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A NONLINEAR MODEL OF A TURBINE BLADE BY ASYMPTOTIC ANALYSIS

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In this paper we obtain a limit model for a turbine blade fixed to a 3D solid. This model is a three-dimensional linear elasticity problem in the 3D part of the piece (the rotor) and a two-dimensional problem (the nonlinear shallow shell equations) in the 2D part (the turbine blade), with junction conditions in the part of the turbine blade fixed to the rotor. To obtain this model, we perform an asymptotic analysis, starting with the nonlinear three-dimensional elasticity equations on all the pieces and taking as a small parameter the thickness of the blade.

Keywords: asymptotic analysis, nonlinear elasticity, junctions, shallow shells, multistructures

1. Introduction

The objective of this article is to mathematically justify a coupled 3D-2D model for a turbine blade under centrifugal and pressure forces. Let us suppose that the turbine blade can be modelled by a shallow shell, as defined by (Ciarlet and Miara, 1992).

We shall assume that the turbine blade is made of a Saint Venant-Kirchhoff material. We shall need the following notation:

Let $\mathcal{O}$ be a bounded and connected open set, $\mathcal{O} \subset \mathbb{R}^3$, with boundary $\partial\mathcal{O}$ smooth enough (for instance, a Lipschitz boundary composed of a finite number of parts of class C^1). Let $\Gamma_0 \subset \partial\mathcal{O}$ have positive measure ($\text{meas}(\Gamma_0) > 0$).

Here and subsequently, we will use the following symbols:

$$\omega \subset \mathbb{R}^2, \quad \Omega = \omega \times]-1,1[, \quad \Omega^\varepsilon = \omega \times]-\varepsilon,\varepsilon[,$$

ω a bounded and connected open set with $\partial\omega$
a Lipschitz boundary composed of
a finite number of curves of class C^1,

$$\theta : (x_1,x_2) \in \overline{\omega} \longrightarrow \mathbb{R}, \quad \theta \in C^3(\overline{\omega}),$$

$$\theta^\varepsilon(x_1,x_2) = \varepsilon\theta(x_1,x_2),$$

$$\alpha^\varepsilon(x_1,x_2) = 1 + |\partial_1\theta^\varepsilon|^2 + |\partial_2\theta^\varepsilon|^2,$$

$$\hat{d}^\varepsilon(x_1,x_2) = (\alpha^\varepsilon)^{-1/2}(-\partial_1\theta^\varepsilon, -\partial_2\theta^\varepsilon, 1),$$

$$\pi^\varepsilon : \mathbf{x} \in \overline{\Omega} \longrightarrow \mathbf{x}^\varepsilon = \pi^\varepsilon(\mathbf{x}) = (x_1,x_2,\varepsilon x_3) \in \overline{\Omega^\varepsilon},$$

$$\Theta^\varepsilon : \mathbf{x}^\varepsilon \in \overline{\Omega^\varepsilon} \longrightarrow \hat{\mathbf{x}}^\varepsilon = \Theta^\varepsilon(\mathbf{x}^\varepsilon) \in \overline{\Theta^\varepsilon(\Omega^\varepsilon)},$$

$$\Theta^\varepsilon(\mathbf{x}^\varepsilon) = \big(x_1,x_2,\theta^\varepsilon(x_1,x_2)\big) + x_3^\varepsilon \hat{d}^\varepsilon(x_1,x_2),$$

$$\hat{\Omega}^\varepsilon = \Theta^\varepsilon(\Omega^\varepsilon),$$

$$\hat{\omega}^\varepsilon = \left\{\big(x_1,x_2,\theta^\varepsilon(x_1,x_2)\big) \in \mathbb{R}^3 : (x_1,x_2) \in \omega\right\},$$

$$\mathcal{S}^\varepsilon = \mathcal{O} \cup \hat{\Omega}^\varepsilon, \quad \hat{\Omega}^\varepsilon_\beta = \hat{\Omega}^\varepsilon \cap \mathcal{O}, \quad \mathcal{O}^\varepsilon_\beta = \mathcal{O} - \overline{\hat{\Omega}^\varepsilon_\beta}.$$

We shall use the summation convention for repeated indices, where the Latin indices $(i,j,k,\dots)$ take on values in $\{1,2,3\}$ and the Greek indices $(\alpha,\beta,\gamma,\dots)$ take on values in $\{1,2\}$. For example, $x_\alpha x_\alpha = x_1^2 + x_2^2$, $x_i x_i = x_1^2 + x_2^2 + x_3^2$. Let δ_{ij} be the identity matrix, i.e. $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ij} = 1$ if $i = j$. We shall use the following notation for partial derivatives: ∂_i for $\partial/\partial x_i$, ∂_i^ε for $\partial/\partial x_i^\varepsilon$ and $\hat{\partial}_i^\varepsilon$ for $\partial/\partial \hat{x}_i^\varepsilon$.

From our definitions, we can see that $\mathcal{S}^\varepsilon$ is a complete solid, $\mathcal{O}$ is the 3D part, $\hat{\Omega}^\varepsilon$ is the shallow shell and $\hat{\omega}^\varepsilon$ stands for the middle surface of the shallow shell (a turbine usually has more than one blade, but the results can be easily extended). We shall also use the following notation:

Let $\omega_\beta = \{(x_1,x_2) \in \omega : (x_1,x_2,0) \in \mathcal{O}\}$, and suppose that $\text{meas}(\omega_\beta) > 0$. Then we have $\omega_\beta \neq \emptyset$ and, for a small ε, $\hat{\Omega}^\varepsilon_\beta \neq \emptyset$.

Let $\omega^* = \omega - \overline{\omega_\beta}$, $\gamma^* = \overline{\omega^*} \cap \overline{\omega_\beta}$, $\Gamma^* = \gamma^* \times [-1,1]$, $\Omega_\beta = \omega_\beta \times]-1,1[$, $\Omega^* = \omega^* \times]-1,1[$, $\Gamma_+ = \omega^* \times \{1\}$. The upper face of the shallow shell can be defined by $\hat{\Gamma}^\varepsilon_+ = \partial\left(\hat{\Omega}^\varepsilon - \overline{\hat{\Omega}^\varepsilon_\beta}\right) \cap \Theta^\varepsilon(\omega \times$

$\{\varepsilon\})$. We also introduce $\Omega_\beta(\varepsilon) = (\Theta^\varepsilon \circ \pi^\varepsilon)^{-1}\left(\hat{\Omega}^\varepsilon_\beta\right) \subset \Omega$, $\Gamma_+(\varepsilon) = (\Theta^\varepsilon \circ \pi^\varepsilon)^{-1}\left(\hat{\Gamma}^\varepsilon_+\right)$ and $\omega_\beta(\varepsilon) = \{(x_1,x_2) \in \omega : (x_1,x_2,\varepsilon\theta(x_1,x_2)) \in \mathcal{O}\}$.

2. Problem Posed on $\mathcal{S}^\varepsilon$

We shall consider the nonlinear elasticity problem for a solid $\mathcal{S}^\varepsilon$. We are interested in the case when $\mathcal{S}^\varepsilon$ is a turbine blade subjected to large centrifugal and pressure forces, and we suppose that the turbine is fixed on Γ_0 (we shall choose a reference frame that spins with $\mathcal{S}^\varepsilon$). Let us consider the case when the Lamé coefficients are $\hat{\lambda}^\varepsilon$ and $\hat{\mu}^\varepsilon$ in $\mathcal{S}^\varepsilon$. Let us consider a body force $\hat{\mathbf{f}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = (\hat{f}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon))$ applied in $\mathcal{S}^\varepsilon$ and a surface force $\hat{\mathbf{g}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = (\hat{g}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon))$ applied on $\hat{\Gamma}^\varepsilon_+$. Let us remember that we are interested in the case when $\hat{\mathbf{f}}^\varepsilon$ is a centrifugal force and $\hat{\mathbf{g}}^\varepsilon$ is a pressure, so we have (if $\mathcal{S}^\varepsilon$ spins around the Ox_2 axis):

$$\hat{f}^\varepsilon_1(\hat{\mathbf{x}}^\varepsilon,\hat{\mathbf{u}}^\varepsilon) = \hat{\delta}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)(\omega^\varepsilon)^2\big(\hat{x}^\varepsilon_1 + \hat{u}^\varepsilon_1(\hat{\mathbf{x}}^\varepsilon)\big), \quad \hat{f}^\varepsilon_2(\hat{\mathbf{x}}^\varepsilon) = 0,$$
$$\hat{f}^\varepsilon_3(\hat{\mathbf{x}}^\varepsilon,\hat{\mathbf{u}}^\varepsilon) = \hat{\delta}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)(\omega^\varepsilon)^2\big(\hat{x}^\varepsilon_3 + \hat{u}^\varepsilon_3(\hat{\mathbf{x}}^\varepsilon)\big), \tag{1}$$
$$\hat{g}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon) = -\hat{p}^\varepsilon\left(\hat{\mathbf{x}}^\varepsilon + \hat{\mathbf{u}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\right)\hat{d}^\varepsilon_i\big(\hat{\mathbf{x}}^\varepsilon + \hat{\mathbf{u}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\big),$$

where $\hat{\delta}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)$ is the mass density at point $\hat{\mathbf{x}}^\varepsilon$, ω^ε is the angular velocity, $\hat{p}^\varepsilon$ is the normal pressure to $(I + \hat{\mathbf{u}}^\varepsilon)(\hat{\Gamma}^\varepsilon_+)$ and $\hat{\mathbf{d}}^\varepsilon$ is the unit outward normal to $(I + \hat{\mathbf{u}}^\varepsilon)(\hat{\Gamma}^\varepsilon_+)$.

Then our problem is (see, e.g., Ciarlet, 1990): Find $\hat{\mathbf{u}}^\varepsilon \in V^\varepsilon$ such that

$$\int_{\mathcal{S}^\varepsilon}\left\{\hat{\lambda}^\varepsilon\hat{E}^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon\hat{E}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon)\right\}\hat{e}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon)\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon$$
$$+ \int_{\mathcal{S}^\varepsilon}\left\{\hat{\lambda}^\varepsilon\hat{E}^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon\hat{E}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon)\right\}\hat{\partial}^\varepsilon_i\hat{u}^\varepsilon_k\hat{\partial}^\varepsilon_j\hat{v}^\varepsilon_k\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon$$
$$= \int_{\mathcal{S}^\varepsilon}\hat{f}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon,\hat{\mathbf{u}}^\varepsilon)\hat{v}^\varepsilon_i\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon + \int_{\hat{\Gamma}^\varepsilon_+}\hat{g}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon,\hat{\mathbf{u}}^\varepsilon)\hat{v}^\varepsilon_i\,\mathrm{d}\hat{a}^\varepsilon,$$
$$\forall\,\hat{\mathbf{v}}^\varepsilon \in V^\varepsilon, \tag{2}$$

where

$$\hat{E}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon) = \hat{e}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon) + \frac{1}{2}\hat{\partial}^\varepsilon_i\hat{v}^\varepsilon_k\hat{\partial}^\varepsilon_j\hat{v}^\varepsilon_k$$
$$= \frac{1}{2}(\hat{\partial}^\varepsilon_j\hat{v}^\varepsilon_i + \hat{\partial}^\varepsilon_i\hat{v}^\varepsilon_j) + \frac{1}{2}\hat{\partial}^\varepsilon_i\hat{v}^\varepsilon_k\hat{\partial}^\varepsilon_j\hat{v}^\varepsilon_k \tag{3}$$

and $V^\varepsilon = \big\{\hat{\mathbf{v}}^\varepsilon \in W^{1,4}(\mathcal{S}^\varepsilon) : \hat{\mathbf{v}}^\varepsilon = 0 \text{ on } \Gamma_0\big\}$.

3. Changing to the Reference Sets

Let us now perform two changes of variables to reference open sets in order to be able to pass on to the limit within open sets independent of the parameter ε. Let us consider the following two changes of variables:

A) $\tilde{\mathcal{O}} = \mathcal{O} + \mathbf{r}$, with $\mathbf{r} \in \mathbb{R}^3$ ($\tilde{\mathbf{x}} \in \tilde{\mathcal{O}} \Leftrightarrow \tilde{x}_i = \hat{x}^\varepsilon_i + r_i, \hat{\mathbf{x}}^\varepsilon \in \mathcal{O}$) and let $\tilde{\mathcal{O}}^\varepsilon_\beta = \mathcal{O}^\varepsilon_\beta + \mathbf{r}$, $\tilde{\Omega}^\varepsilon_\beta = \hat{\Omega}^\varepsilon_\beta + \mathbf{r}$, etc.

B) $\mathbf{x} \in \Omega \longrightarrow \mathbf{x}^\varepsilon = \pi^\varepsilon(\mathbf{x}) \in \Omega^\varepsilon$
$\longrightarrow \hat{\mathbf{x}}^\varepsilon = \Theta^\varepsilon\big(\pi^\varepsilon(\mathbf{x})\big) \in \hat{\Omega}^\varepsilon$.

Let us take $\mathbf{r} \in \mathbb{R}^3$ such that $\Omega \cap \tilde{\mathcal{O}} = \emptyset$ and consider now $\nabla^\varepsilon\Theta^\varepsilon(\mathbf{x}^\varepsilon) = \big(\partial^\varepsilon_j\Theta^\varepsilon_i(\mathbf{x}^\varepsilon)\big)$. If we use $(\alpha^\varepsilon)^{-1/2} = 1 + \varepsilon^2 r(\varepsilon,\theta)$, where $r(\varepsilon,\theta)$ is bounded on $\bar{\omega}$ (Ciarlet and Miara, 1992; Ciarlet and Paumier, 1986), we have

$$\nabla^\varepsilon\Theta^\varepsilon(\mathbf{x}^\varepsilon) = \begin{pmatrix} 1 & 0 & -\varepsilon\partial_1\theta \\ 0 & 1 & -\varepsilon\partial_2\theta \\ \varepsilon\partial_1\theta & \varepsilon\partial_2\theta & 1 \end{pmatrix} + \varepsilon^2 M(\varepsilon;\mathbf{x}^\varepsilon), \tag{4}$$

where $\sup_{0<\varepsilon\leq\varepsilon_0}\max_{\mathbf{x}^\varepsilon\in\bar{\Omega}^\varepsilon}|M(\varepsilon;\mathbf{x}^\varepsilon)| < +\infty$.

Consider now the following terms which appear when performing the change of the variables of the problem (2) to the reference sets:

$$\begin{cases} \delta^\varepsilon(\mathbf{x}^\varepsilon) = \det\big(\nabla^\varepsilon\Theta^\varepsilon(\mathbf{x}^\varepsilon)\big), \\ b^\varepsilon_{ij}(\mathbf{x}^\varepsilon) = \Big(\big(\nabla^\varepsilon\Theta^\varepsilon(\mathbf{x}^\varepsilon)\big)^{-1}\Big)_{ij}, \end{cases} \qquad \mathbf{x}^\varepsilon \in \bar{\Omega}^\varepsilon,$$
$$\beta^\varepsilon(\mathbf{x}^\varepsilon) = \big(b^\varepsilon_{3i}(\mathbf{x}^\varepsilon)b^\varepsilon_{3i}(\mathbf{x}^\varepsilon)\big)^{1/2}, \qquad \mathbf{x}^\varepsilon \in \Gamma^\varepsilon_+.$$

We can prove (Ciarlet and Miara, 1992; Ciarlet and Paumier, 1986) that we have

$$\begin{aligned} \delta^\varepsilon(\mathbf{x}^\varepsilon) &= 1 + \varepsilon^2 r_\Delta(\varepsilon;\mathbf{x}^\varepsilon), \\ \beta^\varepsilon(\mathbf{x}^\varepsilon) &= 1 + \varepsilon^2 r_B(\varepsilon;\mathbf{x}^\varepsilon), \\ b^\varepsilon_{\alpha\beta}(\mathbf{x}^\varepsilon) &= \delta_{\alpha\beta} + \varepsilon^2 r_{\alpha\beta}(\varepsilon;\mathbf{x}^\varepsilon), \\ b^\varepsilon_{\alpha3}(\mathbf{x}^\varepsilon) &= \varepsilon\partial_\alpha\theta + \varepsilon^3 r_{\alpha3}(\varepsilon;\mathbf{x}^\varepsilon), \\ b^\varepsilon_{3\alpha}(\mathbf{x}^\varepsilon) &= -\varepsilon\partial_\alpha\theta + \varepsilon^3 r_{3\alpha}(\varepsilon;\mathbf{x}^\varepsilon), \\ b^\varepsilon_{33}(\mathbf{x}^\varepsilon) &= 1 + \varepsilon^2 r_{33}(\varepsilon;\mathbf{x}^\varepsilon), \end{aligned}$$

where

$$\sup_{0<\varepsilon\leq\varepsilon_0}\max_{\mathbf{x}^\varepsilon\in\bar{\Omega}^\varepsilon}\big\{|r_\Delta(\varepsilon;\mathbf{x}^\varepsilon)|, |r_{ij}(\varepsilon;\mathbf{x}^\varepsilon)|, |r_B(\varepsilon;\mathbf{x}^\varepsilon)|\big\} < +\infty.$$

We now use the change of variables to the reference sets in eqn. (2). The change of variables used depends on the integral under consideration as follows:

$$\int_{\mathcal{S}^\varepsilon} \hat{F}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon = \int_{\mathcal{O}} \hat{F}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon + \int_{\hat{\Omega}^\varepsilon - \overline{\hat{\Omega}^\varepsilon_\beta}} \hat{F}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon$$

$$= \int_{\tilde{\mathcal{O}}} \tilde{F}(\varepsilon)(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} + \int_{\Omega - \Omega_\beta(\varepsilon)} F(\varepsilon)(\mathbf{x})\big(1 + \varepsilon^2 r_\Delta(\varepsilon)(\mathbf{x})\big)\mathrm{d}\mathbf{x}, \tag{5}$$

where $\hat{F}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)$ is any function on $\mathcal{S}^\varepsilon$, and $\tilde{F}(\varepsilon)(\tilde{\mathbf{x}})$ and $F(\varepsilon)(\mathbf{x})$ denote their images on $\tilde{\mathcal{O}}$ and Ω through the changes of variables. The same goes for $r_\Delta(\varepsilon)(\mathbf{x})$ and $r_\Delta(\varepsilon; \mathbf{x}^\varepsilon)$.

In order not to have to integrate over a domain $\Omega - \Omega_\beta(\varepsilon)$ that depends *a priori* on ε, we make the following simplifying geometric assumption:

$$\Omega - \Omega_\beta(\varepsilon) = \Omega^*, \tag{6}$$

where Ω^* is independent of ε.

Hypothesis (6) is very restrictive and it means (nearly) that the turbine blade ($\hat{\Omega}^\varepsilon$) is fixed to $\mathcal{O}$ in a right angle. This hypothesis simplifies the computations and we shall study later the consequences of eliminating (6).

With hypothesis (6), eqn. (5) becomes

$$\int_{\mathcal{S}^\varepsilon} \hat{F}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon = \int_{\tilde{\mathcal{O}}} \tilde{F}(\varepsilon)(\tilde{\mathbf{x}}) d\tilde{\mathbf{x}} + \int_{\Omega^*} F(\varepsilon)(\mathbf{x})\big(1 + \varepsilon^2 r_\Delta(\varepsilon)(\mathbf{x})\big)\mathrm{d}\mathbf{x}. \tag{7}$$

Performing the change of variables from Ω^ε to Ω in functions $\delta^\varepsilon(\mathbf{x}^\varepsilon)$, $b^\varepsilon_{ij}(\mathbf{x}^\varepsilon)$ and $\beta^\varepsilon(\mathbf{x}^\varepsilon)$, we obtain

$$\begin{aligned}
\delta(\varepsilon)(\mathbf{x}) &= 1 + \varepsilon^2 r_\Delta(\varepsilon)(\mathbf{x}),\\
\beta(\varepsilon)(\mathbf{x}) &= 1 + \varepsilon^2 r_B(\varepsilon)(\mathbf{x}),\\
b_{\alpha\beta}(\varepsilon)(\mathbf{x}) &= \delta_{\alpha\beta} + \varepsilon^2 r_{\alpha\beta}(\varepsilon)(\mathbf{x}),\\
b_{\alpha 3}(\varepsilon)(\mathbf{x}) &= \varepsilon\partial_\alpha\theta + \varepsilon^3 r_{\alpha 3}(\varepsilon)(\mathbf{x}),\\
b_{3\alpha}(\varepsilon)(\mathbf{x}) &= -\varepsilon\partial_\alpha\theta + \varepsilon^3 r_{3\alpha}(\varepsilon)(\mathbf{x}),\\
b_{33}(\varepsilon)(\mathbf{x}) &= 1 + \varepsilon^2 r_{33}(\varepsilon)(\mathbf{x}),
\end{aligned} \tag{8}$$

where

$$\sup_{0<\varepsilon\le\varepsilon_0} \max_{\mathbf{x}\in\bar{\Omega}} \big\{ |r_\Delta(\varepsilon)(\mathbf{x})|, |r_{ij}(\varepsilon)(\mathbf{x})|, |r_B(\varepsilon)(\mathbf{x})| \big\} < +\infty.$$

Now, for each $\hat{\mathbf{v}}^\varepsilon \in W^{1,4}(\mathcal{S}^\varepsilon)$, we define the pair $(\tilde{\mathbf{v}}, \mathbf{v}) \in W^{1,4}(\mathcal{O}) \times W^{1,4}(\Omega)$, given by the following scalings:

$$\begin{aligned}
&\hat{v}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^2 \tilde{v}_i(\tilde{\mathbf{x}}), \quad \hat{\mathbf{x}}^\varepsilon \in \mathcal{O},\\
&\hat{v}^\varepsilon_\alpha(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^2 v_\alpha(\mathbf{x}), \ \ \hat{v}^\varepsilon_3(\hat{\mathbf{x}}^\varepsilon) = \varepsilon v_3(\mathbf{x}), \ \ \hat{\mathbf{x}}^\varepsilon \in \hat{\Omega}^\varepsilon.
\end{aligned} \tag{9}$$

Then we have

$$\begin{aligned}
&\hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^2 \tilde{\partial}_j \tilde{v}_i(\tilde{\mathbf{x}}), && \hat{\mathbf{x}}^\varepsilon \in \mathcal{O},\\
&\hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon) = \partial^\varepsilon_k v^\varepsilon_i(\mathbf{x}^\varepsilon) b^\varepsilon_{kj}(\mathbf{x}^\varepsilon), && \hat{\mathbf{x}}^\varepsilon \in \hat{\Omega}^\varepsilon,
\end{aligned} \tag{10}$$

where $\partial^\varepsilon_\alpha v^\varepsilon_\beta(\mathbf{x}^\varepsilon) = \varepsilon^2 \partial_\alpha v_\beta(\mathbf{x})$, $\partial^\varepsilon_\alpha v^\varepsilon_3(\mathbf{x}^\varepsilon) = \varepsilon\partial_\alpha v_3(\mathbf{x})$, $\partial^\varepsilon_3 v^\varepsilon_\beta(\mathbf{x}^\varepsilon) = \varepsilon\partial_3 v_\beta(\mathbf{x})$, $\partial^\varepsilon_3 v^\varepsilon_3(\mathbf{x}^\varepsilon) = \partial_3 v_3(\mathbf{x})$ and $b^\varepsilon_{kj}(\mathbf{x}^\varepsilon) = b_{kj}(\varepsilon)(\mathbf{x})$.

To proceed as with (7), let us consider the $\mathcal{O}$ part of the integral on the left-hand side of (2). Let us suppose that the Lamé constants satisfy $\hat{\lambda}^\varepsilon = \varepsilon^{-t}\hat{\lambda}$ and $\hat{\mu}^\varepsilon = \varepsilon^{-t}\hat{\mu}$.

Then we have, since $\hat{\mathbf{u}}^\varepsilon$ fulfils relations (9),

$$\begin{aligned}
&\int_{\mathcal{O}} \Big\{ \hat{\lambda}^\varepsilon \hat{E}^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon \hat{E}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon) \Big\} \hat{e}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon)\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon\\
&+ \int_{\mathcal{O}} \Big\{ \hat{\lambda}^\varepsilon \hat{E}^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon \hat{E}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon) \Big\} \hat{\partial}^\varepsilon_i \hat{u}^\varepsilon_k \hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_k\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon\\
&= \varepsilon^{4-t}\mathcal{A}^\varepsilon\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big),
\end{aligned} \tag{11}$$

where

$$\mathcal{A}^\varepsilon\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big) = \mathcal{A}_0\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big) + \varepsilon^2\mathcal{A}_2\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big) + \varepsilon^4\mathcal{A}_4\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big) \tag{12}$$

and

$$\mathcal{A}_0\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big) = \int_{\tilde{\mathcal{O}}} \Big\{ \hat{\lambda} e_{pp}\big(\tilde{\mathbf{u}}(\varepsilon)\big)\delta_{ij} + 2\hat{\mu} e_{ij}\big(\tilde{\mathbf{u}}(\varepsilon)\big) \Big\} e_{ij}(\tilde{\mathbf{v}})\,\mathrm{d}\tilde{\mathbf{x}}, \tag{13}$$

$$\begin{aligned}
\mathcal{A}_2\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big) = {}& \frac{1}{2}\int_{\tilde{\mathcal{O}}} \Big\{ \hat{\lambda}\partial_p\tilde{u}_k(\varepsilon)\partial_p\tilde{u}_k(\varepsilon)\delta_{ij} + 2\hat{\mu}\partial_i\tilde{u}_k(\varepsilon)\partial_j\tilde{u}_k(\varepsilon) \Big\} e_{ij}(\tilde{\mathbf{v}})\,\mathrm{d}\tilde{\mathbf{x}}\\
&+ \int_{\tilde{\mathcal{O}}} \Big\{ \hat{\lambda} e_{pp}\big(\tilde{\mathbf{u}}(\varepsilon)\big)\delta_{ij} + 2\hat{\mu} e_{ij}\big(\tilde{\mathbf{u}}(\varepsilon)\big) \Big\}\\
&\times \partial_i\tilde{u}_k(\varepsilon)\partial_j\tilde{v}_k\,\mathrm{d}\tilde{\mathbf{x}},
\end{aligned} \tag{14}$$

$$\begin{aligned}
\mathcal{A}_4\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big) = {}& \frac{1}{2}\int_{\tilde{\mathcal{O}}} \Big\{ \hat{\lambda}\partial_p\tilde{u}_k(\varepsilon)\partial_p\tilde{u}_k(\varepsilon)\delta_{ij} + 2\hat{\mu}\partial_i\tilde{u}_k(\varepsilon)\\
&\times \partial_j\tilde{u}_k(\varepsilon) \Big\} \partial_i\tilde{u}_k(\varepsilon)\partial_j\tilde{v}_k\,\mathrm{d}\tilde{\mathbf{x}}.
\end{aligned} \tag{15}$$

We are now going to do the same with $\hat{\Omega}^\varepsilon$. We must study the behaviour of $\hat{e}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon)$ and $\hat{\partial}^\varepsilon_i \hat{w}^\varepsilon_k \hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_k$ under the change of variables. For this reason, we are going to define the following functionals on $W^{1,4}(\Omega)$ and $W^{1,4}(\Omega)\times W^{1,4}(\Omega)$:

$$P_{ij}(\varepsilon)(\mathbf{v})(\mathbf{x}) = \frac{1}{\varepsilon^2}\hat{e}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon)(\hat{\mathbf{x}}^\varepsilon), \tag{16}$$

$$Q_{ij}(\varepsilon)(\mathbf{w},\mathbf{v})(\mathbf{x}) = \frac{1}{\varepsilon^2}\left\{\hat{\partial}^\varepsilon_i \hat{w}^\varepsilon_k \hat{\partial}^\varepsilon_j \hat{v}^\varepsilon_k\right\}(\hat{\mathbf{x}}^\varepsilon), \tag{17}$$

respectively. The dependence of (16) and (17) on ε can be studied if we use (8)–(10), and then we obtain

$$\begin{aligned}
\hat{e}^\varepsilon_{\alpha\beta}(\hat{\mathbf{v}}^\varepsilon) &= \frac{1}{2}\big(\partial^\varepsilon_\gamma v^\varepsilon_\alpha b^\varepsilon_{\gamma\beta} + \partial^\varepsilon_\gamma v^\varepsilon_\beta b^\varepsilon_{\gamma\alpha} + \partial^\varepsilon_3 v^\varepsilon_\alpha b^\varepsilon_{3\beta} + \partial^\varepsilon_3 v^\varepsilon_\beta b^\varepsilon_{3\alpha}\big)\\
&= \frac{\varepsilon^2}{2}\Big[\partial_\gamma v_\alpha\big(\delta_{\gamma\beta} + \varepsilon^2 r_{\gamma\beta}(\varepsilon)\big)\\
&\quad + \partial_\gamma v_\beta\big(\delta_{\gamma\alpha} + \varepsilon^2 r_{\gamma\alpha}(\varepsilon)\big)\\
&\quad + \varepsilon^{-1}\partial_3 v_\alpha\big(-\varepsilon\partial_\beta\theta + \varepsilon^3 r_{3\beta}(\varepsilon)\big)\\
&\quad + \varepsilon^{-1}\partial_3 v_\beta\big(-\varepsilon\partial_\alpha\theta + \varepsilon^3 r_{3\alpha}(\varepsilon)\big)\Big]\\
&= \frac{\varepsilon^2}{2}\big(\partial_\beta v_\alpha + \partial_\alpha v_\beta - \partial_3 v_\alpha \partial_\beta\theta - \partial_3 v_\beta\partial_\alpha\theta\big)\\
&\quad + \frac{\varepsilon^4}{2}\big(\partial_\gamma v_\alpha r_{\gamma\beta}(\varepsilon) + \partial_\gamma v_\beta r_{\gamma\alpha}(\varepsilon)\\
&\quad + \partial_3 v_\alpha r_{3\beta}(\varepsilon) + \partial_3 v_\beta r_{3\alpha}(\varepsilon)\big),
\end{aligned}$$

so we have

$$P_{\alpha\beta}(\varepsilon)(\mathbf{v}) = e^\theta_{\alpha\beta}(\mathbf{v}) + \varepsilon^2 e^\sharp_{\alpha\beta}(\varepsilon,\theta;\mathbf{v}), \tag{18}$$

where $e^\sharp_{\alpha\beta}(\varepsilon,\theta;\mathbf{v})$ depends on neither θ nor v_3:

$$\begin{aligned}
e^\theta_{\alpha\beta}(\mathbf{v}) &= e_{\alpha\beta}(\mathbf{v}) - \frac{1}{2}\left(\partial_\beta\theta\partial_3 v_\alpha + \partial_\alpha\theta\partial_3 v_\beta\right),\\
e_{\alpha\beta}(\mathbf{v}) &= \frac{1}{2}\left(\partial_\beta v_\alpha + \partial_\alpha v_\beta\right),\\
e^\sharp_{\alpha\beta}(\varepsilon,\theta;\mathbf{v}) &= \frac{1}{2}\big(\partial_i v_\alpha r_{i\beta}(\varepsilon) + \partial_i v_\beta r_{i\alpha}(\varepsilon)\big).
\end{aligned}$$

In a similar way, we obtain

$$P_{\alpha 3}(\varepsilon)(\mathbf{v}) = \frac{1}{\varepsilon}e^\theta_{\alpha 3}(\mathbf{v}) + \varepsilon e^\sharp_{\alpha 3}(\varepsilon,\theta;\mathbf{v}), \tag{19}$$

where

$$\begin{aligned}
e^\theta_{\alpha 3}(\mathbf{v}) &= e_{\alpha 3}(\mathbf{v}) - \frac{1}{2}\partial_\alpha\theta\partial_3 v_3,\\
e_{\alpha 3}(\mathbf{v}) &= \frac{1}{2}\left(\partial_3 v_\alpha + \partial_\alpha v_3\right),\\
e^\sharp_{\alpha 3}(\varepsilon,\theta;\mathbf{v}) &= \frac{1}{2}\big(\partial_\gamma v_\alpha\partial_\gamma\theta + \partial_i v_3 r_{i\alpha}(\varepsilon) + \partial_3 v_\alpha r_{33}(\varepsilon)\big)\\
&\quad + \frac{\varepsilon^2}{2}\big(\partial_\gamma v_\alpha r_{\gamma 3}(\varepsilon)\big).
\end{aligned}$$

We know that $\hat{e}^\varepsilon_{\alpha 3}(\hat{\mathbf{v}}^\varepsilon) = \hat{e}^\varepsilon_{3\alpha}(\hat{\mathbf{v}}^\varepsilon)$, so we have

$$P_{3\alpha}(\varepsilon)(\mathbf{v}) = \frac{1}{\varepsilon}e^\theta_{3\alpha}(\mathbf{v}) + \varepsilon e^\sharp_{3\alpha}(\varepsilon,\theta;\mathbf{v}), \tag{20}$$

where $e^\theta_{3\alpha}(\mathbf{v}) = e^\theta_{\alpha 3}(\mathbf{v})$, $e_{3\alpha}(\mathbf{v}) = e_{\alpha 3}(\mathbf{v})$, $e^\sharp_{3\alpha}(\varepsilon,\theta;\mathbf{v}) = e^\sharp_{\alpha 3}(\varepsilon,\theta;\mathbf{v})$.

We also have

$$\begin{aligned}
P_{33}(\varepsilon)(\mathbf{v}) &= \frac{1}{\varepsilon^2}\partial_3 v_3 + \partial_\gamma v_3\partial_\gamma\theta + \partial_3 v_3 r_{33}(\varepsilon)\\
&\quad + \varepsilon^2 e^\sharp_{33}(\varepsilon,\theta;\mathbf{v}),
\end{aligned} \tag{21}$$

where $e^\theta_{33}(\mathbf{v}) = e_{33}(\mathbf{v}) = \partial_3 v_3$, $e^\sharp_{33}(\varepsilon,\theta;\mathbf{v}) = \partial_\gamma v_3 r_{\gamma 3}(\varepsilon)$.

We can clearly see that there exists a constant C independent of ε, such that

$$\max_{i,j}\left|e^\sharp_{ij}(\varepsilon,\theta;\mathbf{v})\right|_{0,\Omega} \le C\|\mathbf{v}\|_{1,\Omega}.$$

Similarly, we can find the expressions for $Q_{ij}(\varepsilon)(\mathbf{w},\mathbf{v})$ after the change of variables. We thus obtain

$$\begin{aligned}
Q_{\alpha\beta}(\varepsilon)(\mathbf{w},\mathbf{v}) &= (\partial_\alpha w_3 - \partial_\alpha\theta\partial_3 w_3)(\partial_\beta v_3 - \partial_\beta\theta\partial_3 v_3)\\
&\quad + \varepsilon^2 q^\sharp_{\alpha\beta}(\varepsilon,\theta;\mathbf{w},\mathbf{v}),
\end{aligned} \tag{22}$$

and also

$$\begin{aligned}
Q_{\alpha 3}(\varepsilon)(\mathbf{w},\mathbf{v}) &= \frac{1}{\varepsilon}(\partial_\alpha w_3 - \partial_\alpha\theta\partial_3 w_3)\partial_3 v_3\\
&\quad + \varepsilon q^\sharp_{\alpha 3}(\varepsilon,\theta;\mathbf{w},\mathbf{v}).
\end{aligned} \tag{23}$$

From (17), it is clear that $Q_{3\alpha}(\varepsilon)(\mathbf{w},\mathbf{v}) = Q_{\alpha 3}(\varepsilon)(\mathbf{v},\mathbf{w})$, and then we have

$$\begin{aligned}
Q_{3\alpha}(\varepsilon)(\mathbf{w},\mathbf{v}) &= \frac{1}{\varepsilon}\partial_3 w_3(\partial_\alpha v_3 - \partial_\alpha\theta\partial_3 v_3)\\
&\quad + \varepsilon q^\sharp_{3\alpha}(\varepsilon,\theta;\mathbf{w},\mathbf{v}),
\end{aligned} \tag{24}$$

where $q^\sharp_{3\alpha}(\varepsilon,\theta;\mathbf{w},\mathbf{v}) = q^\sharp_{\alpha 3}(\varepsilon,\theta;\mathbf{v},\mathbf{w})$.

Finally, we also have

$$\begin{aligned}
Q_{33}(\varepsilon)(\mathbf{w},\mathbf{v}) &= \frac{1}{\varepsilon^2}\partial_3 w_3\partial_3 v_3 + \partial_3 w_\rho\partial_3 v_\rho\\
&\quad + \partial_3 w_3\big(\partial_\eta v_3\partial_\eta\theta + \partial_3 v_3 r_{33}(\varepsilon)\big)\\
&\quad + \partial_3 v_3\big(\partial_\gamma w_3\partial_\gamma\theta + \partial_3 w_3 r_{33}(\varepsilon)\big)\\
&\quad + \varepsilon^2 q^\sharp_{33}(\varepsilon,\theta;\mathbf{w},\mathbf{v}).
\end{aligned} \tag{25}$$

For the remainders $q^\sharp_{ij}$, we have the following bound:

$$\max_{i,j}\left|q^\sharp_{ij}(\varepsilon,\theta;\mathbf{w},\mathbf{v})\right|_{0,\Omega} \le C\|\mathbf{w}\|_{W^{1,4}(\Omega)}\|\mathbf{v}\|_{W^{1,4}(\Omega)}.$$

We are now able to proceed as in (11), but on $\hat{\Omega}^\varepsilon$ instead of $\mathcal{O}$. For that purpose, let us introduce the following expressions:

$$\begin{aligned} b_{\alpha\beta}(\mathbf{w},\mathbf{v}) &= e^\theta_{\alpha\beta}(\mathbf{v}) \\ &\quad +(\partial_\alpha w_3 - \partial_\alpha\theta\partial_3 w_3)(\partial_\beta v_3 - \partial_\beta\theta\partial_3 v_3), \\ b_{\alpha 3}(\mathbf{w},\mathbf{v}) &= e^\theta_{\alpha 3}(\mathbf{v}) + (\partial_\alpha w_3 - \partial_\alpha\theta\partial_3 w_3)\partial_3 v_3, \\ b_{3\beta}(\mathbf{w},\mathbf{v}) &= e^\theta_{3\beta}(\mathbf{v}) + \partial_3 w_3(\partial_\beta v_3 - \partial_\beta\theta\partial_3 v_3), \\ b_{33}(\mathbf{w},\mathbf{v}) &= \partial_3 v_3 + \partial_3 w_3\partial_3 v_3. \end{aligned} \tag{26}$$

We also introduce

$$d^\sharp_{ij}(\varepsilon;\mathbf{w},\mathbf{v}) = e^\sharp_{ij}(\varepsilon,\theta;\mathbf{v}) + q^\sharp_{ij}(\varepsilon,\theta;\mathbf{w},\mathbf{v}), \tag{27}$$

and

$$\begin{aligned} c_{33}(\varepsilon;\mathbf{w},\mathbf{v}) &= \partial_\alpha v_3\partial_\alpha\theta + \partial_3 v_3 r_{33}(\varepsilon) + \partial_3 w_\rho\partial_3 v_\rho \\ &\quad + \partial_3 w_3\big(\partial_\eta v_3\partial_\eta\theta + \partial_3 v_3 r_{33}(\varepsilon)\big) \\ &\quad + \partial_3 v_3\big(\partial_\gamma w_3\partial_\gamma\theta + \partial_3 w_3 r_{33}(\varepsilon)\big). \end{aligned} \tag{28}$$

Remark 1. If $\partial_3 v_3 = 0$, then $c_{33}(\varepsilon;\mathbf{w},\mathbf{v})$ does not depend on ε, and we have

$$\begin{aligned} c_{33}(\varepsilon;\mathbf{w},\mathbf{v}) &= c^*_{33}(\mathbf{w},\mathbf{v}) \\ &= \partial_\alpha v_3\partial_\alpha\theta + \partial_3 w_\rho\partial_3 v_\rho + \partial_3 w_3\partial_\eta v_3\partial_\eta\theta. \end{aligned}$$

Now we can proceed as in (11). Using the last expressions and hypothesis (6), we obtain

$$\begin{aligned} &\int_{\hat{\Omega}^\varepsilon - \overline{\hat{\Omega}^\varepsilon_\beta}} \Big\{\hat{\lambda}^\varepsilon\hat{E}^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon\hat{E}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon)\Big\}\hat{e}^\varepsilon_{ij}(\hat{\mathbf{v}}^\varepsilon)\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon \\ &+ \int_{\hat{\Omega}^\varepsilon - \overline{\hat{\Omega}^\varepsilon_\beta}} \Big\{\hat{\lambda}^\varepsilon\hat{E}^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon\hat{E}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon)\Big\}\hat{\partial}^\varepsilon_i\hat{u}^\varepsilon_k\hat{\partial}^\varepsilon_j\hat{v}^\varepsilon_k\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon \\ &\quad = \varepsilon^{5-t}\mathcal{B}^\varepsilon\big(\mathbf{u}(\varepsilon),\mathbf{v}\big), \end{aligned} \tag{29}$$

where

$$\begin{aligned} \mathcal{B}^\varepsilon\big(\mathbf{u}(\varepsilon),\mathbf{v}\big) &= \int_{\Omega^*}\Big\{\hat{\lambda}B^\varepsilon_0\big(\tfrac{1}{2}\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)\big)B^\varepsilon_0\big(\mathbf{u}(\varepsilon),\mathbf{v}\big) \\ &\quad + 2\hat{\mu}B^\varepsilon_{ij}\big(\tfrac{1}{2}\mathbf{u}(\varepsilon),\mathbf{u}(\varepsilon)\big)B^\varepsilon_{ij}\big(\mathbf{u}(\varepsilon),\mathbf{v}\big)\Big\} \\ &\quad \times\big(1+\varepsilon^2 r_\Delta(\varepsilon)\big)\,\mathrm{d}\mathbf{x} \end{aligned} \tag{30}$$

with

$$\begin{aligned} B^\varepsilon_0(\mathbf{w},\mathbf{v}) &= \frac{1}{\varepsilon^2}b_{33}(\mathbf{w},\mathbf{v}) + c_{33}(\varepsilon;\mathbf{w},\mathbf{v}) + b_{\alpha\alpha}(\mathbf{w},\mathbf{v}) \\ &\quad + \varepsilon^2 d^\sharp_{pp}(\varepsilon;\mathbf{w},\mathbf{v}), \\ B^\varepsilon_{\alpha\beta}(\mathbf{w},\mathbf{v}) &= b_{\alpha\beta}(\mathbf{w},\mathbf{v}) + \varepsilon^2 d^\sharp_{\alpha\beta}(\varepsilon;\mathbf{w},\mathbf{v}), \\ B^\varepsilon_{\alpha 3}(\mathbf{w},\mathbf{v}) &= \frac{1}{\varepsilon}b_{\alpha 3}(\mathbf{w},\mathbf{v}) + \varepsilon d^\sharp_{\alpha 3}(\varepsilon;\mathbf{w},\mathbf{v}), \\ B^\varepsilon_{3\beta}(\mathbf{w},\mathbf{v}) &= \frac{1}{\varepsilon}b_{3\beta}(\mathbf{w},\mathbf{v}) + \varepsilon d^\sharp_{3\beta}(\varepsilon;\mathbf{w},\mathbf{v}), \\ B^\varepsilon_{33}(\mathbf{w},\mathbf{v}) &= \frac{1}{\varepsilon^2}b_{33}(\mathbf{w},\mathbf{v}) + c_{33}(\varepsilon;\mathbf{w},\mathbf{v}) \\ &\quad + \varepsilon^2 d^\sharp_{33}(\varepsilon;\mathbf{w},\mathbf{v}). \end{aligned}$$

Remark 2. If we do not consider hypothesis (6), we find that in (30) the integral on Ω^* is then on $\Omega - \Omega_\beta(\varepsilon)$. Computations are then harder and we must know exactly what is $\Omega - \Omega_\beta(\varepsilon)$ (unless terms with an integral on $\Omega_\beta(\varepsilon)$ are small enough to not affect our computations). Removing hypothesis (6) would then require more complicated computations, but yield similar results.

The last step is to perform the same change of variable on the right part (the force contribution) of (2). We must then know the asymptotic behaviour of forces (1). Let us suppose that the mass density and the angular velocity have the following global asymptotic behaviour: $\hat{\delta}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)(\omega^\varepsilon)^2 = \varepsilon^{-\gamma}\omega_0$. That is, we are supposing large angular velocities (larger as γ grows).

Then, after the change of variables, in $\tilde{\mathcal{O}}$ we have

$$\hat{f}^\varepsilon_i(\hat{\mathbf{x}}^\varepsilon,\hat{\mathbf{u}}^\varepsilon) = \varepsilon^{-\gamma}\tilde{f}_i(\varepsilon)\big(\tilde{\mathbf{x}},\tilde{\mathbf{u}}(\varepsilon)\big), \tag{31}$$

with $\tilde{f}_1(\varepsilon)\big(\tilde{\mathbf{x}},\tilde{\mathbf{u}}(\varepsilon)\big) = \omega_0(\tilde{x}_1 - r_1) + \varepsilon^2\omega_0\tilde{u}_1(\varepsilon)$, $\tilde{f}_2(\varepsilon)\big(\tilde{\mathbf{x}},\tilde{\mathbf{u}}(\varepsilon)\big) = 0$ and $\tilde{f}_3(\varepsilon)\big(\tilde{\mathbf{x}},\tilde{\mathbf{u}}(\varepsilon)\big) = \omega_0(\tilde{x}_3 - r_3) + \varepsilon^2\omega_0\tilde{u}_3(\varepsilon)$.

If we perform the change of variables given on Ω, we obtain

$$\begin{aligned} &\hat{f}^\varepsilon_1(\hat{\mathbf{x}}^\varepsilon,\hat{\mathbf{u}}^\varepsilon) = \varepsilon^{-\gamma}f_1(\varepsilon)\big(\mathbf{x},\mathbf{u}(\varepsilon)\big), \quad \hat{f}^\varepsilon_2(\hat{\mathbf{x}}^\varepsilon,\hat{\mathbf{u}}^\varepsilon) = 0, \\ &\hat{f}^\varepsilon_3(\hat{\mathbf{x}}^\varepsilon,\hat{\mathbf{u}}^\varepsilon) = \varepsilon^{-\gamma+1}f_3(\varepsilon)\big(\mathbf{x},\mathbf{u}(\varepsilon)\big), \end{aligned} \tag{32}$$

with $f_1(\varepsilon)(\mathbf{x},\mathbf{u}(\varepsilon)) = \omega_0 x_1 + \varepsilon^2\omega_0\big(-x_3\partial_1\theta(\alpha^\varepsilon)^{-1/2} + u_1(\varepsilon)\big)$ and $f_3(\varepsilon)(\mathbf{x},\mathbf{u}(\varepsilon)) = \omega_0\big(x_3 + \theta(x_1,x_2) + u_3(\varepsilon)\big) + \varepsilon^2\omega_0 r(\varepsilon,\theta)$, where we bring to mind that $(\alpha^\varepsilon)^{-1/2} = 1 + \varepsilon^2 r(\varepsilon,\theta)$.

Now, we must do the same study with pressure $\hat{\mathbf{g}}^\varepsilon$. We must thus consider $\hat{p}^\varepsilon(\hat{\mathbf{x}}^\varepsilon + \hat{\mathbf{u}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon))$ and also the surface $(I + \hat{\mathbf{u}}^\varepsilon)(\hat{\Gamma}^\varepsilon_+)$. This surface is described by the function $(x_1,x_2) \in \omega^* = \Gamma_+ \longrightarrow G^\varepsilon(x_1,x_2) \in (I + \hat{\mathbf{u}}^\varepsilon)(\hat{\Gamma}^\varepsilon_+)$, where

$$\begin{aligned} G^\varepsilon(x_1,x_2) &= (\Theta^\varepsilon\circ\pi^\varepsilon)(x_1,x_2,1) \\ &\quad + \hat{\mathbf{u}}^\varepsilon\big((\Theta^\varepsilon\circ\pi^\varepsilon)(x_1,x_2,1)\big). \end{aligned} \tag{33}$$

Using the scalings, we have

$$\begin{aligned} \hat{u}^\varepsilon_\alpha\big((\Theta^\varepsilon\circ\pi^\varepsilon)(x_1,x_2,1)\big) &= \varepsilon^2 u_\alpha(\varepsilon)(x_1,x_2,1), \\ \hat{u}^\varepsilon_3\big((\Theta^\varepsilon\circ\pi^\varepsilon)(x_1,x_2,1)\big) &= \varepsilon u_3(\varepsilon)(x_1,x_2,1), \end{aligned} \tag{34}$$

so we obtain $\partial_1 G_1^\varepsilon = 1 - \varepsilon^2\partial_1\left((\alpha^\varepsilon)^{-1/2}\partial_1\theta\right) + \varepsilon^2\partial_1 u_1(\varepsilon)$, $\partial_1 G_2^\varepsilon = -\varepsilon^2\partial_1\left((\alpha^\varepsilon)^{-1/2}\partial_2\theta\right) + \varepsilon^2\partial_1 u_2(\varepsilon)$, $\partial_1 G_3^\varepsilon = \varepsilon\partial_1\theta + \varepsilon\partial_1[(\alpha^\varepsilon)^{-1/2}] + \varepsilon\partial_1 u_3(\varepsilon)$, and then $\partial_1 G^\varepsilon = (1, 0, \varepsilon(\partial_1\theta + \partial_1 u_3(\varepsilon)) + \varepsilon^2(a_1^\varepsilon + \partial_1 u_1(\varepsilon), a_2^\varepsilon + \partial_1 u_2(\varepsilon), \varepsilon a_3^\varepsilon)$, where $|a_i^\varepsilon|$ is bounded in $L^2(\Gamma_+)$. In the same way, we obtain $\partial_2 G^\varepsilon = (0, 1, \varepsilon(\partial_2\theta + \partial_2 u_3(\varepsilon)) + \varepsilon^2(b_1^\varepsilon + \partial_2 u_1(\varepsilon), b_2^\varepsilon + \partial_2 u_2(\varepsilon), \varepsilon b_3^\varepsilon)$, where $|b_i^\varepsilon|$ is bounded in $L^2(\Gamma_+)$.

Then the unit outward normal is

$$\begin{aligned}\hat{\mathbf{d}}^\varepsilon\big(\hat{\mathbf{x}}^\varepsilon + \hat{\mathbf{u}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\big) = \Big(&-\varepsilon\big(\partial_1\theta + \partial_1 u_3(\varepsilon)\big),\\ &-\varepsilon\big(\partial_2\theta + \partial_2 u_3(\varepsilon)\big), 1\Big)\\ &+\varepsilon^2\big(\varepsilon m_1(\varepsilon), \varepsilon m_2(\varepsilon), m_3(\varepsilon)\big),\end{aligned}$$

where $|m_i(\varepsilon)|_{0,\Omega} \le C\left(1 + \|\mathbf{u}(\varepsilon)\|_{1,\Omega}\right)$.

Consider now an arbitrary point of $\hat{\Gamma}_+^\varepsilon$ after a displacement and the change of variables. Let $\hat{\mathbf{x}}^\varepsilon \in \hat{\Gamma}_+^\varepsilon$. We have (recall that $(\alpha^\varepsilon)^{-1/2} = 1 + \varepsilon^2 r(\varepsilon, \theta)$):

$$\begin{aligned}&\hat{\mathbf{x}}^\varepsilon + \hat{\mathbf{u}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\\ &= \big(x_1, x_2, \varepsilon\big(1 + \theta(x_1,x_2) + u_3(\varepsilon)(x_1,x_2,1)\big)\big)\\ &\quad + \varepsilon^2\big(-\partial_1\theta(x_1,x_2) + u_1(\varepsilon)(x_1,x_2,1),\\ &\quad - \partial_2\theta(x_1,x_2) + u_2(\varepsilon)(x_1,x_2,1), 0\big)\\ &\quad + \varepsilon^3\big(0,0,r(\varepsilon,\theta)(x_1,x_2)\big)\\ &\quad + \varepsilon^4\big(-\partial_1\theta(x_1,x_2)r(\varepsilon,\theta)(x_1,x_2),\\ &\quad - \partial_2\theta(x_1,x_2)r(\varepsilon,\theta)(x_1,x_2), 0\big).\end{aligned}$$

We are now going to study the asymptotic behaviour of $\hat{p}^\varepsilon$. We have $G^\varepsilon(x_1,x_2) = \hat{\mathbf{x}}^\varepsilon + \hat{\mathbf{u}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)$ on $\hat{\Gamma}_+^\varepsilon$, so it is easy to see that G^ε tends to $(x_1,x_2,0)$ as ε goes to zero in a space X if $\mathbf{u}(\varepsilon)$ is bounded in the same space X. For this reason, it is useful to define a scaled pressure on $\omega^* = \Gamma_+$. Let

$$p(\varepsilon)(x_1,x_2) = \varepsilon^\eta\left(\hat{p}^\varepsilon \circ G^\varepsilon\right)(x_1,x_2), \tag{35}$$

for $(x_1,x_2) \in \omega^*$. This definition allows us to make the following hypothesis about the asymptotic behaviour of $p(\varepsilon)$:

$$p(\varepsilon)(x_1,x_2) = p(0)(x_1,x_2) + \varepsilon\gamma(\varepsilon)(x_1,x_2), \tag{36}$$

with $p(0) \in L^2(\Gamma_+)$, $\gamma(\varepsilon) \in L^2(\Gamma_+)$, and $|\gamma(\varepsilon)|_{0,\Gamma_+} \le C$.

Then $\hat{p}^\varepsilon$ satisfies

$$\hat{p}^\varepsilon\big(\hat{\mathbf{x}}^\varepsilon + \hat{\mathbf{u}}^\varepsilon(\hat{\mathbf{x}}^\varepsilon)\big) = \varepsilon^{-\eta}p(0)(x_1,x_2) + \varepsilon^{-\eta+1}\gamma(\varepsilon)(x_1,x_2). \tag{37}$$

That is, we assume large pressures (larger as η increases).

We can now write the expression for $\hat{\mathbf{g}}^\varepsilon$ after the change of variables:

$$\hat{g}_\alpha^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{-\eta+1}g_\alpha(\varepsilon)(\mathbf{x}), \quad \hat{g}_3^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{-\eta}g_3(\varepsilon)(\mathbf{x}), \tag{38}$$

with $g_\alpha(\varepsilon)(\mathbf{x}) = \big(\partial_\alpha\theta + \partial_\alpha u_3(\varepsilon)\big)p(0) + \varepsilon\tilde{m}_\alpha(\varepsilon)$, $g_3(\varepsilon)(\mathbf{x}) = -p(0) - \varepsilon\tilde{m}_3(\varepsilon)$ and $|\tilde{m}_i(\varepsilon)|$ bounded in $L^2(\Gamma_+)$.

Now we can rewrite the expression on the right hand-side of (2), which takes the form

$$\begin{aligned}&\int_{\mathcal{S}^\varepsilon}\hat{f}_i^\varepsilon\hat{v}_i^\varepsilon\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon}\hat{g}_i^\varepsilon\hat{v}_i^\varepsilon\,\mathrm{d}\hat{a}^\varepsilon\\ &= \int_{\tilde{\mathcal{O}}}\varepsilon^{-\gamma+2}\tilde{f}_i(\varepsilon)\tilde{v}_i\,\mathrm{d}\tilde{\mathbf{x}}\\ &\quad + \int_{\Omega^*}\varepsilon^{-\gamma+3}f_i(\varepsilon)v_i\delta(\varepsilon)\,\mathrm{d}\mathbf{x}\\ &\quad + \int_{\Gamma_+}\left[\varepsilon^{-\eta+3}g_\alpha(\varepsilon)v_\alpha + \varepsilon^{-\eta+1}g_3(\varepsilon)v_3\right]\delta(\varepsilon)\beta(\varepsilon)\,\mathrm{d}a.\end{aligned}$$

If we take into account the last equation as well as (31), (32), (38) and (8), we conclude that

$$\begin{aligned}&\int_{\mathcal{S}^\varepsilon}\hat{f}_i^\varepsilon\hat{v}_i^\varepsilon\,\mathrm{d}\hat{\mathbf{x}}^\varepsilon + \int_{\hat{\Gamma}_+^\varepsilon}\hat{g}_i^\varepsilon\hat{v}_i^\varepsilon\,\mathrm{d}\hat{a}^\varepsilon\\ &= \varepsilon^{-\gamma+2}\Big[L_0^1(\tilde{\mathbf{v}}) + \varepsilon^2 L_2^1\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big)\Big]\\ &\quad + \varepsilon^{-\gamma+3}\Big[L_0^2\big(\mathbf{u}(\varepsilon), \mathbf{v}\big) + \varepsilon^2 L_2^2\big(\varepsilon; \mathbf{u}(\varepsilon), \mathbf{v}\big)\Big]\\ &\quad + \varepsilon^{-\eta+1}\Big[L_0^3(\mathbf{v}) + \varepsilon L_1^3(\varepsilon; \mathbf{v}) + \varepsilon^2 L_2^3(\varepsilon; \mathbf{v})\Big],\end{aligned} \tag{39}$$

where

$$L_0^1(\tilde{\mathbf{v}}) = \int_{\tilde{\mathcal{O}}}\omega_0\big[(\tilde{x}_1 - r_1)\tilde{v}_1 + (\tilde{x}_3 - r_3)\tilde{v}_3\big]\,\mathrm{d}\tilde{\mathbf{x}},$$

$$L_2^1\big(\tilde{\mathbf{u}}(\varepsilon), \tilde{\mathbf{v}}\big) = \int_{\tilde{\mathcal{O}}}\omega_0\big[\tilde{u}_1(\varepsilon)\tilde{v}_1 + \tilde{u}_3(\varepsilon)\tilde{v}_3\big]\,\mathrm{d}\tilde{\mathbf{x}},$$

$$\begin{aligned}L_0^2\big(\mathbf{u}(\varepsilon), \mathbf{v}\big) = \int_{\Omega^*}\omega_0\big[&x_1 v_1 + \big(x_3 + \theta(x_1,x_2)\\ &+ u_3(\varepsilon)\big)v_3\big]\,\mathrm{d}\mathbf{x},\end{aligned}$$

$$\begin{aligned}L_2^2\big(\varepsilon; \mathbf{u}(\varepsilon), \mathbf{v}\big) = &\int_{\Omega^*}\omega_0\big[x_1 v_1 + \big(x_3 + \theta(x_1,x_2)\\ &\quad + u_3(\varepsilon)\big)v_3\big]r_\Delta(\varepsilon)\,\mathrm{d}\mathbf{x}\\ &+ \int_{\Omega^*}\omega_0\Big[\big(-x_3\partial_1\theta(\alpha^\varepsilon)^{-1/2} + u_1(\varepsilon)\big)v_1\\ &\quad + r(\varepsilon,\theta)v_3\Big]\delta(\varepsilon)\,\mathrm{d}\mathbf{x},\end{aligned}$$

$$L_0^3(\mathbf{v}) = -\int_{\Gamma_+} p(0) v_3 \,\mathrm{d}a,$$

$$L_1^3(\varepsilon;\mathbf{v}) = -\int_{\Gamma_+} \tilde{m}_3(\varepsilon) v_3 \,\mathrm{d}a,$$

$$L_2^3(\varepsilon;\mathbf{v}) = \int_{\Gamma_+} g_\alpha(\varepsilon) v_\alpha \delta(\varepsilon)\beta(\varepsilon)\,\mathrm{d}a + \int_{\Gamma_+} g_3(\varepsilon) v_3 \big(r_\Delta(\varepsilon) + r_B(\varepsilon) + \varepsilon^2 r_\Delta(\varepsilon) r_B(\varepsilon)\big)\,\mathrm{d}a.$$

Finally, we use eqns. (11), (29) and (39) to deduce that (2) is equivalent to the problem:

Find $\big(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)\big) \in V(\varepsilon)$ such that

$$\begin{aligned} &\varepsilon^{4-t}\mathcal{A}^\varepsilon\big(\tilde{\mathbf{u}}(\varepsilon),\tilde{\mathbf{v}}\big) + \varepsilon^{5-t}\mathcal{B}^\varepsilon\big(\mathbf{u}(\varepsilon),\mathbf{v}\big) \\ &\quad = \varepsilon^{-\gamma+2}\Big[L_0^1(\tilde{\mathbf{v}}) + \varepsilon^2 L_2^1\big(\tilde{\mathbf{u}}(\varepsilon),\tilde{\mathbf{v}}\big)\Big] \\ &\qquad + \varepsilon^{-\gamma+3}\Big[L_0^2\big(\mathbf{u}(\varepsilon),\mathbf{v}\big) + \varepsilon^2 L_2^2\big(\varepsilon;\mathbf{u}(\varepsilon),\mathbf{v}\big)\Big] \\ &\qquad + \varepsilon^{-\eta+1}\Big[L_0^3(\mathbf{v}) + \varepsilon L_1^3(\varepsilon;\mathbf{v}) + \varepsilon^2 L_2^3(\varepsilon;\mathbf{v})\Big], \\ &\qquad \forall\,(\tilde{\mathbf{v}},\mathbf{v}) \in V(\varepsilon), \end{aligned} \tag{40}$$

where $V(\varepsilon)$ is the space obtained after the change of variables from V^ε, and then

$$\begin{aligned} V(\varepsilon) = \Big\{&(\tilde{\mathbf{v}},\mathbf{v}) \in W^{1,4}(\tilde{\mathcal{O}}) \times W^{1,4}(\Omega); \\ &\tilde{\mathbf{v}} = 0 \text{ on } \tilde{\Gamma}_0, \quad \tilde{v}_\alpha\left(\Theta^\varepsilon\big(\pi^\varepsilon(\tilde{\mathbf{x}})\big) + \mathbf{r}\right) = v_\alpha(\mathbf{x}), \\ &\varepsilon\tilde{v}_3\left(\Theta^\varepsilon\big(\pi^\varepsilon(\tilde{\mathbf{x}})\big) + \mathbf{r}\right) = v_3(\mathbf{x}),\ \forall\,\mathbf{x}\in\Omega_\beta\Big\}. \end{aligned} \tag{41}$$

We must now choose values for parameters t, γ and η. There are infinitely many different possibilities which will give us different limit models. The best choice is the same as in the linear case (Rodriguez, 1997; 1999)

$$\eta > 0, \quad t = 4 + \eta, \quad \gamma = 2 + \eta. \tag{42}$$

In this way we preserve the most interesting effects and the nonlinear model is going to remain ‘close’ to the linear model. Then (40) becomes

$$\begin{aligned} &\mathcal{A}^\varepsilon\big(\tilde{\mathbf{u}}(\varepsilon),\tilde{\mathbf{v}}\big) + \varepsilon\mathcal{B}^\varepsilon\big(\mathbf{u}(\varepsilon),\mathbf{v}\big) \\ &\quad = \Big[L_0^1(\tilde{\mathbf{v}}) + \varepsilon^2 L_2^1\big(\tilde{\mathbf{u}}(\varepsilon),\tilde{\mathbf{v}}\big)\Big] \\ &\qquad + \varepsilon\Big[L_0^2\big(\mathbf{u}(\varepsilon),\mathbf{v}\big) + \varepsilon^2 L_2^2\big(\varepsilon;\mathbf{u}(\varepsilon),\mathbf{v}\big)\Big] \\ &\qquad + \varepsilon\Big[L_0^3(\mathbf{v}) + \varepsilon L_1^3(\varepsilon;\mathbf{v}) + \varepsilon^2 L_2^3(\varepsilon;\mathbf{v})\Big], \\ &\qquad \forall\,(\tilde{\mathbf{v}},\mathbf{v}) \in V(\varepsilon). \end{aligned} \tag{43}$$

4. Asymptotic Expansion

We are going to suppose now, as usually in this kind of methods, that it is possible to write the solution $\big(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)\big) \in V(\varepsilon)$ as an asymptotic expansion in $W^{1,4}(\tilde{\mathcal{O}}) \times W^{1,4}(\Omega)$, i.e. there exist pairs $(\tilde{\mathbf{u}}^m, \mathbf{u}^m)$ in space $W^{1,4}(\tilde{\mathcal{O}}) \times W^{1,4}(\Omega)$ such that

$$\big(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)\big) = \sum_{m=0}^{k} \varepsilon^m (\tilde{\mathbf{u}}^m, \mathbf{u}^m) + \cdots . \tag{44}$$

The solution $\big(\tilde{\mathbf{u}}(\varepsilon), \mathbf{u}(\varepsilon)\big)$ belongs to space $V(\varepsilon)$, so we have (cf. (41))

$$\begin{aligned} &\tilde{u}_\alpha(\varepsilon)\left(\Theta^\varepsilon\big(\pi^\varepsilon(\mathbf{x})\big) + \mathbf{r}\right) = u_\alpha(\varepsilon)(\mathbf{x}), \\ &\varepsilon\tilde{u}_3(\varepsilon)\left(\Theta^\varepsilon\big(\pi^\varepsilon(\mathbf{x})\big) + \mathbf{r}\right) = u_3(\varepsilon)(\mathbf{x}),\ \forall\,\mathbf{x}\in\Omega_\beta. \end{aligned} \tag{45}$$

We also have the following injections:

$$W^{1,4}(\tilde{\mathcal{O}}) \subset C^0(\overline{\tilde{\mathcal{O}}}), \quad W^{1,4}(\Omega) \subset C^0(\overline{\Omega}). \tag{46}$$

As $(\alpha^\varepsilon)^{-1/2} = 1 + \varepsilon^2 r(\varepsilon,\theta)$, we can write

$$\begin{aligned} &\Theta^\varepsilon\big(\pi^\varepsilon(\mathbf{x})\big) + \mathbf{r} \\ &\quad = \mathbf{r} + \big(x_1, x_2, \varepsilon\theta(x_1,x_2)\big) + \varepsilon x_3(-\varepsilon\partial_1\theta, -\varepsilon\partial_2\theta, 1) \\ &\qquad + \varepsilon^3 r(\varepsilon,\theta) x_3(-\varepsilon\partial_1\theta, -\varepsilon\partial_2\theta, 1). \end{aligned} \tag{47}$$

If we take limits in (45), keeping in mind (46) and (47), then we obtain

$$\begin{aligned} &\tilde{u}_\alpha^0(x_1 + r_1, x_2 + r_2, r_3) = u_\alpha^0(x_1, x_2, x_3), \\ &u_3^0(\mathbf{x}) = 0, \quad \forall\,\mathbf{x} \in \Omega_\beta, \end{aligned} \tag{48}$$

i.e. we have

$$u^0_{\alpha|\Omega_\beta} = \tilde{u}^0_{\alpha|\tilde{\omega}_\beta}, \quad u^0_{3|\Omega_\beta} = 0. \tag{49}$$

We must now substitute (44) into (43) and then equate the terms multiplied by the same power of ε. If we do so, we obtain

$$\mathcal{A}^\varepsilon\big(\tilde{\mathbf{u}}(\varepsilon),\tilde{\mathbf{v}}\big) = A_0 + \varepsilon A_1 + \varepsilon^2 A_2 + \cdots,$$

$$\varepsilon\mathcal{B}^\varepsilon\big(\mathbf{u}(\varepsilon),\mathbf{v}\big) = \varepsilon^{-3} B_{-3} + \varepsilon^{-2} B_{-2} + \cdots,$$

$$\begin{aligned} &\Big[L_0^1(\tilde{\mathbf{v}}) + \varepsilon^2 L_2^1\big(\tilde{\mathbf{u}}(\varepsilon),\tilde{\mathbf{v}}\big)\Big] \\ &\quad + \varepsilon\Big[L_0^2\big(\mathbf{u}(\varepsilon),\mathbf{v}\big) + \varepsilon^2 L_2^2\big(\varepsilon;\mathbf{u}(\varepsilon),\mathbf{v}\big)\Big] \\ &\quad + \varepsilon\Big[L_0^3(\mathbf{v}) + \varepsilon L_1^3(\varepsilon;\mathbf{v}) + \varepsilon^2 L_2^3(\varepsilon;\mathbf{v})\Big] \\ &\quad = L_0 + \varepsilon L_1 + \varepsilon^2 L_2(\varepsilon) + \cdots, \end{aligned}$$

where, for example,

$$A_0 = \int_{\tilde{\mathcal{O}}} \Big\{ \hat{\lambda} e_{pp}(\tilde{\mathbf{u}}^0)\delta_{ij} + 2\hat{\mu} e_{ij}(\tilde{\mathbf{u}}^0) \Big\} e_{ij}(\tilde{\mathbf{v}})\,\mathrm{d}\tilde{\mathbf{x}},$$

$$B_{-3} = \int_{\Omega^*} \Big\{ \hat{\lambda} b_{33}\Big(\frac{1}{2}\mathbf{u}^0, \mathbf{u}^0\Big) b_{33}(\mathbf{u}^0, \mathbf{v}) + 2\hat{\mu} b_{33}\Big(\frac{1}{2}\mathbf{u}^0, \mathbf{u}^0\Big) b_{33}(\mathbf{u}^0, \mathbf{v}) \Big\}\,\mathrm{d}\mathbf{x},$$

$$L_0 = \int_{\tilde{\mathcal{O}}} \omega_0 \big[(\tilde{x}_1 - r_1)\tilde{v}_1 + (\tilde{x}_3 - r_3)\tilde{v}_3 \big]\,\mathrm{d}\tilde{\mathbf{x}}.$$

We have written these terms as examples. Next, we are going to equate the terms multiplied by the same power of ε and we write explicitly the terms needed in each case (see (Rodriguez, 1997) for more detailed steps):

B_{-3} is the only term multiplied by ε^{-3}, so we have $B_{-3} = 0$, i.e.

$$\int_{\Omega^*} \Big\{ \hat{\lambda} b_{33}\Big(\frac{1}{2}\mathbf{u}^0, \mathbf{u}^0\Big) b_{33}(\mathbf{u}^0, \mathbf{v}) + 2\hat{\mu} b_{33}\Big(\frac{1}{2}\mathbf{u}^0, \mathbf{u}^0\Big) b_{33}(\mathbf{u}^0, \mathbf{v}) \Big\}\,\mathrm{d}\mathbf{x} = 0$$

for all $(\tilde{\mathbf{v}}, \mathbf{v})$ in $V(\varepsilon)$. Then we have (cf. (26))

$$(\hat{\lambda} + 2\hat{\mu}) \int_{\Omega^*} \partial_3 u_3^0 \Big(1 + \frac{1}{2}\partial_3 u_3^0\Big)(1 + \partial_3 u_3^0)\partial_3 v_3\,\mathrm{d}\mathbf{x} = 0 \tag{50}$$

for all $\mathbf{v}$ in $W^{1,4}(\Omega^*)$. As in (Ciarlet, 1990), we can use the following result in (50):

Let $w \in L^2(\Omega^*)$ *satisfy* $\int_{\Omega^*} w \partial_3 v\,\mathrm{d}\mathbf{x} = 0$

for all $v \in C^\infty(\overline{\Omega^*})$ *such that* $v = 0$ *on* $\gamma^* \times [-1, 1]$.

Then $w = 0$. (51)

Accordingly, we have $\partial_3 u_3^0(1+\frac{1}{2}\partial_3 u_3^0)(1+\partial_3 u_3^0) = 0$ in Ω^*. From (49) we deduce that $u_3^0 = 0$ on Γ^*, and, if we make the hypothesis $\partial_3 u_3^0 \in C^0(\overline{\Omega^*})$, we obtain

$$\partial_3 u_3^0 = 0 \ \text{ in } \ \Omega^*. \tag{52}$$

The unique term multiplied by ε^{-2} is B_{-2}, so we have $B_{-2} = 0$. Using (52) we obtain $(\hat{\lambda} + 2\hat{\mu}) \times \int_{\Omega^*}\partial_3 u_3^1 \partial_3 v_3\,\mathrm{d}\mathbf{x} = 0$, and then (cf. (51))

$$\partial_3 u_3^1 = 0 \ \text{ in } \ \Omega^*. \tag{53}$$

The unique term multiplied by ε^{-1} is B_{-1}, so we have $B_{-1} = 0$ and then we deduce (in a similar way as before) that

$$e^\theta_{\alpha 3}(\mathbf{u}^0) = 0, \ \text{ in } \ \Omega^*, \tag{54}$$

and

$$(\hat{\lambda} + 2\hat{\mu}) \Big[\partial_3 u_3^2 + \partial_\alpha u_3^0 \partial_\alpha \theta + \frac{1}{2}\partial_3 u_\rho^0 \partial_3 u_\rho^0 \Big] + \hat{\lambda}\Big[e^\theta_{\alpha\alpha}(\mathbf{u}^0) + \frac{1}{2}\partial_\alpha u_3^0 \partial_\alpha u_3^0 \Big] = 0. \tag{55}$$

Now, from (52) and (54) we deduce that $e^\theta_{3\alpha}(\mathbf{u}^0) = 0$, $\partial_3 u_3^0 = 0$, and then

$$\mathbf{u}^0 \in V_{KL}(\Omega^*) = \big\{ \mathbf{v} \in W^{1,4}(\Omega^*);\ e_{3i}(\mathbf{v}) = 0 \text{ in } \Omega^* \big\}. \tag{56}$$

The space $V_{KL}(\Omega^*)$ can be also written in the following way (Ciarlet, 1990):

$$V_{KL}(\Omega^*) = \Big\{ \mathbf{v} \in W^{1,4}(\Omega^*);\ v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3,\ v_3 = \eta_3 \quad \text{and } \eta_\alpha \in W^{1,4}(\omega^*),\ \eta_3 \in W^{2,4}(\omega^*) \Big\}. \tag{57}$$

But from (49), $\mathbf{u}^0$ satisfies

$$u_\alpha^0 = \zeta_\alpha - x_3 \partial_\alpha \zeta_3,\ u_3^0 = \zeta_3 \tag{58}$$

with $\zeta_\alpha \in W^{1,4}(\omega^*)$, $\zeta_3 \in W^{2,4}(\omega^*)$, and

$$\zeta_{\alpha|\gamma^*} = \big(\tilde{u}^0_{\alpha|\tilde{\omega}_\beta}\big)_{|\gamma^*}, \quad \zeta_3 = \partial_\nu \zeta_3 = 0 \text{ on } \gamma^*, \tag{59}$$

where $\nu = (\nu_1, \nu_2)$ is the unit outward normal to ω^* on γ^*.

If we take terms multiplied by ε^0, we obtain

$$A_0 + B_0 = L_0, \tag{60}$$

where, after the steps done before, we have

$$A_0 = \int_{\tilde{\mathcal{O}}} \Big\{ \hat{\lambda} e_{pp}(\tilde{\mathbf{u}}^0)\delta_{ij} + 2\hat{\mu} e_{ij}(\tilde{\mathbf{u}}^0) \Big\} e_{ij}(\tilde{\mathbf{v}})\,\mathrm{d}\tilde{\mathbf{x}}, \tag{61}$$

$$\begin{aligned} B_0 = {} & 2\hat{\mu} \int_{\Omega^*} e^\theta_{\alpha 3}(\mathbf{u}^1) \left[2 e^\theta_{3\alpha}(\mathbf{v}) + \partial_\alpha u_3^0 \partial_3 v_3 \right]\,\mathrm{d}\mathbf{x} \\ & + \int_{\Omega^*} \Big[(\hat{\lambda} + 2\hat{\mu}) \left(\partial_3 u_3^3 + \partial_\alpha u_3^1 \partial_\alpha \theta + \partial_3 u_\rho^1 \partial_3 u_\rho^0 \right) \\ & + \hat{\lambda} \left(e^\theta_{\alpha\alpha}(\mathbf{u}^1) + \partial_\alpha u_3^0 \partial_\alpha u_3^1 \right) \Big] \partial_3 v_3\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{62}$$

$$L_0 = \int_{\tilde{\mathcal{O}}} \omega_0 \left[(\tilde{x}_1 - r_1)\tilde{v}_1 + (\tilde{x}_3 - r_3)\tilde{v}_3 \right]\,\mathrm{d}\tilde{\mathbf{x}}. \tag{63}$$

Now, if we consider $\tilde{\mathbf{v}} = 0$, $\mathbf{v}_{|\Omega_\beta} = 0$, $\mathbf{v}_{|\Gamma^*} = 0$, $\mathbf{v} \in W^{1,4}(\Omega^*)$, we have $(\tilde{\mathbf{v}}, \mathbf{v}) \in V(\varepsilon)$ for all $\varepsilon > 0$ and we can take $(\tilde{\mathbf{v}}, \mathbf{v}) \in V(\varepsilon)$ as a test function in (60). Taking first $v_3 = 0$, we obtain $e^\theta_{\alpha 3}(\mathbf{u}^1) = 0$ in Ω^*, and with (53) this gives

$$\mathbf{u}^1 \in V_{KL}(\Omega^*). \tag{64}$$

Taking now $v_\alpha = 0$, we obtain

$$(\hat{\lambda} + 2\hat{\mu}) \left(\partial_3 u_3^3 + \partial_\alpha u_3^1 \partial_\alpha \theta + \partial_3 u_\rho^1 \partial_3 u_\rho^0 \right) + \hat{\lambda} \left(e^\theta_{\alpha\alpha}(\mathbf{u}^1) + \partial_\alpha u_3^0 \partial_\alpha u_3^1 \right) = 0. \tag{65}$$

Equations (64) and (65) give $B_0 = 0$ and then (60) becomes $A_0 = L_0$ for all $(\tilde{\mathbf{v}}, \mathbf{v}) \in V(\varepsilon)$, i.e.

$$\int_{\tilde{\mathcal{O}}} \left\{ \hat{\lambda} e_{pp}(\tilde{\mathbf{u}}^0)\delta_{ij} + 2\hat{\mu} e_{ij}(\tilde{\mathbf{u}}^0) \right\} e_{ij}(\tilde{\mathbf{v}})\,\mathrm{d}\tilde{\mathbf{x}} = \int_{\tilde{\mathcal{O}}} \left[\tilde{f}_1^0 \tilde{v}_1 + \tilde{f}_3^0 \tilde{v}_3 \right] \mathrm{d}\tilde{\mathbf{x}}, \quad \forall\, \tilde{\mathbf{v}} \in H^1(\tilde{\mathcal{O}}), \quad \tilde{\mathbf{v}}_{|\tilde{\Gamma}_0} = 0, \tag{66}$$

where $\tilde{f}_1^0 = \omega_0(\tilde{x}_1 - r_1)$ and $\tilde{f}_3^0 = \omega_0(\tilde{x}_3 - r_3)$.

Then we deduce that $\tilde{\mathbf{u}}^0$ *is the unique solution to the linearized elasticity problem* (66).

Remark 3. In (66) as a test function we have taken $\tilde{\mathbf{v}}$ in $H^1(\tilde{\mathcal{O}})$ and not in $W^{1,4}(\tilde{\mathcal{O}})$. This is correct since $W^{1,4}(\tilde{\mathcal{O}})$ is dense in $H^1(\tilde{\mathcal{O}})$. Now we only have $\tilde{\mathbf{u}}^0 \in H^1(\tilde{\mathcal{O}})$ and we suppose that $\tilde{\mathbf{u}}^0 \in W^{1,4}(\tilde{\mathcal{O}})$.

Actually, if we have $\tilde{f}_1^0, \tilde{f}_3^0 \in L^2(\tilde{\mathcal{O}})$ (and, with our choice, this is the case) and also $\tilde{\mathcal{O}}$ with smooth boundary or convex, then we have $\tilde{\mathbf{u}}^0 \in H^2(\tilde{\mathcal{O}}) \subset W^{1,4}(\tilde{\mathcal{O}})$.

The terms containing the factor of ε give the equation

$$A_1 + B_1 = L_1, \quad \forall\, (\tilde{\mathbf{v}}, \mathbf{v}) \in V(\varepsilon), \tag{67}$$

where

$$A_1 = \int_{\tilde{\mathcal{O}}} \left\{ \hat{\lambda} e_{pp}(\tilde{\mathbf{u}}^1)\delta_{ij} + 2\hat{\mu} e_{ij}(\tilde{\mathbf{u}}^1) \right\} e_{ij}(\tilde{\mathbf{v}})\,\mathrm{d}\tilde{\mathbf{x}}, \tag{68}$$

$$\begin{aligned} B_1 = \int_{\Omega^*} \Big\{ & \hat{\lambda}\Big(\partial_3 u_3^2 + e_{\alpha\alpha}^\theta(\mathbf{u}^0) + \frac{1}{2}\partial_\alpha u_3^0 \partial_\alpha u_3^0 + \partial_\alpha u_3^0 \partial_\alpha \theta \\ & + \frac{1}{2}\partial_3 u_\rho^0 \partial_3 u_\rho^0 \Big) \left(e_{\alpha\alpha}^\theta(\mathbf{v}) + \partial_\alpha u_3^0 (\partial_\alpha v_3 - \partial_\alpha \theta \partial_3 v_3) \right) \\ & + 2\hat{\mu}\Big[\Big(e_{\alpha\beta}^\theta(\mathbf{u}^0) + \frac{1}{2}\partial_\alpha u_3^0 \partial_\beta u_3^0 \Big) \\ & \times \Big(e_{\alpha\beta}^\theta(\mathbf{v}) + \partial_\alpha u_3^0 (\partial_\beta v_3 - \partial_\beta \theta \partial_3 v_3) \Big) \\ & + \Big(e_{\alpha 3}^\theta(\mathbf{u}^2) + \frac{1}{2}\partial_3 u_3^2 \partial_\alpha u_3^0 + d_{\alpha 3}^\sharp\Big(\varepsilon; \frac{1}{2}\mathbf{u}^0, \mathbf{u}^0\Big) \Big) e_{\alpha 3}^\theta(\mathbf{v}) \\ & + \Big(e_{3\alpha}^\theta(\mathbf{u}^2) + \frac{1}{2}\partial_3 u_3^2 \partial_\alpha u_3^0 + d_{3\alpha}^\sharp\Big(\varepsilon; \frac{1}{2}\mathbf{u}^0, \mathbf{u}^0\Big) \Big) \\ & \times e_{3\alpha}^\theta(\mathbf{v}) \Big] \Big\}\,\mathrm{d}\mathbf{x} \\ & + \int_{\Omega^*} \Big\{ \hat{\lambda}\Big[\partial_3 u_3^4 + \frac{1}{2}\partial_3 u_3^2 \partial_3 u_3^2 + e_{\alpha\alpha}^\theta(\mathbf{u}^2) \\ & + \frac{1}{2}\partial_\alpha u_3^1 \partial_\alpha u_3^1 + \partial_\alpha u_3^0 (\partial_\alpha u_3^2 - \partial_\alpha \theta \partial_3 u_3^2) \\ & + \partial_\alpha u_3^2 \partial_\alpha \theta + \partial_3 u_3^2 r_{33}(\varepsilon) + \partial_3 u_\rho^0 \partial_3 u_\rho^2 \\ & + \frac{1}{2}\partial_3 u_\rho^1 \partial_3 u_\rho^1 + \frac{1}{2}\partial_3 u_3^2 \partial_\eta u_3^0 \partial_\eta \theta + d_{pp}^\sharp\Big(\varepsilon; \frac{1}{2}\mathbf{u}^0, \mathbf{u}^0\Big)\Big] \\ & + 2\hat{\mu}\Big[\partial_3 u_3^4 + \frac{1}{2}\partial_3 u_3^2 \partial_3 u_3^2 + \partial_\alpha u_3^2 \partial_\alpha \theta + \partial_3 u_3^2 r_{33}(\varepsilon) \\ & + \partial_3 u_\rho^0 \partial_3 u_\rho^2 + \frac{1}{2}\partial_3 u_\rho^1 \partial_3 u_\rho^1 + \frac{1}{2}\partial_3 u_3^2 \partial_\eta u_3^0 \partial_\eta \theta \\ & + d_{33}^\sharp\Big(\varepsilon; \frac{1}{2}\mathbf{u}^0, \mathbf{u}^0\Big)\Big] \Big\} \partial_3 v_3\,\mathrm{d}\mathbf{x}, \end{aligned} \tag{69}$$

$$L_1 = \int_{\Omega^*} \omega_0 \left[x_1 v_1 + (x_3 + \theta + u_3^0) v_3 \right] \mathrm{d}\mathbf{x} - \int_{\Gamma_+} p(0) v_3\,\mathrm{d}a. \tag{70}$$

Using in (67) a test function $(\tilde{\mathbf{v}}, \mathbf{v}) \in V(\varepsilon)$ such that $\tilde{\mathbf{v}} = 0$ and $\mathbf{v} \in W^{1,4}(\Omega^*)$ with $\mathbf{v}_{|\Gamma^*} = 0$, we obtain

$$B_1 = L_1, \quad \forall\, \mathbf{v} \in W^{1,4}(\Omega^*);\ \mathbf{v}_{|\Gamma^*} = 0. \tag{71}$$

If we also take $\mathbf{v}$ in $V_{KL}(\Omega^*)$, use (55) and define

$$\gamma_{\alpha\beta}^\theta(\mathbf{u}^0) = e_{\alpha\beta}^\theta(\mathbf{u}^0) + \frac{1}{2}\partial_\alpha u_3^0 \partial_\beta u_3^0, \tag{72}$$

$$\tilde{m}_{\alpha\beta}^\theta(\mathbf{u}^0) = -\left\{ \frac{2\hat{\lambda}\hat{\mu}}{\hat{\lambda} + 2\hat{\mu}} \gamma_{\rho\rho}^\theta(\mathbf{u}^0)\delta_{\alpha\beta} + 2\hat{\mu}\gamma_{\alpha\beta}^\theta(\mathbf{u}^0) \right\}, \tag{73}$$

then (71) becomes

$$-\int_{\Omega^*} \tilde{m}_{\alpha\beta}^\theta(\mathbf{u}^0)(e_{\alpha\beta}^\theta(\mathbf{v}) + \partial_\alpha u_3^0 \partial_\beta v_3)\,\mathrm{d}\mathbf{x} = \int_{\Omega^*} \left[f_1^0 v_1 + (f_3^0 + \omega_0 u_3^0) v_3 \right] \mathrm{d}\mathbf{x} - \int_{\Gamma_+} p(0) v_3\,\mathrm{d}a, \quad \forall\, \mathbf{v} \in V_{KL}(\Omega^*);\ \mathbf{v}_{|\Gamma^*} = 0, \tag{74}$$

where $f_1^0 = \omega_0 x_1$ and $f_3^0 = \omega_0(x_3 + \theta)$.

Now, we are going to see that (74) is a two-dimensional problem, using the fact that $\mathbf{u}^0$, $\mathbf{v} \in V_{KL}(\Omega^*)$. We have then (58), (59) and from (57) (with $\mathbf{v}_{|\Gamma^*} = 0$) we also have

$$v_\alpha = \eta_\alpha - x_3 \partial_\alpha \eta_3,\ v_3 = \eta_3,\ \text{with}\ \eta_\alpha \in W^{1,4}(\omega^*), \quad \eta_3 \in W^{2,4}(\omega^*)\ \text{and}\ \eta_i = \partial_\nu \eta_3 = 0 \text{ on } \gamma^*. \tag{75}$$

Then we get

$$e_{\alpha\beta}^\theta(\mathbf{v}) = e_{\alpha\beta}^\theta(\eta) - x_3 \partial_{\alpha\beta}\eta_3, \tag{76}$$

where $e_{\alpha\beta}^\theta(\eta) = e_{\alpha\beta}(\eta) + \frac{1}{2}(\partial_\beta \theta \partial_\alpha \eta_3 + \partial_\alpha \theta \partial_\beta \eta_3)$.

We also have $\partial_\alpha u_3^0 \partial_\beta v_3 = \partial_\alpha \zeta_3 \partial_\beta \eta_3$, and

$$\gamma_{\alpha\beta}^\theta(\mathbf{u}^0) = e_{\alpha\beta}^\theta(\zeta) - x_3 \partial_{\alpha\beta}\zeta_3 + \frac{1}{2}\partial_\alpha \zeta_3 \partial_\beta \zeta_3. \tag{77}$$

Then we obtain

$$\tilde{m}^{\theta}_{\alpha\beta}(\mathbf{u}^0) = -\Big\{ \frac{2\hat{\lambda}\hat{\mu}}{\hat{\lambda}+2\hat{\mu}} \Big(e^{\theta}_{\rho\rho}(\zeta) + \frac{1}{2}\partial_\rho\zeta_3\partial_\rho\zeta_3 \Big) \delta_{\alpha\beta} + 2\hat{\mu} \Big(e^{\theta}_{\alpha\beta}(\zeta) + \frac{1}{2}\partial_\alpha\zeta_3\partial_\beta\zeta_3 \Big) \Big\} + x_3 \Big\{ \frac{2\hat{\lambda}\hat{\mu}}{\hat{\lambda}+2\hat{\mu}} (\Delta\zeta_3)\,\delta_{\alpha\beta} + 2\hat{\mu}\partial_{\alpha\beta}\zeta_3 \Big\}. \quad (78)$$

Performing the integration with respect to x_3 in (74) and introducing the notation

$$m^{\theta}_{\alpha\beta}(\zeta) = -\Big\{ \frac{4\hat{\lambda}\hat{\mu}}{3(\hat{\lambda}+2\hat{\mu})} (\Delta\zeta_3)\,\delta_{\alpha\beta} + \frac{4}{3}\hat{\mu}\partial_{\alpha\beta}\zeta_3 \Big\}, \quad (79)$$

$$n^{\theta}_{\alpha\beta}(\zeta) = \frac{4\hat{\lambda}\hat{\mu}}{\hat{\lambda}+2\hat{\mu}} \Big(e^{\theta}_{\rho\rho}(\zeta) + \frac{1}{2}\partial_\rho\zeta_3\partial_\rho\zeta_3 \Big) \delta_{\alpha\beta} + 4\hat{\mu} \Big(e^{\theta}_{\alpha\beta}(\zeta) + \frac{1}{2}\partial_\alpha\zeta_3\partial_\beta\zeta_3 \Big), \quad (80)$$

$$e^{\theta}_{\alpha\beta}(\eta) = \frac{1}{2}(\partial_\alpha\eta_\beta + \partial_\beta\eta_\alpha) + \frac{1}{2}(\partial_\alpha\theta\partial_\beta\eta_3 + \partial_\beta\theta\partial_\alpha\eta_3), \quad (81)$$

we prove the following result:

Theorem 1. *The function $\zeta = (\zeta_1, \zeta_2, \zeta_3)$ gives $\mathbf{u}^0$ through (58) and is a solution to the following variational problem :*

$$\zeta_\alpha \in W^{1,4}(\omega^*),\ \zeta_3 \in W^{2,4}(\omega^*), \quad \zeta_{\alpha|\gamma^*} = \big(\tilde{u}^0_{\alpha|\tilde{\omega}_\beta}\big)_{\,|\gamma^*},$$

$$\zeta_3 = \partial_\nu\zeta_3 = 0 \text{ on } \gamma^*,$$

$$-\int_{\omega^*} m^{\theta}_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\eta_3 \,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\omega^*} n^{\theta}_{\alpha\beta}(\zeta)\big(e^{\theta}_{\alpha\beta}(\eta) + \partial_\alpha\zeta_3\partial_\beta\eta_3\big)\,\mathrm{d}x_1\,\mathrm{d}x_2$$
$$= \int_{\omega^*} \Big\{ \Big(\int_{-1}^{1} f^0_1 \,\mathrm{d}x_3\Big)\eta_1 - \Big(\int_{-1}^{1} x_3 f^0_1 \,\mathrm{d}x_3\Big)\partial_1\eta_3 + \Big[\Big(\int_{-1}^{1} f^0_3 \,\mathrm{d}x_3\Big) + 2\omega_0\zeta_3 - p(0)\Big]\eta_3 \Big\}\,\mathrm{d}x_1\,\mathrm{d}x_2,$$
$$\forall\, (\eta_\alpha, \eta_3) \in W^{1,4}(\omega^*)^2 \times W^{2,4}(\omega^*);$$
$$\eta_i = \partial_\nu\eta_3 = 0 \text{ on } \gamma^*.$$

Remark 4. The last problem is a nonlinear shallow shell problem of the same type as that found in (Ciarlet and Paumier, 1986).

If we take first $\eta_\alpha = 0,\ \eta_3 \in W^{2,4}(\omega^*)$ and then $\eta_3 = 0,\ \eta_\alpha \in W^{1,4}(\omega^*)$ in the last variational problem, we obtain

$$-\int_{\omega^*} m^{\theta}_{\alpha\beta}(\zeta)\partial_{\alpha\beta}\eta_3 \,\mathrm{d}x_1\,\mathrm{d}x_2 + \int_{\omega^*} n^{\theta}_{\alpha\beta}(\zeta)(\partial_\alpha\theta + \partial_\alpha\zeta_3)\partial_\beta\eta_3 \,\mathrm{d}x_1\,\mathrm{d}x_2$$
$$= \int_{\omega^*} \Big\{ -\Big(\int_{-1}^{1} x_3 f^0_1 \,\mathrm{d}x_3\Big)\partial_1\eta_3 + \Big[\Big(\int_{-1}^{1} f^0_3 \,\mathrm{d}x_3\Big) + 2\omega_0\zeta_3 - p(0)\Big]\eta_3 \Big\}\,\mathrm{d}x_1\,\mathrm{d}x_2,$$
$$\forall\, \eta_3 \in W^{2,4}(\omega^*); \quad \eta_3 = \partial_\nu\eta_3 = 0 \text{ on } \gamma^*, \quad (82)$$

$$\int_{\omega^*} n^{\theta}_{\alpha\beta}(\zeta)\partial_\beta\eta_\alpha \,\mathrm{d}x_1\,\mathrm{d}x_2 = \int_{\omega^*} \Big[\Big(\int_{-1}^{1} f^0_1 \,\mathrm{d}x_3\Big)\eta_1\Big]\,\mathrm{d}x_1\,\mathrm{d}x_2,$$
$$\forall\, \eta_\alpha \in W^{1,4}(\omega^*); \quad \eta_\alpha = 0 \text{ on } \gamma^*. \quad (83)$$

If the solution ζ to the problem (82), (83) is smooth enough (e.g. $\zeta_\alpha \in H^3(\omega^*)$ and $\zeta_3 \in H^4(\omega^*)$), we can apply Green's formula to these problems and obtain the following result:

Theorem 2. *If ζ is a sufficiently smooth solution to the variational problem (82) and (83), then it is also a solution to the strong problem*

$$-\partial_{\alpha\beta}m^{\theta}_{\alpha\beta}(\zeta) - \partial_\beta\big(n^{\theta}_{\alpha\beta}(\zeta)\partial_\alpha(\theta + \zeta_3)\big) = \int_{-1}^{1} f^0_3 \,\mathrm{d}x_3 + \partial_1\Big(\int_{-1}^{1} x_3 f^0_1 \,\mathrm{d}x_3\Big) + 2\omega_0\zeta_3 - p(0) \text{ in } \omega^*,$$

$$-\partial_\beta n^{\theta}_{\alpha\beta}(\zeta) = \Big(\int_{-1}^{1} f^0_1 \,\mathrm{d}x_3\Big)\delta_{\alpha 1} \text{ in } \omega^*,$$

$$\zeta_{\alpha|\gamma^*} = \big(\tilde{u}^0_{\alpha|\tilde{\omega}_\beta}\big)_{\,|\gamma^*}, \quad \zeta_3 = \partial_\nu\zeta_3 = 0 \text{ on } \gamma^*,$$

$$\big(\partial_\beta m^{\theta}_{\alpha\beta}(\zeta)\big)\nu_\alpha + n^{\theta}_{\alpha\beta}(\zeta)\partial_\alpha(\theta+\zeta_3)\nu_\beta + \partial_\tau\big(m^{\theta}_{\alpha\beta}(\zeta)\tau_\alpha\nu_\beta\big) = -\Big(\int_{-1}^{1} x_3 f^0_1 \,\mathrm{d}x_3\Big)\nu_1 \text{ on } \partial\omega^* - \gamma^*,$$

$$m^{\theta}_{\alpha\beta}(\zeta)\nu_\alpha\nu_\beta = 0 \quad \text{on } \partial\omega^* - \gamma^*,$$
$$n^{\theta}_{\alpha\beta}(\zeta)\nu_\beta = 0 \quad \text{on } \partial\omega^* - \gamma^*.$$

Remark 5. Let us point out that $\tau = (\tau_1, \tau_2)$ is the tangent on γ^* and that with our choice of forces we have

$f_1^0 = \omega_0 x_1,\ f_3^0 = \omega_0(x_3+\theta)$, and then in the last theorem

$$\int_{-1}^{1} f_1^0 \,\mathrm{d}x_3 = 2\omega_0 x_1, \quad \int_{-1}^{1} f_3^0 \,\mathrm{d}x_3 = 2\omega_0 \theta,$$

$$\int_{-1}^{1} x_3 f_1^0 \,\mathrm{d}x_3 = 0.$$

We should determine now $\tilde{\mathbf{u}}^1$. From (67) we have

$$\int_{\tilde{\mathcal{O}}} \Big\{ \hat{\lambda} e_{pp}(\tilde{\mathbf{u}}^1)\delta_{ij} + 2\hat{\mu} e_{ij}(\tilde{\mathbf{u}}^1) \Big\} e_{ij}(\tilde{\mathbf{v}}) \,\mathrm{d}\tilde{\mathbf{x}} = L_1 - B_1, \quad \forall\,(\tilde{\mathbf{v}},\mathbf{v}) \in V(\varepsilon), \tag{84}$$

and (71) gives $L_1 - B_1 = 0$ if $\tilde{\mathbf{v}} \in W^{1,4}(\tilde{\mathcal{O}})$, and there exists $\mathbf{v} \in W^{1,4}(\Omega)$ such that $(\tilde{\mathbf{v}},\mathbf{v}) \in V(\varepsilon)$ and $\mathbf{v}_{|\Gamma^*} = 0$. We cannot guarantee the existence of $\mathbf{v}$ satisfying these conditions for each $\tilde{\mathbf{v}}$, so $L_1 - B_1$ can be non-zero. In this way, we cannot compute $\tilde{\mathbf{u}}^1$, that is in general non-zero.

5. Problem in the Original Domain

In the previous section, we found the zeroth-order problem in reference sets. We must now perform a change of variables to the original domain to obtain a zeroth-order problem in the original domain. Let us consider the following functions defined on $\mathcal{S}^\varepsilon$:

$$\tilde{u}_i^\varepsilon(0)(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^2 \tilde{u}_i^0(\tilde{\mathbf{x}}), \quad \forall\,\tilde{\mathbf{x}} \in \tilde{\mathcal{O}},$$
$$u_\alpha^\varepsilon(0)(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^2 u_\alpha^0(\mathbf{x}), \quad u_3^\varepsilon(0)(\hat{\mathbf{x}}^\varepsilon) = \varepsilon u_3^0(\mathbf{x}), \quad \forall\,\mathbf{x} \in \Omega. \tag{85}$$

If we set

$$\zeta_\alpha^\varepsilon(x_1,x_2) = \varepsilon^2 \zeta_\alpha(x_1,x_2), \quad \zeta_3^\varepsilon(x_1,x_2) = \varepsilon \zeta_3(x_1,x_2), \quad \forall\,(x_1,x_2) \in \omega, \tag{86}$$

then we have

$$\begin{aligned} u_\alpha^\varepsilon(0)(\hat{\mathbf{x}}^\varepsilon) &= \zeta_\alpha^\varepsilon(x_1,x_2) - x_3^\varepsilon \partial_\alpha \zeta_3^\varepsilon(x_1,x_2), \\ u_3^\varepsilon(0)(\hat{\mathbf{x}}^\varepsilon) &= \zeta_3^\varepsilon(x_1,x_2). \end{aligned} \tag{87}$$

The forces $\tilde{f}_i^0$, f_i^0 and $p(0)$ are only approximations of order zero of forces $\tilde{f}_i(\varepsilon)$, $f_i(\varepsilon)$ and $p(\varepsilon)$, so when we perform the change of variables to the original domain $\mathcal{S}^\varepsilon$, we do not recover the original forces $\hat{f}_i^\varepsilon$ and $\hat{p}^\varepsilon$, and we must define

$$\tilde{F}_1^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{-\gamma} \tilde{f}_1^0(\tilde{\mathbf{x}}), \quad \tilde{F}_2^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = 0,$$
$$\tilde{F}_3^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{-\gamma} \tilde{f}_3^0(\tilde{\mathbf{x}}),$$

$$\bar{F}_1^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{-\gamma} f_1^0(\mathbf{x}), \quad \bar{F}_2^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = 0,$$
$$\bar{F}_3^\varepsilon(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{-\gamma+1}\big(f_3^0(\mathbf{x}) + \omega_0 u_3^0(\mathbf{x})\big),$$

$$P^\varepsilon\big(G^\varepsilon(x_1,x_2)\big) = \varepsilon^{-\eta} p(0)(x_1,x_2).$$

We must also scale $m_{\alpha\beta}^\theta(\zeta)$, $n_{\alpha\beta}^\theta(\zeta)$ and $e_{\alpha\beta}^\theta(\zeta)$, and we define

$$m_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon) = -\varepsilon^3 \left\{ \frac{4\hat{\lambda}^\varepsilon \hat{\mu}^\varepsilon}{3(\hat{\lambda}^\varepsilon + 2\hat{\mu}^\varepsilon)} (\Delta\zeta_3^\varepsilon)\,\delta_{\alpha\beta} + \frac{4}{3}\hat{\mu}^\varepsilon \partial_{\alpha\beta}\zeta_3^\varepsilon \right\},$$

$$\begin{aligned} n_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon) = \varepsilon \Bigg\{ &\frac{4\hat{\lambda}^\varepsilon \hat{\mu}^\varepsilon}{\hat{\lambda}^\varepsilon + 2\hat{\mu}^\varepsilon} \left(e_{\rho\rho}^{\theta^\varepsilon}(\zeta^\varepsilon) + \frac{1}{2}\partial_\rho \zeta_3^\varepsilon \partial_\rho \zeta_3^\varepsilon \right) \delta_{\alpha\beta} \\ &+ 4\hat{\mu}^\varepsilon \left(e_{\alpha\beta}^{\theta^\varepsilon}(\zeta^\varepsilon) + \frac{1}{2}\partial_\alpha \zeta_3^\varepsilon \partial_\beta \zeta_3^\varepsilon \right) \Bigg\}, \end{aligned}$$

$$e_{\alpha\beta}^{\theta^\varepsilon}(\zeta^\varepsilon) = \frac{1}{2}(\partial_\alpha \zeta_\beta^\varepsilon + \partial_\beta \zeta_\alpha^\varepsilon) + \frac{1}{2}(\partial_\alpha \theta^\varepsilon \partial_\beta \zeta_3^\varepsilon + \partial_\beta \theta^\varepsilon \partial_\alpha \zeta_3^\varepsilon).$$

Then, using (66) and Theorem 2, and after the change of scale, we obtain the following result:

Theorem 3. *The function* $\tilde{\mathbf{u}}^\varepsilon(0)$ *is the unique solution to the following problem:*

$$\tilde{\mathbf{u}}^\varepsilon(0) \in H^1(\mathcal{O}), \quad \tilde{\mathbf{u}}^\varepsilon(0)_{|\Gamma_0} = 0,$$
$$-\partial_j \left\{ \hat{\lambda}^\varepsilon e_{pp}\big(\tilde{\mathbf{u}}^\varepsilon(0)\big)\delta_{ij} + 2\hat{\mu}^\varepsilon e_{ij}\big(\tilde{\mathbf{u}}^\varepsilon(0)\big) \right\} = \tilde{F}_i^\varepsilon \ \text{ in } \mathcal{O},$$
$$\left\{ \hat{\lambda}^\varepsilon e_{pp}\big(\tilde{\mathbf{u}}^\varepsilon(0)\big)\delta_{ij} + 2\hat{\mu}^\varepsilon e_{ij}\big(\tilde{\mathbf{u}}^\varepsilon(0)\big) \right\} n_j = 0 \ \text{ on } \partial\mathcal{O} - \Gamma_0,$$

and ζ^ε *is a solution to the problem*

$$\begin{aligned} -\partial_{\alpha\beta} m_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon) &- \partial_\beta\big(n_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon)\partial_\alpha(\theta^\varepsilon + \zeta_3^\varepsilon)\big) \\ &= \int_{-\varepsilon}^{\varepsilon} \bar{F}_3^\varepsilon \,\mathrm{d}x_3^\varepsilon + \partial_1\Big(\int_{-\varepsilon}^{\varepsilon} x_3^\varepsilon \bar{F}_1^\varepsilon \,\mathrm{d}x_3^\varepsilon\Big) - P^\varepsilon \ \text{ in } \omega^*, \end{aligned}$$

$$-\partial_\beta n_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon) = \Big(\int_{-\varepsilon}^{\varepsilon} \bar{F}_1^\varepsilon \,\mathrm{d}x_3^\varepsilon\Big)\delta_{\alpha 1} \ \text{ in } \omega^*,$$

$$\zeta_{\alpha|\gamma^*}^\varepsilon = \big(\tilde{u}_\alpha^\varepsilon(0)_{|\tilde{\omega}_\beta}\big)_{|\gamma^*}, \quad \zeta_3^\varepsilon = \partial_\nu \zeta_3^\varepsilon = 0 \ \text{ on } \gamma^*,$$

$$\begin{aligned} \big(\partial_\beta m_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon)\big)\nu_\alpha &+ n_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon)\partial_\alpha(\theta^\varepsilon + \zeta_3^\varepsilon)\nu_\beta \\ &+ \partial_\tau\big(m_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon)\tau_\alpha\nu_\beta\big) \\ &= -\Big(\int_{-\varepsilon}^{\varepsilon} x_3^\varepsilon \bar{F}_1^\varepsilon \,\mathrm{d}x_3^\varepsilon\Big)\nu_1 \ \text{ on } \partial\omega^* - \gamma^*, \end{aligned}$$

$$m_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon)\nu_\alpha\nu_\beta = 0, \ n_{\alpha\beta}^{\theta^\varepsilon,\varepsilon}(\zeta^\varepsilon)\nu_\beta = 0 \ \text{ on } \partial\omega^* - \gamma^*.$$

The forces $\tilde{F}_i^\varepsilon$, $\bar{F}_i^\varepsilon$ and P^ε are 'at order zero' equal to forces $\hat{f}_i^\varepsilon$ and $\hat{p}^\varepsilon$ (i.e. the first terms of their respective asymptotic expansions are equal), so if we change in Theorem 3 ones for the others, we obtain an approximation of the same order of the exact solution $\mathbf{u}^\varepsilon$. This will allow us to write a model without referring to the scalings in ε. In the same way (as we did in (Rodriguez, 1999)), we can change the boundary conditions of ζ^ε for conditions of the 'same order'. Then we can change the boundary conditions

$$\zeta^\varepsilon_{\alpha|\gamma^*} = \left(\tilde{u}^\varepsilon_\alpha(0)_{|\tilde{\omega}_\beta}\right)_{|\gamma^*}, \quad \zeta_3^\varepsilon = \partial_\nu \zeta_3^\varepsilon = 0 \text{ on } \gamma^*,$$

for

$$\zeta^\varepsilon_{i\,|\gamma^*} = \left(\tilde{u}^\varepsilon_i(0)_{|\tilde{\omega}_\beta}\right)_{|\gamma^*}, \quad \partial_\nu \zeta^\varepsilon_{3|\gamma^*} = \left(\left(\partial_\nu \tilde{u}^\varepsilon_3(0)\right)_{|\tilde{\omega}_\beta}\right)_{|\gamma^*},$$

where ν is the unit outward normal to ω^* on γ^*.

Following these considerations, we can introduce the following model:

Proposed model:

$$-\partial_j\left\{\hat{\lambda}^\varepsilon e_{pp}\big(\tilde{\mathbf{u}}^\varepsilon(0)\big)\delta_{ij} + 2\hat{\mu}^\varepsilon e_{ij}\big(\tilde{\mathbf{u}}^\varepsilon(0)\big)\right\} = \hat{f}_i^\varepsilon \text{ in } \mathcal{O},$$

$$\tilde{\mathbf{u}}^\varepsilon(0) = 0 \text{ on } \Gamma_0,$$

$$\left\{\hat{\lambda}^\varepsilon e_{pp}\big(\tilde{\mathbf{u}}^\varepsilon(0)\big)\delta_{ij} + 2\hat{\mu}^\varepsilon e_{ij}\big(\tilde{\mathbf{u}}^\varepsilon(0)\big)\right\} n_j = 0 \text{ on } \partial\mathcal{O} - \Gamma_0,$$

$$-\partial_{\alpha\beta} m^{\theta^\varepsilon,\varepsilon}_{\alpha\beta}(\zeta^\varepsilon) - \partial_\beta\big(n^{\theta^\varepsilon,\varepsilon}_{\alpha\beta}(\zeta^\varepsilon)\partial_\alpha(\theta^\varepsilon + \zeta_3^\varepsilon)\big)$$
$$= \int_{-\varepsilon}^{\varepsilon} \hat{f}_3^\varepsilon \,\mathrm{d}x_3^\varepsilon + \partial_1\Big(\int_{-\varepsilon}^{\varepsilon} x_3^\varepsilon \hat{f}_1^\varepsilon \,\mathrm{d}x_3^\varepsilon\Big) - \hat{p}^\varepsilon \text{ in } \omega^*,$$

$$-\partial_\beta n^{\theta^\varepsilon,\varepsilon}_{\alpha\beta}(\zeta^\varepsilon) = \Big(\int_{-\varepsilon}^{\varepsilon} \hat{f}_1^\varepsilon \,\mathrm{d}x_3^\varepsilon\Big)\delta_{\alpha 1} \text{ in } \omega^*,$$

$$\zeta^\varepsilon_{i\,|\gamma^*} = \left(\tilde{u}^\varepsilon_i(0)_{|\tilde{\omega}_\beta}\right)_{|\gamma^*}, \quad \partial_\nu \zeta^\varepsilon_{3|\gamma^*} = \left(\left(\partial_\nu \tilde{u}^\varepsilon_3(0)\right)_{|\tilde{\omega}_\beta}\right)_{|\gamma^*},$$

$$\big(\partial_\beta m^{\theta^\varepsilon,\varepsilon}_{\alpha\beta}(\zeta^\varepsilon)\big)\nu_\alpha + n^{\theta^\varepsilon,\varepsilon}_{\alpha\beta}(\zeta^\varepsilon)\partial_\alpha(\theta^\varepsilon + \zeta_3^\varepsilon)\nu_\beta$$
$$+\partial_\tau\big(m^{\theta^\varepsilon,\varepsilon}_{\alpha\beta}(\zeta^\varepsilon)\tau_\alpha\nu_\beta\big)$$
$$= -\Big(\int_{-\varepsilon}^{\varepsilon} x_3^\varepsilon \hat{f}_1^\varepsilon \,\mathrm{d}x_3^\varepsilon\Big)\nu_1 \text{ on } \partial\omega^* - \gamma^*,$$

$$m^{\theta^\varepsilon,\varepsilon}_{\alpha\beta}(\zeta^\varepsilon)\nu_\alpha\nu_\beta = 0, \; n^{\theta^\varepsilon,\varepsilon}_{\alpha\beta}(\zeta^\varepsilon)\nu_\beta = 0 \text{ on } \partial\omega^* - \gamma^*.$$

Remark 6. The proposed model is a 3D model of linear elasticity in solid $\mathcal{O}$ and a nonlinear model of shallow shells in ω^*, where boundary conditions are determined by a displacement of junction in $\mathcal{O}$. This nonlinear model of shallow shells is of the same type as that found in (Ciarlet and Paumier, 1986).

Remark 7. As was done in (Rodriguez, 1999), we can change the boundary conditions

$$\zeta^\varepsilon_{i\,|\gamma^*} = \left(\tilde{u}^\varepsilon_i(0)_{|\tilde{\omega}_\beta}\right)_{|\gamma^*}, \quad \partial_\nu \zeta^\varepsilon_{3|\gamma^*} = \left(\left(\partial_\nu \tilde{u}^\varepsilon_3(0)\right)_{|\tilde{\omega}_\beta}\right)_{|\gamma^*},$$

for

$$\zeta^\varepsilon_{i\,|\gamma^*} = \left(\left[\tilde{u}^\varepsilon_i(0)\circ\Theta^\varepsilon\right]_{|\tilde{\omega}_\beta}\right)_{|\gamma^*},$$
$$\partial_\nu \zeta^\varepsilon_{3|\gamma^*} = \left(\partial_\nu\left[\tilde{u}^\varepsilon_3(0)\circ\Theta^\varepsilon\right]_{|\tilde{\omega}_\beta}\right)_{|\gamma^*},$$

and then we have a model of same order as exposed, but giving also the continuity of the deformation of $\Theta^\varepsilon(\omega)$.

6. Stress Approximation

We shall use Hooke's law to compute the stresses:

$$\hat{\sigma}^\varepsilon_{ij}(\hat{\mathbf{x}}^\varepsilon) = \left\{\hat{\lambda}^\varepsilon \hat{E}^\varepsilon_{pp}(\hat{\mathbf{u}}^\varepsilon)\delta_{ij} + 2\hat{\mu}^\varepsilon \hat{E}^\varepsilon_{ij}(\hat{\mathbf{u}}^\varepsilon)\right\}(\hat{\mathbf{x}}^\varepsilon). \quad (88)$$

We are going to use asymptotic expansion to obtain the stresses, so we shall need a scaling of the same type as in (Ciarlet, 1990):

$$\hat{\sigma}^\varepsilon_{ij}(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{2-t}\tilde{\sigma}_{ij}(\varepsilon)(\tilde{\mathbf{x}}), \quad \forall\, \tilde{\mathbf{x}} \in \tilde{\mathcal{O}}, \quad (89)$$

$$\hat{\sigma}^\varepsilon_{\alpha\beta}(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{2-t}\sigma_{\alpha\beta}(\varepsilon)(\mathbf{x}), \quad \hat{\sigma}^\varepsilon_{\alpha 3}(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{3-t}\sigma_{\alpha 3}(\varepsilon)(\mathbf{x}),$$

$$\hat{\sigma}^\varepsilon_{33}(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{4-t}\sigma_{33}(\varepsilon)(\mathbf{x}), \quad \forall\, \mathbf{x} \in \Omega. \quad (90)$$

We have $\tilde{\mathbf{u}}(\varepsilon) = \tilde{\mathbf{u}}^0 + \varepsilon\tilde{\mathbf{u}}^1 + \varepsilon^2\tilde{\mathbf{u}}^2 + \cdots$ in $\tilde{\mathcal{O}}$ and $\mathbf{u}(\varepsilon) = \mathbf{u}^0 + \varepsilon\mathbf{u}^1 + \varepsilon^2\mathbf{u}^2 + \cdots$ in Ω. From (88)–(90) and (3), (16), (17), we obtain

$$\tilde{\sigma}_{ij}(\varepsilon) = \tilde{\sigma}^0_{ij} + \varepsilon\tilde{\sigma}^1_{ij} + \varepsilon^2\tilde{\sigma}^2_{ij} + \cdots \quad \text{in } \tilde{\mathcal{O}},$$

$$\sigma_{\alpha\beta}(\varepsilon) = \sigma^0_{\alpha\beta} + \varepsilon\sigma^1_{\alpha\beta} + \cdots \quad \text{in } \Omega,$$

$$\sigma_{\alpha 3}(\varepsilon) = \varepsilon^{-1}\sigma^{-1}_{33} + \sigma^0_{\alpha 3} + \varepsilon\sigma^1_{\alpha 3} + \cdots \quad \text{in } \Omega,$$

$$\sigma_{33}(\varepsilon) = \varepsilon^{-2}\sigma^{-2}_{33} + \varepsilon^{-1}\sigma^{-1}_{33} + \sigma^0_{33} + \cdots \quad \text{in } \Omega.$$

Using expressions (18)–(25), (52)-(54) and (64), we obtain (see (Rodriguez, 1997) for details)

$$\tilde{\sigma}^0_{ij} = \hat{\lambda} e_{pp}(\tilde{\mathbf{u}}^0)\delta_{ij} + 2\hat{\mu} e_{ij}(\tilde{\mathbf{u}}^0) \quad \text{in } \tilde{\mathcal{O}},$$

$$\sigma^0_{\alpha\beta} = \frac{1}{2} n^\theta_{\alpha\beta}(\zeta) + \frac{3}{2} x_3 m^\theta_{\alpha\beta}(\zeta) \quad \text{in } \Omega^*,$$

$$\sigma^{-1}_{\alpha 3} = \sigma^{-2}_{33} = \sigma^{-1}_{33} = 0 \quad \text{in } \Omega^*.$$

Now we must return to the original domains $\mathcal{O}$ and $\hat{\Omega}^\varepsilon$, where we propose the following approximations:

$$\tilde{\sigma}^\varepsilon_{ij}(0)(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{2-t}\tilde{\sigma}^0_{ij}(\tilde{\mathbf{x}}), \quad \tilde{\mathbf{x}} \in \tilde{\mathcal{O}},$$

$$\sigma^\varepsilon_{\alpha\beta}(0)(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{2-t}\sigma^0_{\alpha\beta}(\mathbf{x}), \quad \mathbf{x} \in \Omega^*,$$

$$\sigma^\varepsilon_{\alpha 3}(0)(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{3-t}\sigma^0_{\alpha 3}(\mathbf{x}), \; \sigma^\varepsilon_{33}(0)(\hat{\mathbf{x}}^\varepsilon) = \varepsilon^{4-t}\sigma^0_{33}(\mathbf{x}),$$
$$\mathbf{x} \in \Omega^*,$$

i.e. we have

$$\tilde{\sigma}^{\varepsilon}_{ij}(0) = \Big\{\hat{\lambda}^{\varepsilon}\hat{e}^{\varepsilon}_{pp}\big(\tilde{\mathbf{u}}^{\varepsilon}(0)\big)\delta_{ij} + 2\hat{\mu}^{\varepsilon}\hat{e}^{\varepsilon}_{ij}\big(\tilde{\mathbf{u}}^{\varepsilon}(0)\big)\Big\}, \quad (91)$$

$$\sigma^{\varepsilon}_{\alpha\beta}(0) = \frac{1}{2\varepsilon}n^{\theta^{\varepsilon},\varepsilon}_{\alpha\beta}(\zeta^{\varepsilon}) + \frac{3}{2\varepsilon^3}x^{\varepsilon}_3 m^{\theta^{\varepsilon},\varepsilon}_{\alpha\beta}(\zeta^{\varepsilon}). \quad (92)$$

Remark 8. The zeroth-order approximation of stress in $\mathcal{O}$ of (91) is a linear approximation. Equation (92) is a classic equation for plane stress (Ciarlet, 1990), where $n^{\theta^{\varepsilon},\varepsilon}_{\alpha\beta}(\zeta^{\varepsilon})$ and $m^{\theta^{\varepsilon},\varepsilon}_{\alpha\beta}(\zeta^{\varepsilon})$ are the stress resultants and bending moments, respectively. We also have

$$n^{\theta^{\varepsilon},\varepsilon}_{\alpha\beta}(\zeta^{\varepsilon}) = \int_{-\varepsilon}^{\varepsilon}\sigma^{\varepsilon}_{\alpha\beta}(0)\,\mathrm{d}x^{\varepsilon}_3,$$

$$m^{\theta^{\varepsilon},\varepsilon}_{\alpha\beta}(\zeta^{\varepsilon}) = \int_{-\varepsilon}^{\varepsilon}x^{\varepsilon}_3\sigma^{\varepsilon}_{\alpha\beta}(0)\,\mathrm{d}x^{\varepsilon}_3.$$

As regards $\sigma^{\varepsilon}_{\alpha 3}(0)$ and $\sigma^{\varepsilon}_{33}(0)$, we only know that they are negligible with respect to $\sigma^{\varepsilon}_{\alpha\beta}(0)$ (because $\sigma^{\varepsilon}_{\alpha\beta}(0)$ is of order $O(\varepsilon^{2-t})$ and $\sigma^{\varepsilon}_{\alpha 3}(0)$, $\sigma^{\varepsilon}_{33}(0)$ are of orders $O(\varepsilon^{3-t})$ and $O(\varepsilon^{4-t})$, respectively).

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