A REALIZATION PROBLEM FOR POSITIVE CONTINUOUS–TIME SYSTEMS WITH REDUCED NUMBERS OF DELAYS

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A realization problem for positive, continuous-time linear systems with reduced numbers of delays in state and in control is formulated and solved. Sufficient conditions for the existence of positive realizations with reduced numbers of delays of a given proper transfer function are established. A procedure for the computation of positive realizations with reduced numbers of delays is presented and illustrated by an example.

Keywords: positive realization, continuous-time system, delay, existence, computation

1. Introduction

In positive systems inputs, state variables and outputs take only nonnegative values. Examples of positive systems are industrial processes involving chemical reactors, heat exchangers and distillation columns, storage systems, compartmental systems, water and atmospheric pollution models. A variety of models having positive linear systems behaviour can be found in engineering, management science, economics, social sciences, biology and medicine, etc.

Positive linear systems are defined on cones and not on linear spaces. Therefore, the theory of positive systems is more complicated and less advanced. An overview of the state of the art in positive systems theory is given in the monographs (Farina and Rinaldi, 2000; Kaczorek, 2002). Recent developments in positive systems theory and some new results are given in (Kaczorek, 2003). Realization problems of positive linear systems without time-delays were considered in many papers and books (Benvenuti and Farina, 2004; Farina and Rinaldi, 2000; Kaczorek, 2002).

An explicit solution of equations describing discrete-time systems with time-delay was given in (Busłowicz, 1982). Recently, the reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays were considered in (Busłowicz and Kaczorek, 2004; Kaczorek and Busłowicz, 2006; Xie and Wang, 2003). The realization problem for positive multivariable discrete-time systems with one time-delay was formulated and solved in (Kaczorek, 2004) and (Kaczorek and Busłowicz, 2004). The realization problem for positive continuous-time systems with delays in state was considered in (Kaczorek, 2005a), and with delays in state and control in (Kaczorek, 2005b). The methods presented in (Kaczorek, 2005a; 2005b) enable us to find positive realizations with the numbers of delays equal to the highest powers of the variables s and w in given transfer functions.

The main purpose of this paper is to present a method for the computation of positive realizations for positive continuous-time systems with reduced numbers of time-delays in state and control. Sufficient conditions for the solvability of the realization problem will be established and a procedure for the computation of a positive realization of a proper transfer function will be presented. To the best of the author's knowledge, the realization problem for positive continuous-time linear systems with reduced numbers of delays in the state vector and control has not been considered yet.

2. Preliminaries and Problem Formulation

Consider the multivariable continuous-time system with h delays in state and q delays in control:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=0}^{h} A_i x(t-id) + \sum_{j=0}^{q} B_j u(t-jd),\\ y(t) &= Cx(t) + Du(t),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input (control) and output vectors, respectively, and $A_i \in \mathbb{R}^{n\times n}$, $i = 0, 1, \dots, h$, $B_j \in \mathbb{R}^{n\times m}$, $j = 0, 1, \dots, q$, $C \in \mathbb{R}^{p\times n}$, $D \in \mathbb{R}^{p\times m}$ and $d > 0$ is a delay. The initial conditions for (1) are given by

$$x_0(t) \text{ for } t \in [-hd,\, 0] \text{ and } u_0(t) \text{ for } t \in [-hq,\, 0). \quad (2)$$

Let $\mathbb{R}_+^{m\times n}$ be the set of $m \times n$ real matrices with nonnegative entries and $\mathbb{R}_+^m = \mathbb{R}_+^{m\times 1}$.

Definition 1. The system (1) is called (internally) *positive* if for every $x_0(t) \in \mathbb{R}_+^n$, $t \in [-hd, 0]$, $u_0(t) \in \mathbb{R}_+^m$, $t \in [-qh, 0)$ and all inputs $u(t) \in \mathbb{R}_+^m$, $t \geq 0$ we have $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$ for $t \geq 0$.

Let M_n be the set of $n \times n$ Metzler matrices, i.e., the set of $n \times n$ real matrices with nonnegative off diagonal entries.

Theorem 1. *The system (1) is positive if and only if A_0 is a Metzler matrix and the matrices A_i, $i = 1, \dots, h$, B_j, $j = 0, 1, \dots, q$, C, D have nonnegative entries, i.e.,*

$$\begin{aligned} A_0 \in M_n, \quad A_i \in \mathbb{R}_+^{n\times n}, \quad i = 1, \dots, h, \\ B_j \in \mathbb{R}_+^{n\times m}, \quad j = 0, 1, \dots, q, \\ C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m}. \end{aligned} \quad (3)$$

Proof. The proof is given in (Kaczorek, 2005a). ■

The transfer function of the system (1) is given by

$$\begin{aligned} T(s,w) &= C[I_n s - A_0 - A_1 w - \cdots - A_h w^h]^{-1} \\ &\quad \times [B_0 + B_1 w + \cdots + B_q w^q] + D, \\ w &= e^{-ds}. \end{aligned} \quad (4)$$

Definition 2. The matrices (3) are called a *positive realization* of a given transfer function $T(s,w)$ if they satisfy the equality (4). A realization is called minimal if the dimension $n \times n$ of the matrices A_i, $i = 0, 1, \dots, h$ is minimal among all realizations of $T(s,w)$.

The positive realization problem under consideration can be stated as follows: Given a proper transfer matrix $T(s,w)$, find a positive realization with reduced numbers of delays of $T(s,w)$. In this paper, sufficient conditions for the solvability of the problem will be established and a procedure for the computation of a positive realization with reduced numbers of delays will be proposed.

3. Problem Solution

The transfer matrix (4) can be rewritten in the form

$$\begin{aligned} T(s,w) &= \frac{C\big(\operatorname{Adj} H(s,w)\big)(B_0 + B_1 w + \cdots + B_q w^q)}{\det H(s,w)} + D \\ &= \frac{N(s,w)}{d(s,w)} + D, \end{aligned} \quad (5)$$

where

$$H(s,w) = [I_n s - A_0 - A_1 w - \cdots - A_h w^h], \quad (6)$$

$$\begin{aligned} N(s,w) &= C\big(\operatorname{Adj} H(s,w)\big)(B_0 + B_1 w + \cdots + B_q w^q), \\ d(s,w) &= \det H(s,w). \end{aligned} \quad (7)$$

From (5) we have

$$D = \lim_{s\to\infty} T(s,w) \quad (8)$$

since $\lim_{s\to\infty} H^{-1}(s,w) = 0$. The strictly proper part of $T(s,w)$ is given by

$$T_{sp}(s,w) = T(s,w) - D = \frac{N(s,w)}{d(s,w)}. \quad (9)$$

Therefore, the positive realization problem has been reduced to finding matrices

$$\begin{aligned} A_0 \in M_n, \quad A_i \in \mathbb{R}_+^{n\times n}, \quad i = 1, \dots, h, \\ B_j \in \mathbb{R}_+^{n\times m}, \quad j = 0, 1, \dots, q, \quad \in \mathbb{R}_+^{p\times n} \end{aligned} \quad (10)$$

for a given strictly proper transfer matrix (9).

To simplify the notation, we shall consider a single-input single-output (SISO) system described by Eqn. (1) for $m = p = 1$. Let a given strictly proper, irreducible transfer function have the form

$$T(s,w) = \frac{n(s,w)}{d(s,w)}, \quad (11a)$$

where

$$\begin{aligned} n(s,w) &= b_{n-1}(w)s^{n-1} + \cdots + b_1(w)s + b_0(w) \\ b_k(w) &= b_{km}w^m + \cdots + b_{k1}w + b_{k0}, \\ &\qquad k = 0, 1, \dots, n-1 \end{aligned} \quad (11b)$$

$$\begin{aligned} d(s,w) &= s^n - a_{n-1}(w)s^{n-1} - \cdots - a_1(w)s - a_0(w) \\ a_k(w) &= a_{km}w^m + \cdots + a_{k1}w + a_{k0}, \\ &\qquad k = 0, 1, \dots, n-1. \end{aligned} \quad (11c)$$

The solution of the positive realization problem for (11) is based on the following two lemmas:

Lemma 1. *Let $p_k = p_k(w)$ for $k = 1, \dots, 2n-1$ be some polynomials in w with nonnegative coefficients and*

$$P(w) = \begin{bmatrix} 0 & 0 & \cdots & 0 & p_n \\ p_1 & 0 & \cdots & 0 & p_{n+1} \\ 0 & p_2 & \cdots & 0 & p_{n+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & p_{n-1} & p_{2n-1} \end{bmatrix}. \quad (12)$$

Then

$$\det\left[I_n s - P(w)\right] = s^n - p_{2n-1}s^{n-1} - p_{n-1}p_{2n-2}s^{n-2} - \cdots - p_2p_3\cdots p_{n-1}p_{n+1}s - p_1p_2\cdots p_n. \quad (13)$$

Proof. The expansion of the determinant with respect to the n-th column yields

$$\det\left[I_n s - P(w)\right] = \begin{vmatrix} s & 0 & \cdots & 0 & -p_n \\ -p_1 & s & \cdots & 0 & -p_{n+1} \\ 0 & -p_2 & \cdots & 0 & -p_{n+2} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & s & -p_{2n-2} \\ 0 & 0 & \cdots & -p_{n-1} & s - p_{2n-1} \end{vmatrix}$$

$$= s^n - p_{2n-1}s^{n-1} - p_{n-1}p_{2n-2}s^{n-2} - \cdots - p_2p_3\cdots p_{n-1}p_{n+1}s - p_1p_2\cdots p_n.$$

■

Lemma 2. *Let $R_n(w)$ be the n-th row of the adjoint matrix* $\operatorname{Adj}[I_n s - P(w)]$. *Then*

$$R_n(w) = [p_1p_2\cdots p_{n-1},\ p_2p_3\cdots p_{n-1}s,\ p_3p_4\cdots p_{n-1}s^2, \ldots, p_{n-1}s^{n-2}, s^{n-1}]. \quad (14)$$

Proof. Using the well-known equality $(\operatorname{Adj}[I_n s - P(w)]) \times [I_n s - P(w)] = I_n \det[I_n s - P(w)]$ and (14), it is easy to verify that

$$R_n(w)\left[I_n s - P(w)\right] = [\,0\ \cdots\ 0\ 1\,]\det\left[I_n s - P(w)\right]. \quad (15)$$

From Lemma 1 and 2 it follows that if

$$P(w) = \begin{bmatrix} 0 & 0 & \cdots & 0 & p_2 \\ p_1 & 0 & \cdots & 0 & p_3 \\ 0 & p_1 & \cdots & 0 & p_4 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & p_1 & p_{n+1} \end{bmatrix}, \quad (16)$$

then

$$\det\left[I_n s - P(w)\right] = s^n - p_{n+1}s^{n-1} - \cdots - p_3p_1^{n-2}s - p_2p_1^{n-1} \quad (17)$$

and

$$R_n(w) = [p_1^{n-1}, p_1^{n-2}s, \ldots, p_1s^{n-2}, s^{n-1}]. \quad (18)$$

It is assumed that for a given denominator (11c) there exist polynomials

$$p_k = p_k(w) = p_{kh}w^h + \cdots + p_{k1}w + p_{k0}, \quad k = 1, \ldots, 2n-1, \quad (19)$$

with nonnegative coefficients $p_{kj},\ j = 0, 1, \ldots, h$, such that

$$a_{n-1}(w) = p_{2n-1},\ a_{n-2}(w) = p_{n-1}p_{2n-2}, \ldots, a_1(w) = p_2p_3\cdots p_{n-1}p_{n+1},$$
$$a_0(w) = p_1p_2\cdots p_n. \quad (20)$$

In particular, if the matrix $P(w)$ has the form (16), then (20) takes the form

$$a_k(w) = p_1^{n-k-1}p_{k+2} \text{ for } k = 0, 1, \ldots, n-1. \quad (21)$$

Note that, if the assumption (20) is satisfied, then for a given denominator $d(s, w)$ of (11a) we may find the matrix (12) and next the corresponding matrices $A_i \in \mathbb{R}_+^{n\times n},\ i = 0, 1, \ldots, h$ since

$$I_n s - P(w) = I_n s - \sum_{i=0}^{h} A_i w^i. \quad (22)$$

The matrix C is chosen in the form

$$C = [\,0\ \cdots\ 0\ 1\,] \in \mathbb{R}^{1\times n}. \quad (23)$$

Taking into account (14), (22) and (7), we may write

$$\begin{aligned} &C\left(\operatorname{Adj}\left[I_n s - P(w)\right]\right)(B_0 + B_1w + \cdots + B_qw^q) \\ &= R_n(w)(B_0 + B_1w + \cdots + B_qw^q) \\ &= \left[p_1p_2\cdots p_{n-1},\ p_2p_3\cdots p_{n-1}s, \ldots, p_{n-1}s^{n-2}, s^{n-1}\right] \\ &\quad \times (B_0 + B_1w + \cdots + B_qw^q) = n(s, w). \end{aligned} \quad (24)$$

Comparing the coefficients at the same powers of s and w of (24), we obtain the following set of algebraic equations:

$$Hx = f, \quad (25)$$

where $H \in \mathbb{R}^{N\times M},\ x \in \mathbb{R}^M,\ f \in \mathbb{R}^N$.

The entries of H depend on the matrices $A_i,\ i = 0, 1, \ldots, h$, the components of f depend on the coefficients b_{kj} $(k = 0, 1, \ldots, n-1,\ j = 0, 1, \ldots, m)$ of the polynomial $n(s, w)$, and the components of x are the entries of $B_l,\ l = 0, 1, \ldots, q$.

If

$$\operatorname{rank}[H, f] = \operatorname{rank} H, \quad (26)$$

then Eqn. (25) has a nonnegative solution $x \in \mathbb{R}_+^M$ if (Kaczorek, 2004):

$$\sum_{i=1}^{r} \frac{u_i^T H^T f u_i}{s_i} \geq 0 \ \text{ for all } \ s_i > 0, \quad i = 1, \ldots, r \quad (27)$$

$(r = \operatorname{rank} HH^T)$, where s_i is an eigenvalue of H^TH and u_i is the i-th eigenvector associated with s_i, i.e.,

$$H^THu_i = s_iu_i \quad (\|u_i\| = 1). \quad (28)$$

From (24) it follows that the minimal number q of delays in control should satisfy the condition

$$q \deg_w R_n(w) \geq \deg_w n(s,w)$$
$$\big(\text{if } \deg_w R_n(w) = 0 \text{ then } q = \deg_w n(s,w)\big), \quad (29)$$

where $\deg_w(\cdot)$ denotes the degree of $(\cdot)$ with respect to w.

Theorem 2. *Let the assumption (20) be satisfied. Then there exists a positive realization (3) with $m = p = 1$ of $T(s,w)$ if the following conditions are satisfied:*

(i) $\lim_{s\to\infty} T(s,w) \in \mathbb{R}_+$.

(ii) The coefficients a_{ij} $(i = 0,1,\dots,n-1,\ j = 0,1,\dots,m)$ of the polynomial $d(s,w)$ are nonnegative except $a_{n-1,0}$, which can be arbitrary.

(iii) The conditions (26) and (27) are satisfied.

Proof. From (8) it follows that the condition (i) implies $D \in \mathbb{R}_+$. If the assumption (20) and the condition (ii) are satisfied, then from (22) we have $A_i \in \mathbb{R}_+^{n\times n}$ for $i = 1,\dots,h$, and A_0 is a Metzler matrix for arbitrary $a_{n-1,0}$. If (26) and (27) are met, then (25) has a nonnegative solution $x \in \mathbb{R}_+^M$ and $B_j \in \mathbb{R}_+^n$ for $j = 0,1,\dots,q$. The matrix C of the form (23) has always nonnegative entries. ∎

If the conditions of Theorem 2 are satisfied, then a positive realization (3) of $T(s,w)$ can be found with the use of the following procedure:

Procedure.

Step 1. Using (8) and (9), find D and the strictly proper transfer function $T_{sp}(s,w)$.

Step 2. For a given denominator $d(s,w)$ of (11a), find the polynomials (19) and the matrices $A_i \in \mathbb{R}_+^{n\times n}$, $i = 1,\dots,h$ and $A_0 \in M$.

Step 3. Using (29), choose q and, equating the coefficients at the same powers of s and w of the equality (24), find the entries of H and f.

Step 4. Find the solution $x \in \mathbb{R}_+^M$ of (25), the matrices $B_j \in \mathbb{R}_+^n$ for $j = 0,1,\dots,q$ and C of the form (23).

Theorem 3. *Let $p_1 = 1$ in (16). Then there exists a positive realization (3) of $T(s,w)$ if*

(i) The first two conditions of Theorem 2 are satisfied.

(ii) The coefficients b_{ij} $(i = 0,1,\dots,n-1,\ j = 0,1,\dots,m)$ of the numerator $n(s,w)$ of (11) are nonnegative.

Proof. If $p_1 = 1$, then $R_n(w)$ defined by (18) has the form $R_n(w) = [1, s, \dots, s^{n-1}]$. In this case, from (29) we have $q = \deg_w n(s,w)$ and $B_j \in \mathbb{R}_+^n$, $j = 0,1,\dots,q$ if the coefficients of $n(s,w)$ are nonnegative. ∎

4. Example

Find a positive realization (3) of the transfer function

$$T(s,w) = \frac{n(s,w)}{d(s,w)}, \quad (30)$$

where $n(s,w) = (3w^2+w+2)s^2+(w^3+2w^2+3w+2)s+2w^4+3w^3+w^2$, $d(s,w) = s^3-(2w^2+3w-1)s^2-(w^3+3w^2+2w)s-(w^5+2w^4+3w^3+2w^2)$.

In this case

$$\begin{aligned} d(s,w) &= s^3 - a_2(w)s^2 - a_1(w)s - a_0, \\ n(s,w) &= b_2(w)s^2 + b_1(w)s + b_0(w), \\ a_2(w) &= 2w^2 + 3w - 1, \\ a_1(w) &= w^3 + 3w^2 + 2w, \\ a_0(w) &= w^5 + 2w^4 + 3w^3 + 2w^2, \\ b_2(w) &= 3w^2 + w + 2, \\ b_1(w) &= w^3 + 2w^2 + 3w + 2, \\ b_0(w) &= 2w^4 + 3w^3 + w^2. \end{aligned}$$

We shall consider three cases of the choice of the polynomials (19).

Case 1.

Step 1. The transfer function (30) is strictly proper. Thus $D = 0$ and $T_{sp}(s,w) = T(s,w)$.

Step 2. In this case we choose the polynomials (19) of the form

$$\begin{aligned} &p_1 = w^2, \quad p_2 = w+1, \quad p_3 = w^2+w+2, \\ &p_4 = w^2+2w, \quad p_5 = 2w^2+3w-1, \end{aligned} \quad (31)$$

and the matrix (12) is equal to

$$P(w) = \begin{bmatrix} 0 & 0 & p_3 \\ p_1 & 0 & p_4 \\ 0 & p_2 & p_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & w^2+w+2 \\ w^2 & 0 & w^2+2w \\ 0 & w+1 & 2w^2+3w-1 \end{bmatrix}. \quad (32)$$

Using (32), we obtain

$$P(w) = A_0 + A_1 w + A_2 w^2,$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 2 \\ 0 & 1 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}. \tag{33}$$

Step 3. Using (29) and taking into account the fact that

$$R_n(w) = [p_1 p_2,\ p_2 s,\ s^2] = [w^3 + w^2,\ (w+1)s,\ s^2],$$

we choose $q = 2$. Thus, using (24) and (30), we obtain

$$[w^3+w^2,\ (w+1)s,\ s^2] \begin{bmatrix} b_{01}+b_{11}w+b_{21}w^2 \\ b_{02}+b_{12}w+b_{22}w^2 \\ b_{03}+b_{13}w+b_{23}w^2 \end{bmatrix} = (3w^2+w+2)s^2+(w^3+2w^2+3w+2)s+2w^4+3w^3+w^2. \tag{34}$$

Equating the coefficients at the same powers of s and w in (34), we obtain

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&1 \\ 0&0&0&0&0&1&0&0&0 \\ 0&0&1&0&0&0&0&0&0 \\ 0&0&0&0&0&0&0&1&0 \\ 0&0&0&0&1&0&0&1&0 \\ 0&1&0&0&1&0&0&0&0 \\ 0&1&0&0&0&0&0&0&0 \\ 0&0&0&0&0&0&1&0&0 \\ 0&0&0&1&0&0&1&0&0 \\ 1&0&0&1&0&0&0&0&0 \\ 1&0&0&0&0&0&0&0&0 \end{bmatrix} \begin{bmatrix} b_{01} \\ b_{02} \\ b_{03} \\ b_{11} \\ b_{12} \\ b_{13} \\ b_{21} \\ b_{22} \\ b_{23} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 2 \\ 1 \\ 2 \\ 3 \\ 2 \\ 0 \\ 2 \\ 3 \\ 1 \end{bmatrix}. \tag{35}$$

Step 4. The solution of (35) is

$$b_{01} = 1, \quad b_{02} = 2, \quad b_{03} = 2,$$
$$b_{11} = 2, \quad b_{12} = 1, \quad b_{13} = 1,$$
$$b_{21} = 0, \quad b_{22} = 1, \quad b_{23} = 3.$$

Therefore, $j = 0, 1, 2$,

$$B_0 = \begin{bmatrix} b_{01} \\ b_{02} \\ b_{03} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} b_{21} \\ b_{22} \\ b_{23} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \tag{36}$$

and

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{37}$$

The desired positive realization of (30) is given by (33), (36), (37) and $D = 0$.

Case 2. If we choose the polynomials (19) of the forms

$$\begin{aligned} p_1 &= p_2 = w, \\ p_3 &= w^3 + 2w^2 + 3w + 2, \\ p_4 &= w^2 + 3w + 2, \\ p_5 &= 2w^2 + 3w - 1, \end{aligned} \tag{38}$$

then

$$P(w) = \begin{bmatrix} 0 & 0 & p_3 \\ p_1 & 0 & p_4 \\ 0 & p_1 & p_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & w^3 + 2w^2 + 3w + 2 \\ w & 0 & w^2 + 3w + 2 \\ 0 & w & 2w^2 + 3w - 1 \end{bmatrix} = A_0 + A_1 w + A_2 w^2 + A_3 w^3,$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 2 \\ 0 & 0 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 3 \\ 1 & 0 & 3 \\ 0 & 1 & 3 \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case, using (29) and (24), we obtain $q = 2$ and

$$\begin{bmatrix} w^2 & ws & s^2 \end{bmatrix} \begin{bmatrix} b_{01} + b_{11}w + b_{21}w^2 \\ b_{02} + b_{12}w + b_{22}w^2 \\ b_{03} + b_{13}w + b_{23}w^2 \end{bmatrix} = (3w^2+w+2)s^2+(w^3+2w^2+3w+2)s+2w^4+3w^3+w^2.$$

The comparison of the polynomials in w at s yields the equality

$$w\left[b_{02} + b_{12}w + b_{22}w^2\right] = w^3 + 2w^2 + 3w + 2,$$

which cannot be satisfied for any w.

Therefore, under the choice (38) of the polynomials (19), a positive realization (3) of the transfer function (30) does not exist.

Case 3. If we choose the polynomials (19) as follows:

$$p_1 = p_2 = 1, \qquad p_3 = w^5 + 2w^4 + 3w^3 + 2w^2,$$
$$p_4 = w^3 + 3w^2 + 2w, \quad p_5 = 2w^2 + 3w - 1, \tag{39}$$

then

$$P(w) = \begin{bmatrix} 0 & 0 & p_3 \\ 1 & 0 & p_4 \\ 0 & 1 & p_5 \end{bmatrix} = \begin{bmatrix} 0 & 0 & w^5 + 2w^4 + 3w^3 + 2w^2 \\ 1 & 0 & w^3 + 3w^2 + 2w \\ 0 & 1 & 2w^2 + 3w - 1 \end{bmatrix} = A_0 + A_1 w + A_2 w^2 + A_3 w^3 + A_4 w^4 + A_5 w^5,$$

where

$$A_0 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}, \quad A_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 3 \\ 0 & 0 & 2 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \tag{40}$$
$$A_4 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad A_5 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

In this case, $q = 4$ and

$$\begin{bmatrix} 1 & s & s^2 \end{bmatrix} \begin{bmatrix} b_{01} + b_{11}w + b_{21}w^2 + b_{31}w^3 + b_{41}w^4 \\ b_{02} + b_{12}w + b_{22}w^2 + b_{32}w^3 + b_{42}w^4 \\ b_{03} + b_{13}w + b_{23}w^2 + b_{33}w^3 + b_{43}w^4 \end{bmatrix} = (3w^2+w+2)s^2+(w^3+2w^2+3w+2)s+2w^4+3w^3+w^2. \tag{41}$$

The comparison of the coefficients at the same powers of s and w of (41) yields

$$B_0 = \begin{bmatrix} b_{01} \\ b_{02} \\ b_{03} \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} b_{11} \\ b_{12} \\ b_{13} \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix},$$
$$B_2 = \begin{bmatrix} b_{21} \\ b_{22} \\ b_{23} \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \quad B_3 = \begin{bmatrix} b_{31} \\ b_{32} \\ b_{33} \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix},$$
$$B_4 = \begin{bmatrix} b_{41} \\ b_{42} \\ b_{43} \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \tag{42}$$

and

$$C = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}. \tag{43}$$

In this case the desired positive realization of (30) is given by (40), (42), (43) and $D = 0$.

5. Concluding Remarks

A method for the computation of positive realizations for continuous-time systems with reduced numbers of delays in state and in control was proposed. Sufficient conditions for the existence of a positive realization for a given proper transfer function were established and a procedure for the computation of positive realization was proposed. The details of the method were presented for single-input single-output systems, but it can be easily extended to multi-input multi-output systems. It is worth underlining that the conditions for the existence of a positive realization with a smaller number of delays are more restrictive than the ones for a larger number of delays (Theorems 2 and 3). The deliberations can be also extended for 2D systems (Gałkowski, 2001).

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