot)$. We try to choose $v(\cdot)$ in such a way that $\epsilon(\tau_i)$ satisfies a difference equation of the following form:

$$\epsilon(\tau_{i+1}) = \epsilon(\tau_i) + \mathrm{O}(\delta^2) + \mathrm{O}(\delta h).$$

If this can be achieved, then it will be possible to proceed as in (Osipov and Kryazhimskii, 1995), in order to prove that the input $v(\cdot)$ so chosen approximates $u_*(\cdot) = Fx(\cdot)$, see Section 2.3.

Let us represent $\epsilon(\tau_{i+1})$ as follows:

$$\begin{aligned}
\epsilon(\tau_{i+1}) = {} & \left\| r(\tau_i) - \tilde{y}(\tau_i) + C\int_{\tau_i}^{\tau_{i+1}} [v(s) - u_*(s)]\,\mathrm{d}s \right\|^2 + \alpha \int_0^{\tau_i} [\|v(s)\|^2 - \|u_*(s)\|^2]\,\mathrm{d}s \\
& + \alpha \int_{\tau_i}^{\tau_{i+1}} [\|v(s)\|^2 - \|u_*(s)\|^2]\,\mathrm{d}s = \epsilon(\tau_i) + \left\| \int_{\tau_i}^{\tau_{i+1}} C[v(s) - u_*(s)]\,\mathrm{d}s \right\|^2 \\
& + \left\{ \int_{\tau_i}^{\tau_{i+1}} \{2\langle C^*[r(\tau_i) - \tilde{y}(\tau_i)], v(s) - u_*(s)\rangle + \alpha[\|v(s)\|^2 - \|u_*(s)\|^2]\}\,\mathrm{d}s \right\} \\
= {} & \epsilon(\tau_i) + \left\| \int_{\tau_i}^{\tau_{i+1}} C[v(s) - u_*(s)]\,\mathrm{d}s \right\|^2 + \int_{\tau_i}^{\tau_{i+1}} 2\langle C^*[\xi_i - \tilde{y}(\tau_i), v(s) - u_*(s)\rangle\,\mathrm{d}s \\
& + \left\{ \int_{\tau_i}^{\tau_{i+1}} \{2\langle C^*[r(\tau_i) - \xi_i], v(s) - u_*(s)\rangle + \alpha[\|v(s)\|^2 - \|u_*(s)\|^2]\}\,\mathrm{d}s \right\}. \qquad (8)
\end{aligned}$$

Here and below the symbol $\langle \cdot, \cdot \rangle$ stands for the scalar product in $\mathbb{R}^q$. We choose $v(\cdot)$ on the interval $[\tau_i, \tau_{i+1})$ in such a way that the quantity in the last braces is negative. Hence, we choose $v(\cdot)$ in such a way that

$$v_{|[\tau_i,\tau_{i+1})}(\cdot) = \arg\min \int_{\tau_i}^{\tau_{i+1}} \{2\langle C^*[r(\tau_i) - \xi_i], v(s)\rangle + \alpha\|v(s)\|^2\}\,\mathrm{d}s.$$

It is easily seen that this minimum exists, since the quadratic functional is coercive, and that

$$v_{|[\tau_i,\tau_{i+1}]}(s) = -\frac{1}{\alpha} C^*[r(\tau_i) - \xi_i], \qquad s \in [\tau_i, \tau_{i+1}]. \qquad (9)$$

This shows that the input $v(\cdot)$, a candidate approximation of $u_*(\cdot)$, is piecewise constant on $[0,T]$.

As has already been stated, the procedure just described is thoroughly analyzed in (Osipov and Kryazhimskii, 1995) and used in several papers, see, e.g. (Maksimov and Pandolfi, 1995), but with an essential difference: in those papers a convex compact set U_* in which $u_*(\cdot)$ took its values was known *a priori*. In that case, the values of $v(\cdot)$ were taken in the same set. (See (Fagnani and Pandolfi, 2000) for a (different) approach, where boundedness is not used.) In turn, in the present setup the input $u_*(\cdot)$ is bounded, but its norm is unknown. So we must give *a priori* estimates on the norm of $v(\cdot)$, only on the basis of the minimization procedure described above.

Lemma 1. *Let $u_*(\cdot)$ be bounded. Then there exists a number N such that*

$$\|\tilde{x}(t;x_0)\| \le N\|x_0\|, \qquad \|\xi_i\| \le N\|x_0\| + h. \tag{10}$$

Moreover, with $\delta = 1/n$, we have

$$\|\tilde{x}(\tau_{i+1};x_0) - \tilde{x}(\tau_i;x_0)\| \le N\delta.$$

Note that as the number N depends on x_0 and $u_*(\cdot)$, it cannot be computed but it does *exist.*

The vector x_0 is fixed, its norm being less than $||\xi_1|| + h$, so that we can replace $N||x_0||$ with a new constant N. In fact, the symbol N in the following will be used to denote a number which may be *unknown,* and which may depend on x_0 and $u_*(\cdot)$. The important thing is its *existence*, and the fact that it does not depend on the values of n, h and α.

In conclusion, we write the inequalities in (10) as $\|x(t;x_0,\tilde{u})\| \le N$ and $\|\xi_i\| \le N$. Now we replace the expression that we found in (9) for $v(\cdot)$ in (7). With $\delta = 1/n$ we obtain

$$w(\tau_{i+1}) = w(\tau_i) - \frac{1}{\alpha}\int_{\tau_i}^{\tau_{i+1}} C^*[r(\tau_i) - \xi_i]\,\mathrm{d}s = w(\tau_i) - \frac{\delta}{\alpha}C^*[r(\tau_i) - \xi_i],$$

so that

$$r(\tau_{i+1}) - \xi_{i+1} = \left[I - \frac{\delta}{\alpha}CC^*\right][r(\tau_i) - \xi_i] + \xi_i - \xi_{i+1}.$$

An orthogonal coordinate transformation in the output space transforms CC^* to the form $CC^* = \mathrm{diag}[H, 0]$, where H is diagonal and its eigenvalues are larger than a certain number $\rho > 0$. This transformation does not affect the norm of the vectors $\eta_i = r(\tau_i) - \xi_i$. We represent $\eta = \mathrm{col}[\eta', \eta'']$ in the same manner and get

$$\eta'(\tau_{i+1}) = \left(1 - \frac{\delta H}{\alpha}\right)\eta'(\tau_i) + \xi_i' - \xi_{i+1}',$$

$$\eta''(\tau_{i+1}) = \eta''(\tau_i) + \xi_i'' - \xi_{i+1}''.$$

This implies

$$\|\eta'(\tau_{i+1})\| \le \|\eta'(0)\| + \sum_{k=0}^{i}\left(1 - \frac{\rho\delta}{\alpha}\right)^{i-k}\|\xi_k' - \xi_{k+1}'\| \le h + N\alpha[1 + nh],$$

$$\|\eta''(\tau_{i+1})\| \le \|\eta''(0)\| + \sum_{k=0}^{i}\|\xi_k' - \xi_{k+1}''\| \le h + Nn(\delta + h) \le h + N(1 + nh).$$

It follows that $\{\eta(\tau_i)\}$ is a bounded sequence uniformly with respect to n, h and α if the reconstruction algorithm is chosen so as to satisfy the following rule:

$$\begin{cases} \text{We fix any } M > 0 \text{ and, for any } n, \\ \text{choose } h = h_n > 0 \text{ such that } nh < M. \\ \text{Moreover, we impose the restriction } h/\alpha < 1. \end{cases} \tag{11}$$

Lemma 2. *Let (11) hold. Then there exists a number N (independent of h, n and α as long as the condition $nh < M$ holds) such that*

$$\|r(\tau_{i+1}) - \xi_i\| \le N, \qquad \|C^*[r(\tau_i) - \xi_i]\| \le h + N\alpha.$$

Lemma 3. *Let (11) hold. The input $v(\cdot)$ constructed from (9) satisfies the following inequality on $[0,T]$:*

$$\|v(s)\| \le N, \tag{12}$$

where N does not depend on n, h and α.

Lemma 4. *Let (11) hold. Then there exists a constant N such that for all $t \in [0,T]$ we have*

$$\|v(t)\| \le N, \qquad \|w(t)\| \le N, \qquad \|w(t) - w(t')\| \le N|t - t'|.$$

Now we complete our estimate of $\epsilon(t)$. For that purpose, we go back to (8). The control $v(\cdot)$ was chosen such that the expression braces is negative. Hence we have

$$\epsilon(\tau_{i+1}) \le \epsilon(\tau_i) + \left\| \int_{\tau_i}^{\tau_{i+1}} C[v(s) - u_*(s)]\,\mathrm{d}s \right\|^2 + \int_{\tau_i}^{\tau_{i+1}} \{2\langle C^*[\xi_i - \tilde{y}(\tau_i), v(s) - u_*(s)\rangle\,\mathrm{d}s.$$

We use estimate (12), condition (11) and the pointwise boundedness of $u_*(\cdot)$ (hence that of $u_*(\cdot)$ too). It follows that the first integral is less than const $\cdot\delta^2$. In turn, the last term is less than const $\cdot\delta h$. This gives the required estimate of $\epsilon(t)$ for $t = \tau_i$.

Consequently, as in (Maksimov and Pandolfi, 1995), we have the following result:

Lemma 5. *There exist positive numbers c and d such that $\epsilon(\tau_i) \le c\delta + dh$. These numbers do not depend on n, h and α as long as (11) holds.*

Now, we extend the previous inequality from the numbers τ_i to every $t \in [0,T]$.

Theorem 1. *Let (11) hold. Then there exist constants c and d which do not depend on n, α and h such that $\epsilon(t) \le c\delta + dh$.*

Proof. We estimate $\epsilon(t)$ for $t \in [\tau_i, \tau_{i+1})$ as follows:

$$\begin{aligned}\epsilon(t) &= \|r(t) - \tilde{y}(t)\|^2 + \alpha \int_0^t [\|v(s)\|^2 - \|u_*(s)\|^2]\,\mathrm{d}s \\ &\le \epsilon(\tau_i) + \| \int_{\tau_i}^t C[v(s) - u_*(s)]\,\mathrm{d}s\|^2 \\ &\quad + 2\Big\langle r(\tau_i) - \tilde{y}(\tau_i), \int_{\tau_i}^t C[v(s) - u_*(s)]\,\mathrm{d}s \Big\rangle + \alpha \int_{\tau_i}^t [\|v(s)]\|^2 - \|u_*(s)\|^2]\,\mathrm{d}s.\end{aligned}$$

The conclusion follows from the boundedness of $v(\cdot)$ and $u_*(\cdot)$. ■

Now we have a certain function $v(\cdot)$. We have presented it as an 'approximant' of $u_*(\cdot)$, but we have not justified this claim yet. This will be done in the next section.

2.3. Input Identification

In the previous section we have constructed a certain input $v(\cdot)$ and asserted that it is an 'approximation' of $u_*(\cdot)$. Now we justify this claim. We maintain our assumption (11) so that we can use all the estimates obtained in the previous section. In particular, we know that $v(\cdot)$ satisfies the estimate (12). From (11) we get the existence of a constant N, which does not depend on n, h and α, such that $\|v(t)\| \le N$, see Lemma 4, and that $u_*(\cdot)$ is bounded too. Hence the integral $\int_0^t [\|v(t)\|^2 - \|u_*(t)\|^2]\,\mathrm{d}t \ge -e$ (where $e \ge 0$) does not depend on n, h and α, so that we have the following result:

Theorem 2. *Let the algorithm satisfy (11). Then, for each* $t \in [0,T]$, *we have*

$$\|r(t) - \tilde{y}(t)\|^2 \le c\delta + dh + e\alpha, \tag{13}$$

$$\int_0^t \{\|v(s)\|^2 - \|u_*(s)\|^2\}\,\mathrm{d}s \le (c\delta + dh)\alpha^{-1}.$$

The previous formulae can be used as in (Maksimov and Pandolfi, 1995) in order to prove that if $\alpha = \alpha_n \to 0$ and $h = h_n \to 0$, then $v_n(\cdot) = v(\,\cdot\,; 1/n, h_n, \alpha_n)$ converges to $u_*(\cdot)$ in $L^2(0,T)$, provided that the algorithm satisfies the additional condition

$$(\delta + h_n){\alpha_n}^{-1} \to 0. \tag{14}$$

But we want to obtain some quantitative estimates. Thus, at this point, we use the further property that $u_*(\cdot)$ is continuously differentiable with bounded derivatives. This is clearly satisfied if $\tilde{u}(\cdot)$ is of class C^1, since the regularity of $\tilde{u}(\cdot)$ is inherited by its projection $u_*(\cdot)$ on $[\ker C]^\perp$. We note explicitly that this regularity assumption on $\tilde{u}(\cdot)$ has not been used yet. Of course, the assumption of differentiability is satisfied when the input has a feedback form.

As we are looking for convergence estimates, we can work on any suitable basis in both the state and output spaces. In fact, invertible changes of coordinates correspond to the use of different but equivalent norms. Hence we use new reference systems in the state and output spaces such that the output operator C takes the form $C = [\ I \quad 0\ ]$, and we recall that both $v(\cdot)$ and $u_*(\cdot)$ take values in $[\ker C]^\perp$ so that, in the coordinate systems just described, we have $\|Cv(t)\| = \|v(t)\|$ and $\|Cu_*(t)\| = \|u_*(t)\|$.

Now we estimate $\|v(\cdot) - u_*(\cdot)\|^2_{L^2(0,T)}$. We use the same argument as in (Maksimov and Pandolfi, 1995, Sec. 5) and the following lemma, which is a special instance of (Osipov and Kryazhimskii, 1995; Maksimov, 1994):

Lemma 6. *Let* $f(\cdot)$ *and* $g(\cdot)$ *be two vector-valued functions defined on* $[0,T]$. *Let* ν *be a number such that*

$$\left\| \int_0^t f(s)\,\mathrm{d}s \right\| \le \nu.$$

Assume that $g(\cdot)$ *is continuously differentiable, with* $\|g(t)\| \le N$ *and* $\|g'(t)\| \le N$ *on* $[0,T]$. *Then we have*

$$\left\| \int_0^t g^*(s) f(s)\,\mathrm{d}s \right\| \le 2N\nu.$$

We use this lemma in order to estimate $\|Cv(\cdot) - Cu_*(\cdot)\|_{L^2(0,T)}$. We obtain

$$\begin{aligned}\|Cv(\cdot)-Cu_*(\cdot)\|^2_{L^2(0,T)} &= \|Cv(\cdot)\|^2_{L^2(0,T)}+\|Cu_*(\cdot)\|^2_{L^2(0,T)}-2\int_0^T [Cu_*(s)]^* Cv(s)\,\mathrm{d}s \\ &\le 2\|Cu_*(\cdot)\|^2_{L^2(0,T)}-2\int_0^T [Cu_*(s)]^* Cv(s)\,\mathrm{d}s+(c\delta+dh)\alpha^{-1} \\ &\le 2\int_0^T [Cu_*(s)]^* C[u_*(s)-v(s)]\,\mathrm{d}s+(c\delta+dh)\alpha^{-1}.\end{aligned}$$

The function $u_*(\cdot)$ is *bounded* and *continuously differentiable,* with *bounded derivative.* Moreover,

$$\int_0^t C[u_*(s)-v(s)]\,\mathrm{d}s = [\tilde{y}(t)-r(t)] - [Cx(0)-Cw(0)].$$

We use (13) and obtain

$$\left\| \int_0^t C[u_*(s)-v(s)]\,\mathrm{d}s \right\| \le \|\tilde{y}(t)-r(t)\| + \|y(0)-r(0)\| \le \sqrt{c\delta+dh+e\alpha}+h.$$

Applying Lemma 6 with $g(s)=u_*(s)$ and $f(s)=u_*(s)-v(s)$, we get the existence of a constant N which does not depend on n, h and α, such that

$$\left\| \int_0^T [u_*(s)-v(s)]^* u_*(s)\,\mathrm{d}s \right\| \le N\{\sqrt{c\delta+dh+e\alpha}+h\},$$

see (Osipov and Kryazhimskii, 1995). We write

$$\sigma^2 = \sqrt{c\delta+dh+e\alpha}+h+\frac{c\delta+dh}{\alpha}. \tag{15}$$

Combining the previous inequalities, we get the following result:

Theorem 3. *Let the algorithm satisfy conditions (11) and (14), and let* $\tilde{u}(\cdot)$ *be of class* C^1. *There exists a constant* N *which does not depend on* n, h *and* α *such that the input* $v(\cdot)$ *constructed in the previous section satisfies*

$$\|v(\cdot)-u_*(\cdot)\|_{L^2(0,T)} \le N\sigma.$$

The previous result holds under the regularity assumption that $\tilde{u}(\cdot)$ is a C^1 function (even a function of bounded variation would do). The fact that $\tilde{u}(\cdot)$ has a feedback form has not been used in its full strength. Now we observe that if $\tilde{u}(\cdot)$ is a feedback, then $u_*(\cdot)$, being the pointwise projection of $\tilde{u}(\cdot)$ on $[\ker C]^\perp$, has a feedback form too. Hence we obtain the following result:

Theorem 4. *Let $u(t) = Fx(t)$ and the chosen algorithm satisfy the prescribed compatibility conditions (11) and (14). Under these conditions, $v_0(\cdot) = \lim v(\cdot)$ exists in $L^2(0,T)$ as $n \to +\infty$, $h \to 0$ and $\alpha \to 0$. Moreover, $v_0(\cdot)$ is a feedback control, $v_0(t) = u_*(t) = PFx(t)$, where P is the orthogonal projection on $[\ker C]^\perp$.*

2.4. The First Algorithm of Reconstructing the Feedback Matrix

In the previous sections we have presented a *recursive* algorithm that identifies the input of a linear finite-dimensional system, on the assumption that the input signal is in a feedback form, see Theorem 4. In this case the identified input is in a feedback form too. The algorithm is based on a fixed evolution of the system, as produced by a fixed (and only partially known) initial condition. Of course, different initial conditions produce different evolutions of the system, so that if it happens that the initial condition x_0 is an eigenvector of the matrix $(A + F)$ in (5), then in fact we identify the 'input' $e^{\lambda t}x_0$, where λ is the corresponding eigenvalue. In order to identify as many coefficients of the feedback matrix as possible, we need to have the possibility of performing several experiments on the system. Hence, in this section, we assume that this is possible, make explicit use of the fact that the input $u_*(\cdot)$ has a feedback form, $u_*(t) = F\tilde{x}(t)$, and present an identification procedure for the matrix F. (In this section we use F for the general feedback matrix. It will be, for example, $u_*(t) = P(A+F)\tilde{x}(t)$, where P projects on $[\ker C]^\perp$, and A, F are the matrices of Section 2.1)

We keep the assumptions (11) and (14) so that we can use the estimates presented in the previous sections. We recall them explicitly as

$$h_n \to 0, \quad \alpha_n \to 0, \quad h_n/\alpha_n \to 0, \quad 1/n\alpha_n \to 0.$$

These conditions imply that $\sigma_n \to 0$, see its definition (15).

We denote by $w_n(\cdot)$ and $v_n(\cdot)$ the functions which are constructed at iteration n. Later on, we must also explicitly indicate the initial value, say w_0, of $w_n(\cdot)$. We shall thus write $w_n(\cdot; w_0)$ and $v_n(\cdot; w_0)$.

For simplicity, we proceed in two steps and, as in the previous sections, we perform coordinate transformations in the state and output spaces, which reduces the matrix C to the form $C = [\ I \quad 0\ ]$. In this way, the system can be represented as

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix} x' \\ x'' \end{bmatrix} = \begin{bmatrix} F' & F'' \\ G' & G'' \end{bmatrix}\begin{bmatrix} x' \\ x'' \end{bmatrix}, \quad y = x', \quad x'(0) = x_0', \ x''(0) = x_0''$$

for suitable matrices F', F'', G', G''. Clearly, we cannot hope to identify *every* entry of the matrix F. Hence we represent the first component of the system in the form

$$\dot{x}' = F'x' + g(t; x_0', x_0''), \qquad y = x'$$

and, correspondingly,

$$\dot{w}' = v', \qquad r = w'.$$

We prove first of all that if it happens that $F''=0$, then we can approximate F'. The analysis of this case is equivalent to that of a system with full-state observation. For notational simplicity, we suppress the primes and denote it as

$$\dot{x} = Fx, \qquad y = x.$$

2.4.1. Full State Observation

Recall the L^2-convergence of $\{v_n(\cdot)\}$ to $u_*(\cdot)$. Moreover, there exists a number N, independent of n, h_n and α_n, and a sequence $\{\chi_n\}$ which converges to 0 such that

$$\|Fw_n(\cdot) - v_n(\cdot)\|_{L^2(0,T)} \le \chi_n. \tag{16}$$

This follows from the estimate

$$\begin{aligned}\|Fw_n(\cdot) - v_n(\cdot)\|_{L^2(0,T)} &\le \|F[w_n(\cdot) - \tilde{x}(\cdot)]\|_{L^2(0,T)} + \|F\tilde{x}(\cdot) - v_n(\cdot)\|_{L^2(0,T)} \\ &\le \|F[w_n(\cdot) - \tilde{x}(\cdot)]\|_{L^2(0,T)} + \|u_*(\cdot) - v_n(\cdot)\|_{L^2(0,T)} \\ &\le N\{\sigma_n + c\delta + dh + e\alpha\} = \chi_n,\end{aligned}$$

from (13), Theorem 3 and the fact that we know $u_*(\cdot) = \tilde{u}(\cdot)$, since we consider $C = I$.

Remark 1. Observe that $\chi_n = \chi_n(x_0)$ depends on the initial condition x_0.

Now we present the reconstruction of the matrix F. We note that the set of the $q\times q$ matrices can be considered as a normed space in many *equivalent* ways. We choose to consider it as the Euclidean space $\mathbb{R}^{q\times q}$.

We compute the matrix $\Phi = \Phi_n$ which is the element of minimal norm in the set of those matrices Φ which satisfy (for example)

$$\|\Phi w_n(\cdot) - v_n(\cdot)\|_{L^2(0,T)} \le \sqrt{\chi_n}. \tag{17}$$

The set of the matrices which satisfy (17) is not empty, it is convex and closed in $\mathbb{R}^{q\times q}$ so that the element Φ_n of minimal norm exists and is unique. Moreover, from (16), we have $\|\Phi_n\| \le \|F\|$ so that the sequence $\{\Phi_n\}$ is bounded.

Theorem 5. *Let $\{\Phi_{n_k}\}$ be a subsequence of $\{\Phi_n\}$ which converges to Φ_0. We have $\Phi_0\tilde{x}(t) = F\tilde{x}(t) = \tilde{u}(t) = u_*(t)$ a.e. on $[0,T]$.*

Proof. Indeed, we have

$$\begin{aligned}\Phi_0\tilde{x}(t) - u_*(t) = [\Phi_0 - \Phi_{n_k}]\tilde{x}(t) + \big\{\Phi_{n_k}[\tilde{x}(t) - w_{n_k}(t)] \\ + [\Phi_{n_k} w_{n_k}(t) - v_{n_k}(t)] + [v_{n_k}(t) - u_*(t)]\big\}.\end{aligned}$$

Each of the four terms tends to zero (the last one in L^2-norm, the previous ones uniformly), which establishes the desired conclusion. ■

The previous result can be interpreted as the assertion that the matrix Φ_0 acts as the matrix F *on the trajectory* $\tilde{x}(\cdot)$ *whose initial condition is* x_0.

Of course, the sequence $\{\Phi_n\}$ will not be convergent in general, and we cannot assert that $\Phi_0 = F$ on the basis of the reconstruction procedure presented above, which uses only *one* trajectory of the system. We shall need several experiments in order to reconstruct the matrix F as follows: we fix a basis $e^{(r)}$, $r = 1, \dots, q$ of $\mathbb{R}^q$ and we repeat the previous construction, i.e. we define Φ_n as the matrix of minimal norm which satisfies the conditions

$$\|\Phi_n w_n(\,\cdot\,;\xi^{(r)}) - v_n(\,\cdot\,;\xi^{(r)})\|_{L^2(0,T)} \le \sqrt{\tilde{\chi}_n}, \qquad r = 1, \dots, q,$$

where $\xi^{(r)}$ is the measure taken on the vector $e^{(r)}$ at time $t = 0$,

$$\|\xi^{(r)} - e^{(r)}\| \le h_n$$

(the vectors $\xi^{(r)}$ are linearly independent if n is large enough), $w_n(\,\cdot\,;\xi^{(r)})$ and $v_n(\,\cdot\,;\xi^{(r)})$ denote the vectors obtained when the initial condition of $\tilde{x}(\cdot)$ is $e^{(r)}$, $\tilde{\chi}_n$ is the maximum of the numbers $\chi_n = \chi_n(e^r)$.

The existence of the matrix Φ_n and the boundedness of the sequence $\{\Phi_n\}$ are seen as above.

Theorem 6. *The sequence* $\{\Phi_n\}$ *constructed above with reference to the basis* $e^{(r)}$ *converges to* F.

Proof. We consider a limit point $\Phi_0(\cdot)$ of the sequence $\{\Phi_n\}$. We see, as in the proof of Theorem 5, that

$$\Phi_0\tilde{x}(t;e^{(r)}) = \tilde{u}(t;e^{(r)}) = u_*(t;e^{(r)}) \quad \text{a.e.} \quad t \in [0,T] \quad \text{and for every } r.$$

It follows that

$$\Phi_0\tilde{x}(t;e^{(r)}) = u_*(t;e^{(r)}) = F\tilde{x}(t;e^{(r)})$$

for each r. Hence $\Phi_0 = F$ since the vectors $\tilde{x}(t;e^{(r)})$, $r \ge 1$ are linearly independent. In particular, the sequence $\{\Phi_n\}$ (is bounded and) has the *unique* limit point F. Thus it converges to F. ■

Remark 2:

- The unicity of the feedback matrix F is explicitly used in this proof.
- Let us discuss the rule (17) for the choice of Φ_n. We choose the condition '$\le \sqrt{\chi_n}$' because we have asymptotically $\sqrt{\chi_n} \le M\chi_n$ for every number M. Hence an exact value of χ_n is not required. If we know a number N such that $\|F\| \le N$, then we can choose equally well $2\chi_n$ in place of $\sqrt{\chi_n}$.

We explicitly note that it is not necessary that each previous experiment be performed starting at time $t = 0$, i.e. that we work with several copies of the same system which evolve in the same interval of time. If we have independent access to the system throughout an additional input, we can leave the system evolving freely on a first interval $[0, T_1]$, starting from condition $e^{(1)}$; then we can use the additional input as a control, and we can transfer the state reached at T_1 to the new 'initial' condition $e^{(2)}$ and observe the evolution on a subsequent interval, and so on.

2.4.2. The General Case

In this section we study the general case of partial state observation. In order to treat this case, we must have some additional information, which we state as follows:

Assumption 1. We assume that we know the initial condition x_0 with a tolerance h, i.e. we assume that we know a vector ξ_0 such that $\|x_0 - \xi_0\| \le h$. Moreover, we assume that we know a number $\hat{N}$ such that $\|F\| \le \hat{N}$.

Remark 3. The assumption that $\hat{N}$ is known is introduced from the beginning of this section only for simplicity of exposition. It is not needed until the proof of Theorem 9, also see Remarks 5 and 6.

After a coordinate transformation as above, it is easily seen that the problem is equivalent to the following one: We have full state observation, $y = x$, but the system has the form

$$\dot{x} = Fx + g(t; x_0', x_0''), \qquad x(0) = x_0', \qquad y = x.$$

The initial condition x_0' is directly observed while the vector x_0'' is known, with a tolerance h, owing to the previous assumption. We consider it as an additional parameter. The function $g(t; x_0', x_0'')$ is linear in x_0' and x_0'', $g(t; x_0', x_0'') = G'(t)x_0' + G''(t)x_0''$. So, the unknown is now the pair consisting of the matrix F and an L^2 matrix-valued function $[\ G'(\cdot) \quad G''(\cdot)\ ]$. We treat the unknowns as elements of $\mathbb{R}^{q\times q} \times L^2(0,T; \mathbb{R}^{q\times q} \times \mathbb{R}^{q\times q})$. The norm in this space is denoted by $\|\cdot\|_H$.

The aim of this section is to identify matrices Φ (constant), $H'(t)$ and $H''(t)$ such that

$$\text{if} \quad \dot{\eta} = \Phi\eta + H'(t)x_0' + H''(t)x_0'', \quad \eta(0) = x_0' \quad \text{then} \quad \eta(t) = y(t) = x(t), \tag{18}$$

$$\|\Phi\| \le \hat{N}, \quad \text{where } \hat{N} \text{ satisfies } \quad \|F\| \le \hat{N}. \tag{19}$$

For clarity and consistency with the previous sections, we use $\tilde{x}$ to denote the solution of

$$\dot{x} = Fx + G'(t)x_0' + G''(t)x_0'', \tag{20}$$

and write $\tilde{y} = \tilde{x}$ for the observation. Moreover, for simplicity, we put now

$$\tilde{u} = Fx + G'(t)x_0' + G''(t)x_0''.$$

The model system is

$$\dot{w} = v.$$

The algorithm in the previous sections identifies a sequence $\{v_n(\cdot)\}$ which converges in $L^2(0,T)$ to $\tilde{u}(\cdot) = F\tilde{x}(\cdot) + G'(\cdot)x_0' + G''(\cdot)x_0''$. The corresponding sequence $\{w_n(\cdot)\}$ converges uniformly to $\tilde{x}(\cdot)$.

Lemma 7. *There exists a sequence* $\{\chi_n\}$ *which converges to zero such that*

$$\|Fw_n(\cdot) + G'(\cdot)x_0' + G''(\cdot)x_0'' - v_n(\cdot)\|_{L_2(0,T)} \leq \chi_n. \tag{21}$$

Proof. In fact, we have

$$\begin{aligned}
&\|Fw_n(\cdot) + G'(\cdot)x_0' + G''(\cdot)x_0'' - v_n(\cdot)\|_{L_2(0,T)} \\
&\quad \leq \|(Fw_n(\cdot) + G'(\cdot)x_0' + G''(\cdot)x_0'') - (F\tilde{x}(\cdot) + G'(\cdot)x_0' + G''(\cdot)x_0'')\|_{L^2(0,T)} \\
&\qquad + \|(F\tilde{x}(\cdot) + G'(\cdot)x_0' + G''(\cdot)x_0'') - v_n(\cdot)\|_{L^2(0,T)} \\
&\quad \leq \|F\|\,\|w_n(\cdot) - \tilde{x}(\cdot)\|_{L^2(0,T)} + \|(F\tilde{x}(\cdot) + G'(\cdot)x_0' + G''(\cdot)x_0'') - v_n(\cdot)\|_{L^2(0,T)} \\
&\quad = \|F\|\,\|w_n(\cdot) - \tilde{x}(\cdot)\|_{L^2(0,T)} + \|\tilde{u}(\cdot) - v_n(\cdot)\|_{L^2(0,T)} \\
&\quad \leq M\left\{\|w_n - \tilde{x}(\cdot)\|_{L^2(0,T)} + \|\tilde{u}(\cdot) - v_n(\cdot)\|_{L^2(0,T)}\right\}.
\end{aligned}$$

Each term in the last part of the inequality converges to zero, as desired (cf. the analogous proof of (16)). ■

Remark 4. We see from above that $\chi_n \geq \|\tilde{u} - v_n\|_{L^2(0,T)}$. Moreover, χ_n depends on the initial condition.

Now we choose an element $(\Phi_n, [\, H_n'(\cdot) \quad H_n''(\cdot)\,])$ of minimal norm in $\mathbb{R}^{q\times q} \times L^2(0,T;\mathbb{R}^{q\times q})$, which satisfies

$$\|(\Phi w_n(\cdot) + H'(\cdot)x_0' + H''(\cdot)x_0'') - v_n(\cdot)\|_{L_2(0,T)} \leq \sqrt{\chi_n}, \tag{22}$$

$$\|\Phi\| \leq \hat{N}. \tag{23}$$

We use Lemma 7 to deduce the boundedness of the sequence $\{(\Phi_n, [H_n'(\cdot) \quad H_n''(\cdot)])\}$ so that we can find a subsequence (for simplicity denoted with the same symbol) such that $\Phi_n \to \Phi_0$ in $\mathbb{R}^{q\times q}$, $[\, H_n'(\cdot) \quad H_n''(\cdot)\,] \to [\, H_0'(\cdot) \quad H_0''(\cdot)\,]$ weakly in $L^2(0,T;\mathbb{R}^{q\times q})$. We can prove that

$$\Phi_0\tilde{x}(t) + H_0'(t)x_0' + H_0''(t)x_0'' = F\tilde{x}(t) + G'(t)x_0' + G''(t)x_0.$$

In fact (cf. the analogous proof of Theorem 5), we have

$$\Phi_0\tilde{x}(\cdot) + H_0'(\cdot)x_0' + H_0''(\cdot)x_0'' - \tilde{u}(\cdot) = (\Phi_0 - \Phi_n)\tilde{x}(\cdot) + [\Phi_n\tilde{x}(\cdot) + H_n'x_0' + H_n''x_0'' - \tilde{u}(\cdot)]$$

Each term on the right-hand side tends to zero (the quantity in the braces only weakly), while the left-hand side does not depend on n. Hence it is zero.

Remark 5. The boundedness of $\{\|\Phi_n\|\}$ follows from the 'artificial' condition (23). We observe, however, that we have the boundedness of $\{\|\Phi_n\|\}$ directly from (21) and (22), even if condition (23) is not imposed.

Summarizing, we can recover the right-hand side of (20), for the moment only as the limit of a weakly convergent sequence. Now we prove that we have in fact *norm convergence* as follows. We know that $\{v_n(\cdot)\}$ converges to $\tilde{u}(\cdot)$ *in norm.* Hence, we compute

$$\begin{aligned}
&\|(H_k'(\cdot)x_0' + H_k''(\cdot)x_0'') - (H_s'(\cdot)x_0' + H_s''(\cdot)x_0'')\|_{L^2(0,T)} \\
&\quad \le \|\Phi_k w_k(\cdot) + H_k'(\cdot)x_0' + H_k''(\cdot)x_0'' - v_k\|_{L^2(0,T)} \\
&\qquad + \|\Phi_s w_s(\cdot) + H_s'(\cdot)x_0' + H_s''(\cdot)x_0'' - v_s\|_{L^2(0,T)} \\
&\qquad + \|v_k(\cdot) - \tilde{u}(\cdot)\|_{L^2(0,T)} + \|v_s(\cdot) - \tilde{u}(\cdot)\|_{L^2(0,T)} \\
&\qquad + \|\Phi_k w_k(\cdot) - \Phi_s w_s(\cdot)\|_{L^2(0,T)}.
\end{aligned}$$

Each term on the right-hand side converges to zero *strongly* so that the sequence $\{H_k'(\cdot)x_0' + H_k''(\cdot)x_0''\}$ is Cauchy in the L^2-norm. Hence we can formulate the following analog of Theorem 5:

Theorem 7. *Let the initial condition x_0 be known (with a certain tolerance h). It is possible to construct a subsequence $\{(\Phi_n, [\; H_n'(\cdot) \quad H_n''(\cdot) \;])\}$ which strongly converges to $\{(\Phi_0, [\; H_0'(\cdot) \quad H_0''(\cdot) \;])\}$ such that (18) holds, i.e. such that*

$$\Phi_0\tilde{x}(t) + \begin{bmatrix} H_0'(\cdot)x_0' & H_0''(\cdot)x_0'' \end{bmatrix} = F\tilde{x}(t) + \begin{bmatrix} G_0'(\cdot)x_0' & G_0''(\cdot)x_0'' \end{bmatrix} = \tilde{u}(\cdot). \quad (24)$$

As in the previous section, we deduce that it is possible to mimic the evolution of a system along a fixed trajectory. Now we investigate the properties of the element $\{(\Phi_0, [\; H_0'(\cdot) \quad H_0''(\cdot) \;])\}$ just constructed.

Theorem 8. *The element $\{(\Phi_0, [H_0'(\cdot) \quad H_0''(\cdot)])\}$ constructed by the previous procedure is the element of minimal norm among those which satisfy condition (22).*

Proof. The weak semicontinuity of the norm implies that

$$\nu = \left\| \left(\Phi_0, \begin{bmatrix} H_0'(\cdot) & H_0''(\cdot) \end{bmatrix} \right) \right\|_H \le \liminf \left\| \left(\Phi_n, \begin{bmatrix} H_n'(\cdot) & H_n''(\cdot) \end{bmatrix} \right) \right\|_H .$$

Let $\{(\tilde{\Phi}_0, [\tilde{H}_0'(\cdot) \quad \tilde{H}_0''(\cdot)])\}$ have a norm less than ν and satisfy (18), i.e. (24). In this case we have

$$\begin{aligned}
&\|\tilde{\Phi}_0 w_n + \tilde{H}_0'(\cdot)x_0' + \tilde{H}_0''(\cdot)x_0'' - v_n(\cdot)\|_{L^2(0,T)} \\
&\quad \le \|\tilde{\Phi}_0\tilde{x}(\cdot) + \tilde{H}_0'(\cdot)x_0' + \tilde{H}_0''(\cdot)x_0'' - \tilde{u}(\cdot)\|_{L^2(0,T)} + \|\tilde{u}(\cdot) - v_n(\cdot)\|_{L^2(0,T)} \\
&\quad = \|\tilde{\Phi}_0\|\|\tilde{x}(\cdot) - w_n(\cdot)\|_{L^2(0,T)} + \|\tilde{u}(\cdot) - v_n(\cdot)\|_{L^2(0,T)} \le \chi_n.
\end{aligned}$$

Hence we also have

$$\left\| \left\{ \left(\Phi_n, \begin{bmatrix} H_n'(\cdot) & H_n''(\cdot) \end{bmatrix} \right) \right\} \right\|_H \le \left\| \left\{ \left(\tilde{\Phi}_0, \begin{bmatrix} \tilde{H}_0'(\cdot) & \tilde{H}_0''(\cdot) \end{bmatrix} \right) \right\} \right\|_H .$$

We pass to the limit and get a contradiction. ■

Finally, we investigate convergence when repeated experiments, with independent initial conditions $e^{(r)}$, are available. The proof of Theorem 6 cannot be repeated because now it is not true that the matrices F, $G'(\cdot)$ and $G''(\cdot)$ are uniquely identified by the evolutions of the system. However, let $\tilde{\chi}_n$ be the minimum of the numbers $\chi_n(e^{(r)})$, as in the previous section, and $\{(\Phi_n, [H'_n(\cdot)\ \ H''_n(\cdot)])\}$ be the element of minimal norm which satisfies inequalities (22) *for every* initial condition $e^{(r)}$. We consider a convergent subsequence. We see that this subsequence fulfills the properties required in Theorem 8, *for every initial condition* $e^{(r)}$. It follows that this subsequence converges to that element $\{(\Phi_0, [H'_0(\cdot)\ \ H''_0(\cdot)])\}$ which satisfies (18) *for every initial condition* and which has minimal norm. This element *is unique* and this proves the convergence of the original sequence to the same element. Hence we can formulate, in place of Theorem 6, the following result:

Theorem 9. *Let Assumption* 1 *hold. The sequence* $\{(\Phi_n, [H'_n(\cdot)\ \ H''_n(\cdot)])\}$ *converges to the element* $\{(\Phi, [H'(\cdot)\ \ H''(\cdot)])\}$ *of* minimal *norm which satisfies conditions (18) and (19).*

Remark 6. We arrive at the same conclusion, without the minimality property, even if the number $\hat{N}$ in Assumption 1 is unknown.

3. The Second Algorithm for Reconstructing the Feedback Matrix

The identification algorithm presented in the previous sections does not assume any *a priori* information on the matrix F. In this section we adapt ideas from (Blizorukova and Maksimov, 1997; Kryazhimskii, 1999; Kryazhimskii and Osipov, 1987; Kryazhimskii *et al.*, 1997) in order to obtain a second identification algorithm which can be used when the feedback F is an element of a known compact convex set $\mathcal{F}_* \subset \mathbb{R}^{q\times q}$. We shall deal with the case when all coordinates are observed, i.e. $y(t) = x(t)$, and system (1), (2) is of the form

$$\dot{x}(t) = Fx(t), \quad t \in [0,T]. \tag{25}$$

We introduce a family of linear continuous operators $S(x_T(\cdot))$ depending on elements $x_T(\cdot) \in C(0,T;\mathbb{R}^q)$ and acting from $\mathbb{R}^{q\times q}$ into $L^2(0,T;\mathbb{R}^q)$. Namely, we define for every $u \in \mathbb{R}^{q\times q}$

$$(S(x_T(\cdot)))(t)u = \mathcal{A}(x(t))u \quad \text{for a.e.} \quad t \in [0,T].$$

Here $\mathcal{A}(x(t))$ is a $(q\times q)\times q$ matrix of the following structure:

$$\mathcal{A}(x(t)) = \left.\overbrace{\begin{pmatrix} x'(t) & 0 & \dots & 0 \\ 0 & x'(t) & \dots & \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & x'(t) \end{pmatrix}}^{q\times q \text{ columns}}\right\} q \text{ rows.}$$

Primes denote transposition (i.e. the symbol $x'(t)$ means the vector-row corresponding to a vector-column $x(t)$). The symbol $x_T(\cdot)$ is used to recall that the function is defined on the interval $[0,T]$.

We introduce the one-to-one mapping Q: $\mathbb{R}_M^{q\times q} \to \mathbb{R}^{q\times q}$ which transforms every matrix

$$F = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1q} \\ a_{21} & a_{22} & \dots & a_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ a_{q1} & a_{q2} & \dots & a_{qq} \end{pmatrix} \tag{26}$$

into the vector-column $u_F = QF = (a_{11},\dots,a_{1q},a_{21},\dots,a_{2q},\dots,a_{q1},\dots,a_{qq})'$. It is evident that the mapping Q preserves the norm, i.e. $\|u_F\| = \|F\|$.

Equation (25) may be written in the form

$$\dot{x}(t) = S(x_T(\cdot))(t)u_F, \quad t \in [0,T],$$

which can be written as a functional equation in the space $L^2(0,T;\mathbb{R}^q)$:

$$x(\cdot) - x_0 = S_*(x_T(\cdot))(\cdot)u_F. \tag{27}$$

The family of linear continuous operators $S_*(x_T(\cdot))$: $\mathbb{R}^{q\times q} \to L^2(0,T;\mathbb{R}^q)$ is defined by the rule

$$S_*(x_T(\cdot))(t)w = \Big(\int_0^t S(x_T(\cdot))(\tau)\,\mathrm{d}\tau\Big)w \quad \text{for a.e.} \quad t\in[0,T], \quad (w\in\mathbb{R}^{q\times q}).$$

Hence (27) amounts to

$$x(t) - x_0 = \Big(\int_0^t S(x_T(\cdot))(\tau)\,\mathrm{d}\tau\Big)u_F \quad \text{for a.e.} \quad t\in[0,T].$$

For notational brevity, we introduce

$$b(t) = x(t) - x_0 \quad \text{for a.e.} \quad t\in[0,T].$$

Let

$$U_1 = \big\{u \in Q\mathcal{F}_* : \ \dot{x}(t) = S(x_T(\cdot))(t)u, \quad t\in[0,T]\big\}.$$

It is easily seen that this set is convex, bounded and closed. Therefore the set

$$U^* = \arg\min\{\|u\| : \ u \in U_1\}$$

contains only one element, $U^* = \{u_0\}$.

To solve the problem, we introduce the dynamical control system

$$\dot{z}(\rho) = v(\rho), \quad z(0) = 0, \qquad z, v \in \mathbb{R}^{q\times q} \tag{28}$$

acting on the time interval $\mathbb{R}_+ = [0, +\infty)$. Here ρ is an 'artificial' time. Our goal now is to compute an input function $v(\cdot)$ such that for the corresponding trajectory $z(\cdot)$ of system (28), the ratio $z(\rho)/\rho$ is close to u_0 for a 'sufficiently large' ρ.

Inputs $v(\rho)$ to system (28) will have a feedback form. Formally, a feedback is identified with a function $V\colon\ \mathbb{R}_+ \times \mathbb{R}^{q\times q} \to \mathcal{F}_*$. For every $\gamma > 0$, we define the γ-trajectory $z_\gamma(\cdot)$ under the action of feedback $U(\rho, z)$ by

$$z_\gamma(0) = 0, \quad z_\gamma(\rho) = z_\gamma(\rho_j) + v_j^\gamma(\rho - \rho_j), \quad \rho \in [\rho_j, \rho_{j+1}),$$
$$\rho_j = j\gamma, \quad v_j^\gamma = V(\rho_j, z_\gamma(\rho_j)).$$

We introduce the functional

$$\Lambda_\alpha(\rho|z_\gamma(\cdot)) = \|S_*(x_T(\cdot))(\cdot)z_\gamma(\rho) - \rho b(\cdot)\|^2_{L^2(0,T)} + \alpha\int_0^\rho \|\dot{z}_\gamma(\tau)\|^2\,\mathrm{d}\tau - \alpha\rho J^0, \tag{29}$$
$$J^0 = \|u_0\|^2.$$

Here and below the symbol $\|\cdot\|_{\mathcal{L}_2}$ stands for the norm in the space of bounded linear operators acting from $\mathbb{R}^{q\times q}$ into $L^2(0,T;\mathbb{R}^q)$, and the symbol $\langle\cdot,\cdot\rangle_{L^2}$ is the corresponding scalar product. The functional Λ_α is analogous to the functional ε (see Section 2.2). We shall specify such a rule for choosing the feedback control $U(\rho, z)$ so that the following inequality holds:

$$\Lambda_\alpha(\rho|z_\gamma(\cdot)) \le \Lambda_\alpha(\rho_j|z_\gamma(\cdot)) + c_1(\rho - \rho_j)\left\{(\rho - \rho_j) + \rho_j(h + \frac{1}{n})\right\} \tag{30}$$

for $\rho \in [\rho_j, \rho_{j+1})$. Here c_1 is a constant which may be explicitly computed.

Introduce

$$R = \sup\{\|F\|:\ F \in \mathcal{F}_*\}, \quad \xi_T(t) = \xi_i \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}), \quad \tau_i = \frac{iT}{n}. \tag{31}$$

Since the function $\xi_T(\cdot)$ is piecewise constant, we have

$$(S_*(\xi_T(\cdot)))(t) = \delta\sum_{i=0}^{i(t)-1} A(\xi_i) + (t - \tau_{i(t)})A(\xi_{i(t)}), \quad t \in [0, T],$$

where

$$i(t) = [t/\delta], \quad \tau_{i(t)} = i(t)T/n \qquad ([a] \text{ denotes the integer part of } a).$$

We introduce the function

$$b_{h,n}(t) = \xi_i - \xi_0 \quad \text{for} \quad t \in [\tau_i, \tau_{i+1}).$$

It depends on h since, by virtue of (3), $\xi_i = \xi_i^h$.

Lemma 8. *The following inequality is valid:*

$$\|b(\cdot) - b_{h,n}(\cdot)\|_{L^2(0,T)} \le d_1(h, 1/n) = \sqrt{T}(2h + RCT/n),$$

where $C = \|x_0\|(1 + RT\exp(RT))$.

Proof. From $\|F\| \le R$ we obtain

$$\|x(t)\| \le \|x_0\| + \int_0^t \|Fx(\tau)\|\,\mathrm{d}\tau \le \|x_0\| + R\int_0^t \|x(\tau)\|\,\mathrm{d}\tau, \quad t \in [0, T].$$

Using Gronwall's lemma, we get

$$\|x(t)\| \le C, \quad \|\dot{x}(t)\| \le RC, \quad t \in [0, T]. \tag{32}$$

Note that for $t \in [\tau_i, \tau_{i+1})$ we have

$$\|x(t) - \xi_i\| \le h + \int_{\tau_i}^t \|\dot{x}(\tau)\|\,\mathrm{d}\tau \le h + RCT/n. \tag{33}$$

Therefore, by virtue of (32) and (33), we get

$$\begin{aligned}\|b(\cdot) - b_{h,n}(\cdot)\|^2_{L^2(0,T)} &= \sum_{i=0}^{n-1} \int_{\tau_i}^{\tau_{i+1}} \|x(t) - x_0 - \xi_i + \xi_0\|^2\,\mathrm{d}t \\ &\le \sum_{i=0}^{n-1} \int_{\tau_i}^{\tau_{i+1}} (2h + RCT/n)^2\,\mathrm{d}t \\ &= \frac{n-1}{n} T(2h + RCT/n)^2 \le T(2h + RCT/n)^2.\end{aligned}$$

The lemma is thus proved. ■

Lemma 9. *The following inequality is valid:*

$$\|S_*(x_T(\cdot)) - S_*(\xi_T(\cdot))\|_{\mathcal{L}_2} \le d_2(h, 1/n) = T\sqrt{\frac{qT}{3}}(h + RCT/n).$$

Here the symbol $\xi_T(\cdot)$ *means a function* $\xi(t)$, $t \in [0, T]$, *defined according to (31).*

Proof. By (33) we get

$$\|(S(x_T(\cdot)) - S(\xi_T(\cdot)))(\tau)u_F\| \le \sqrt{q}(h + RCT/n)\|u_F\|, \quad \tau \in [0, T].$$

Therefore

$$\|S_*(x_T(\cdot)) - S_*(\xi_T(\cdot))\|_{\mathcal{L}_2}$$

$$= \Big(\sup_{\|u_F\| \le 1} \int_0^T \Big\| \int_0^t (S(x_T(\cdot)) - S(\xi_T(\cdot)))(\tau)\, \mathrm{d}\tau u_F \Big\|^2 \mathrm{d}t \Big)^{1/2}$$
$$\le \Big(\int_0^T \Big\| \int_0^t \sqrt{q}(h + RCT/n)\, \mathrm{d}\tau \Big\|^2 \mathrm{d}t \Big)^{1/2}$$
$$\le \sqrt{q}(h + RCT/n) \Big(\int_0^T t^2\, \mathrm{d}t \Big)^{1/2} \le T\sqrt{\frac{qT}{3}}(h + RCT/n).$$

This is our claim. ■

Let $V(\rho, z)$ be defined by the rule

$$V(\rho, z) = V_\alpha(\rho, z)$$
$$= \arg\min\{2\langle S_*(\xi_T(\cdot))(\cdot)z - \rho b_{h,n}(\cdot), S_*(\xi_T(\cdot))(\cdot)u\rangle_{L_2} + \alpha\|u\|^2 :\ u \in Q\mathcal{F}_*\}. \quad (34)$$

Theorem 10. *The feedback input $V(t, z)$ in (34) satisfies inequality (30).*

Proof. For $\rho = 0$, we have

$$\Lambda_\alpha(0|z_\gamma(\cdot)) = 0, \quad (35)$$

and (30) is true. Suppose that (30) is true for all $\rho \in [0, \rho_j]$. Take $\rho \in [\rho_j, \rho_{j+1}]$ and prove (30). We have

$$\|z_\gamma(\nu)\| \le \nu R \quad \nu \ge 0.$$

Introduce the notation

$$s_j(x, z_\gamma) = S_*(x_T(\cdot))(\cdot)z_\gamma(\rho_j) - \rho_j b(\cdot) \in L_2(0, T; \mathbb{R}^q).$$

Using Lemmas 17 and 18, we deduce that

$$\|s_j(x, z_\gamma) - s^j(\xi, z_\gamma)\|_{L^2(0,T)}$$
$$\le \|S_*(x_T(\cdot))(\cdot) - S_*(\xi_T(\cdot))\|_{\mathcal{L}_2}\|z_\gamma(\rho_j)\|$$
$$+ \rho_j\|b(\cdot) - b_{h,n}(\cdot)\|_{L^2(0,T)} \le \rho_j(d_1(h, 1/n) + Rd_2(h, 1/n)), \quad (36)$$

where

$$s^j(\xi, z_\gamma) = S_*(\xi_T(\cdot))(\cdot)z_\gamma(\rho_j) - \rho_j b_{h,n}(\cdot).$$

Referring to (29), we get

$$\Lambda_\alpha(\rho|z_\gamma(\cdot)) = \Lambda_\alpha(\rho_j|z_\gamma(\cdot)) + \mu_j + \nu_j + \alpha(\|v_j^\gamma\|^2 - J^0)(\rho - \rho_j), \quad (37)$$

where

$$\mu_j = 2(\rho - \rho_j)\langle s_j(x, z_\gamma), S_*(x_T(\cdot))(\cdot)v_j^\gamma - b(\cdot)\rangle_{L_2},$$
$$\nu_j = \|S_*(x_T(\cdot))(\cdot)v_j^\gamma - b(\cdot)\|^2_{L^2(0,T)}(\rho - \rho_j)^2.$$

Taking into account

$$S_*(x_T(\cdot))(\cdot)u_0 - b(\cdot) = 0$$

and using (37), we obtain

$$\Lambda_\alpha(\rho|z_\gamma(\cdot)) = \Lambda_\alpha(\rho_j|z_\gamma(\cdot)) + \nu_j + 2(\rho-\rho_j)\Big\{\Big[\langle s_j(x, z_\gamma), S_*(x_T(\cdot))(\cdot)v_j^\gamma - b(\cdot)\rangle_{L_2} + \alpha\|v_j^\gamma\|^2\Big] - \Big[\langle s_j(x, z_\gamma), S_*(x_T(\cdot))(\cdot)u_0 - b(\cdot)\rangle_{L_2} + \alpha\|u_0\|^2\Big]\Big\}.$$

Then we have

$$\|s_j(x, z_\gamma)\|_{L^2(0,T)} \le (d_0R + b_0)\rho_j, \tag{38}$$
$$\|S_*(x_T(\cdot))(\cdot)v_j^\gamma - b(\cdot)\|_{L^2(0,T)} \le d_0R + b_0.$$

Here

$$d_0 = \|S_*(x_T(\cdot))(\cdot)\|_{\mathcal{L}_2}, \quad b_0 = \|b(\cdot)\|_{L^2(0,T)}.$$

Consequently, by virtue of (36) and (38), we get

$$\Lambda_\alpha(\rho|z_\gamma(\cdot)) \le \Lambda_\alpha(\rho_j|z_\gamma(\cdot)) + \nu_j + 2(\rho - \rho_j)\Big\{\Big[\langle s^j(\xi, z_\gamma), S_*(x_T(\cdot))(\cdot)v_j^\gamma - b(\cdot)\rangle_{L_2} + \alpha\|v_j^\gamma\|^2\Big] - \Big[\langle s^j(\xi, z_\gamma), S_*(x_T(\cdot))(\cdot)u_0 - b(\cdot)\rangle_{L_2} + \alpha\|u_0\|^2\Big]\Big\} + 2(\rho - \rho_j)\rho_j d_3(h, 1/n), \tag{39}$$

where

$$d_3(h, 1/n) = (d_1(h, 1/n) + Rd_2(h, 1/n))(d_0R + b_0).$$

From (38) and (36) it follows that

$$\|s^j(\xi, z_\gamma)\|_{L^2(0,T)} \le \tau_j d_4(h, 1/n), \tag{40}$$
$$d_4(h, 1/n) = b_0 + d_0R + d_1(h, 1/n) + Rd_2(h, 1/n).$$

It is clear that

$$\nu_j \le (\rho - \rho_j)^2(b_0 + d_0R)^2.$$

Using again Lemmas 17 and 18, from (39) and (40) we obtain

$$\begin{aligned}\Lambda_\alpha(\rho|z_\gamma(\cdot)) \leq \Lambda_\alpha(\rho_j|z_\gamma(\cdot))& \\ &+ 2(\rho-\rho_j)\Big\{\Big[\langle s^j(\xi,z_\gamma), S_*(\xi_T(\cdot))(\cdot)v_j^\gamma\rangle_{L_2}\alpha\|v_j^\gamma\|^2\Big] \\ &- \Big[\langle s^j(\xi,z_\gamma), S_*(\xi_T(\cdot))(\cdot)u_0\rangle_{L_2} + \alpha\|u_0\|^2\Big]\Big\} \\ &+ 2(\rho-\rho_j)\rho_j d_5(h,1/n) + (\rho-\rho_j)^2(b_0+d_0R)^2,\end{aligned}$$

where

$$d_5(h,1/n) = d_3(h,1/n) + d_4(h,1/n)d_2(h,1/n)R \leq c_0\left(h+\frac{1}{n}\right),$$

c_0 being a constant which may be explicitly written.

The result follows since the expression in the braces is negative when v_j^γ satisfies condition (34). ■

Lemma 10. *The inequalities*

$$\|S_*(x_T(\cdot))(z_\gamma(\rho_j)/\rho_j) - b(\cdot)\|^2_{L^2(0,T)} \leq c_2(\gamma/\rho_j + h_n + 1/n) + 2\alpha R^2/\rho_j, \quad (41)$$

$$\|z_\gamma(\rho)/\rho\|^2 \leq c_1(\gamma/\alpha + \rho_j(h_n+1/n)/\alpha) + J^0 \quad (42)$$

are valid.

Proof. From (30) and (35) we deduce that

$$\Lambda_\alpha(\rho_j|z_\gamma(\cdot)) \leq c_1\left(\gamma\rho_j + \rho_j(h_n+1/n)\right).$$

Thus the inequalities

$$\|S_*(x_T(\cdot))(\cdot)z_\gamma(\rho_j) - \rho_j b(\cdot)\|^2_{L^2(0,T)} \leq c_1\left(\gamma\rho_j + \rho_j^2(h_n+1/n)\right) + 2\alpha R^2\rho_j, \quad (43)$$

$$\int_0^{\rho_j} \|\dot z_\gamma(\tau)\|^2\, d\tau \leq c_1\left(\frac{\gamma\rho_j}{\alpha} + \rho_j^2\frac{h_n+1/n}{\alpha}\right) + \rho_j J^0 \quad (44)$$

are valid. Dividing both the sides of (43) by ρ_j^2, we obtain (41). The convexity of the norm implies

$$\frac{1}{\rho}\int_0^\rho \|\dot z_\gamma(\tau)\|^2\,\mathrm{d}\tau \geq \left\|\frac{1}{\rho}\int_0^\rho \dot z_\gamma(\tau)\,\mathrm{d}\tau\right\|^2 = \|z_\gamma(\rho)/\rho\|^2 \quad \forall\rho>0.$$

Hence, using (44), we get (42). The lemma is proved. ■

Let sequences of positive numbers $\{\alpha_n\}$, $\{h_n\}$, $\{\gamma_n\}$ and $\{j_n\}$ be chosen in such a way that

$$\alpha_n \to 0, \quad h_n \to 0, \quad j_n \to +\infty, \quad (45)$$

$$\alpha_n/\rho_{j_n} \to 0, \quad \gamma_n/\alpha_n \to 0, \quad \rho_{j_n}(h_n+1/n)/\alpha_n \to 0 \quad \text{as} \quad n\to\infty.$$

Lemma 11. *If conditions (45) are fulfilled, then we have*

$$\lim_{n\to+\infty} z_{\gamma_n}(\rho_{j_n})/\rho_{j_n} = u_0\,. \tag{46}$$

Proof. The convergence (46) follows from (41), (42) and equality $\rho_{j_n} = \gamma j_n$. ■

Let

$$\Phi = Q^{-1}u_0.$$

From Lemma 11 we obtain the following result:

Theorem 11. *If conditions (45) are fulfilled, then we have*

$$\lim_{n\to+\infty} Q^{-1}z_{\gamma_n}(\rho_{j_n})/\rho_{j_n} = \Phi.$$

4. Conclusion

In this paper we have studied an identification problem which is of strong interest in the context of fault detection. Namely, we have assumed that the nominal system $\dot{x} = Ax$ was affected by an unwanted disturbance $u = Fx$ which could model a failure of a component. For example, a failure of an interconnection may set to zero the value of an entry of the matrix A.

The identification problem amounts to the solution of the equation

$$y(t) = Cx(t) = Ce^{(A+F)t}x_0, \qquad t \in [0,T].$$

The unknown is F, so this problems seems very difficult. But we observe a very simple case: if $T = +\infty$ and if we know exactly the function $y(t)$, then we can equivalently solve

$$C(\lambda I - A - F)^{-1}x_0 = \hat{y}(\lambda)$$

where $\hat{}$ denotes the Laplace transform. This is a much simpler problem, being affine in the unknown $(\lambda I - A - F)^{-1}$.

What is more, if we can identify or approximately identify both $x(t)$ and $u(t) = Fx(t)$ on $[0,T]$, then the identification problem is reduced to a linear problem directly in the unknown matrix F:

$$Fx(t) = u(t).$$

This is the point of view adopted in this paper.

We have presented two identification algorithms which have a common feature: structural *a-priori* information on the feedback to be identified is not assumed. In other words, we have assumed that we have no *a-priori* information on the component of the system whose integrity we would like to check. An interesting extension of the results in this paper, reserved for future work, is the case when we do have such information, so that we can make use of the powerful structure theory of finite-dimensional linear systems.

Acknowledgments

In the case of the first author, the work was supported by the RFBR (Russia) under Grants Nos. 00–01–00682 and 01–01–00566, the Russian programme for supporting leading scientific schools under Project No. 00–15–96086, and the International Institute for Applied Systems Analysis, Laxenburg, Austria. The second author was also supported by the Italian MURST within programmes of GNAMPA.

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