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A NEW FUZZY LYAPUNOV APPROACH TO NON–QUADRATIC STABILIZATION OF TAKAGI-SUGENO FUZZY MODELS

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In this paper, new non-quadratic stability conditions are derived based on the parallel distributed compensation scheme to stabilize Takagi-Sugeno (T-S) fuzzy systems. We use a non-quadratic Lyapunov function as a fuzzy mixture of multiple quadratic Lyapunov functions. The quadratic Lyapunov functions share the same membership functions with the T-S fuzzy model. The stability conditions we propose are less conservative and stabilize also fuzzy systems which do not admit a quadratic stabilization. The proposed approach is based on two assumptions. The first one relates to a proportional relation between multiple Lyapunov functions and the second one considers an upper bound to the time derivative of the premise membership functions. To illustrate the advantages of our proposal, four examples are given.

Keywords: T-S fuzzy systems, non-quadratic stability conditions, linear matrix inequalities, parallel distributed compensation, stabilization

1. Introduction

Many complex systems are difficult to describe using linearization or identification techniques. Takagi and Sugeno (1985) proposed a multimodel approach to overcome the difficulties of the conventional modeling techniques. The proposed multimodel is called the Takagi-Sugeno fuzzy model, whose construction is based on the identification (fuzzy modeling) using input-output data, or derivation from given non-linear system equations, i.e., the physical properties of the system (Takagi and Sugeno, 1985; Tanaka and Wang, 2001). The procedure of fuzzy modeling consists mainly of two parts, which are structure identification and parameter identification. We mention that many real systems, e.g., mechanical, can be represented and have been represented by T-S fuzzy models. A T-S fuzzy model approximates the system using simple models in each subspace obtained from the decomposition of the input space. During the last decade, several researchers in the control community have come up with different techniques for designing control systems. Researchers did not stop here; they have been looking for new and revolutionary ideas to solve problems that are not accessible by classical controllers. Among these revolutionary ideas, "fuzzy control" is probably one of the most popular ones (Feng, 2002; Takagi and Sugeno, 1985; Tanaka and Sugeno, 1992; Wang *et al.*, 1996), since it can provide an effective solution to the control of plants that are complex, uncertain or ill-defined. For this purpose, the non-linear plant is represented by a T-S fuzzy model, where local dynamics in different state regions are represented by linear models. The overall model of the system is obtained by fuzzy mixing of these local models. The fuzzy control design is carried out using the Parallel Distributed Compensation (PDC) scheme (Tanaka and Sugeno, 1992; Wang *et al.*, 1996).

The main idea of the PDC controller design is to derive each control rule from the corresponding rule of the T-S fuzzy model so as to compensate for it. The resulting overall fuzzy controller, which is non-linear in general, is a fuzzy mixture of individual linear controllers, knowing that the fuzzy controller shares the same fuzzy sets with the fuzzy system. Wang *et al.* (1996) utilized this concept to design fuzzy controllers to stabilize fuzzy systems.

The advantage of the T-S fuzzy model lies in the fact that the stability and performance characteristics of the system represented by a T-S fuzzy model can be analyzed using the Lyapunov function approach (Tanaka and Sugeno, 1992; Zhao, 1995). Tanaka and Sugeno (1992) proved that the stability of a T-S fuzzy model could be shown by finding a common positive definite symmetric matrix P for N subsystems. Generally, most of stability criteria for this fuzzy system derived by the Lyapunov approach need common P to satisfy a set of Lyapunov inequalities (Tanaka *et al.*, 1996; Wang *et al.*, 1996). Various works have been published based on this approach, such as the one by Cao and Lin (2003), who applied the Lyapunov function based approach to the stability analysis of non-linear systems with actuator saturation, and the one by Lee *et al.* (2001), who proposed a robust fuzzy control scheme for non-linear systems in the presence of parametric uncertainties, where sufficient conditions were derived for robust stabilization in the sense of Lyapunov stability. On the other hand, Tsen *et al.* (2001) proposed a fuzzy H_∞ model reference tracking control scheme and discussed the stability of the closed loop non-linear system by the Lyapunov approach. Korba *et al.* (2003) presented a constructive and automated method for the design of a gain-scheduling controller, based on a given T-S fuzzy model and a controller that guarantees the closed loop stability using Lyapunov quadratic functions. However, a possible limitation of their approach is the use of the Lyapunov method, which is conservative. Hence, the stability analysis of fuzzy control systems was discussed in the bulk of literature, e.g., (Bernal and Hušek, 2005; Blanco *et al.*, 2001; Chadli *et al.*, 2000; Chadli *et al.*, 2001; Chadli *et al.*, 2005; Jadbabaie, 1999; Johansson *et al.*, 1999; Ohtake *et al.*, 2003; Tanaka and Sugeno, 1992; Tanaka *et al.*, 2003; Wang *et al.*, 1995; Wang *et al.*, 1996; Wong *et al.*, 1998). Especially, piecewise Lyapunov functions and multiple Lyapunov functions have attracted a lot of attention due to avoiding the conservatism of stability and stabilization problems. However, three cases were defined (Johansson, 1999; Morère, 2001) where the analysis of quadratic stability is conservative, which are, respectively, saturated systems, piecewise linear systems and certain systems that do not accept Lyapunov functions for stability analysis.

In this context, new stability conditions for Takagi-Sugeno fuzzy models are derived in this paper, based on the use of multiple Lyapunov functions that have been discussed (Cao *et al.*, 1997; Chadli *et al.*, 2000; Hadjili, 2002; Jadbabaie, 1999; Tanaka *et al.*, 2001a) due to their properties of conservatism reduction. It is demonstrated that sufficient conditions for the stability and performance of a system are stated in terms of the feasibility of a set of Linear Matrix Inequalities (LMIs) (Boyd *et al.*, 1994; Tanaka and Sugeno, 1992; Tanaka *et al.*, 2001c), where the problem can be numerically solved by convex optimization techniques. On the other hand, piecewise fuzzy Lyapunov functions were employed to avoid the conservatism of stability conditions derived by the common Lyapunov function approach (Jadbabaie, 1999; Johansson *et al.*, 1999; Ohtake *et al.*, 2003). This approach is a result of the extension of the fuzzy Lyapunov function into a piecewise fuzzy Lyapunov function by mirroring the structure of a piecewise T-S fuzzy model. Johansson *et al.* (1999) proposed a novel stability condition for fuzzy systems, which are less conservative, based on Lyapunov functions that are piecewise quadratic. This approach uses the structural information in the rule base to decrease the conservatism of the analysis, and several alternatives were presented that improve the computational efficiency of the approach.

In this paper, new non-quadratic stability conditions are derived based on Parallel Distributed Compensation (PDC) to stabilize T-S fuzzy models. We use a fuzzy Lyapunov function since it is smooth contrary to a piecewise Lyapunov function, thus avoiding the boundary condition problem. Hence we obtain new conditions, shown to be less conservative, that stabilize all fuzzy systems including those that do not admit a quadratic stabilization. Our approach is based on two assumptions. The first one relies on the existence of a proportionality relation between multiple quadratic Lyapunov functions, and the second one considers an upper bound to the time derivative of the premise membership function as assumed by Tanaka *et al.* (2001a; 2001b; 2001c; 2003). Simulation examples demonstrate the effectiveness of our approach even for systems that do not admit a quadratic stabilization.

The reminder of the paper is organized as follows: Section 2 gives an outline of the fuzzy controller based on the PDC concept, and recalls the quadratic stability conditions of Takagi-Sugeno fuzzy models and the basic concept of the non-quadratic stability conditions. Section 3 presents the proposed stabilization approach and derives fuzzy controller design for stabilizing the closed loop system. Section 4 concerns the design examples. In Section 5, concluding remarks are given.

2. T-S Fuzzy Control and Stability Conditions

2.1. T-S Fuzzy Model and Controller. A T-S fuzzy system is described by fuzzy IF-THEN rules that represent locally linear input-output relations of a system. The i-th rule of this fuzzy system is of the following form:

$$\begin{aligned}&\text{Model Rule } i: \text{IF } z_1(t) \text{ is } M_{i1} \text{ and } \dots\ z_p(t) \text{ is } M_{ip}\\&\quad \text{THEN } \dot{x}(t) = A_i x(t) + B_i u(t),\\&\quad y(t) = C_i x(t), \quad i = 1, 2, \dots, r,\end{aligned} \tag{1}$$

where $z(t) = [z_1(t), \dots, z_p(t)]$ is the premise variable vector whose elements may be states, measurable external variables and/or time, $x(t) = [x_1(t), \dots, x_n(t)]$, $u(t) = [u_1(t), \dots, u_m(t)]$, r is the number of IF-THEN rules, and M_{ij} is a fuzzy set.

The final output of the fuzzy system is inferred as follows:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i\left(z(t)\right)\left(A_i x(t) + B_i u(t)\right), \tag{2}$$

where $h_i\left(z(t)\right)$ is the normalized weight for each rule, i.e.,

$$h_i\left(z(t)\right) \geq 0, \quad \sum_{i=1}^{r} h_i\left(z(t)\right) = 1,$$

and is given by

$$h_i\left(z(t)\right) = \frac{w_i(t)}{\sum_{i=1}^{r} w_i(t)}$$

and

$$w_i(t) = \prod_{j=1}^{p} M_{ij}\left(z_j\left(t\right)\right),$$

$M_{ij}\left(z_j(t)\right)$ being the grade of membership of $z_j(t)$ in M_{ij}.

The PDC scheme that stabilizes the Takagi-Sugeno fuzzy system was proposed by Hua *et al.* (1995; 1996) as a design framework comprising a control algorithm and a stability test using optimization involving LMI constraints.

The PDC controller is given by

$$u(t) = -\sum_{i=1}^{r} h_i\left(z(t)\right) F_i x(t). \tag{3}$$

The goal is to find appropriated F_i gains that ensure the closed loop stability.

2.2. Quadratic Stability Conditions. There exist some definitions of Lyapunov stability, among them the following definition:

Definition 1. The system $\dot{x}(t) = f\left(x\left(t\right), u(t)\right)$ is said to be *quadratically stable* if there exists a quadratic function $V\left(x(t)\right) = x^T(t)Px(t)$, $V\left(0\right) = 0$, satisfying the following conditions:

$$V\left(x(t)\right) > 0, \quad \forall\, x(t) \neq 0 \Longleftrightarrow P > 0, \tag{4}$$

$$\dot{V}\left(x(t)\right) < 0, \quad \forall\, x(t) \neq 0. \tag{5}$$

If V exists, it is called a *Lyapunov function*.

Thus, to find a Lyapunov function amounts to finding the appropriate positive definite matrix P.

In this sense and by substituting (3) in (2), we obtain the Takagi-Sugeno closed loop fuzzy system as follows:

$$\dot{x}(t) = \sum_{i=1}^{r}\sum_{j=1}^{r} h_i\left(z(t)\right) h_j\left(z(t)\right)\left[A_i - B_i F_j\right] x(t), \tag{6}$$

which can be rewritten as

$$\begin{aligned}&\dot{x}(t)\\&= \sum_{i=1}^{r}\sum_{i=1}^{r} h_i\left(z(t)\right) h_i\left(z(t)\right) G_{ii} x(t)\\&+2\sum_{i=1}^{r}\sum_{i<j} h_i\left(z(t)\right) h_j\left(z(t)\right)\left\{\frac{G_{ij}+G_{ji}}{2}\right\} x(t),\end{aligned} \tag{7}$$

where $G_{ij} = A_i - B_i F_j$ and $G_{ii} = A_i - B_i F_i$.

The stabilization of a feedback system containing a state feedback fuzzy controller has been extensively considered. The objective is to select F_i to stabilize the closed-loop system. The stability conditions corresponding to a quadratic Lyapunov function were derived by Tanaka and Sugeno in (1992). They gave sufficient conditions for stable fuzzy models based on the Lyapunov approach.

Theorem 1. (Tanaka and Sugeno, 1992) *The fuzzy system (5) can be stabilized via the PDC controller (3) if there exists a common positive definite matrix X and M_i, $i = 1, \dots, r$, such that*

$$\begin{aligned}-XA_i^T - A_i X + M_i^T B_i^T + B_i M_i > 0,&\\-XA_i^T - A_i X - XA_j^T - A_j X&\\+M_j^T B_i^T + B_i M_j + M_i^T B_j^T + B_j M_i \geq 0,&\end{aligned}$$

for all $i < j$ such that $h_i \cap h_j \neq \varnothing$, where

$$X = P^{-1}, \quad M_i = F_i X. \tag{8}$$

The feedback gains F_i and common P are given by

$$P = X^{-1}, \quad F_i = M_i X^{-1}, \tag{9}$$

whereas the single quadratic Lyapunov function is given by

$$V\left(x(t)\right) = x(t) X^{-1} x\left(t\right). \tag{10}$$

However, the larger the number of rules, the weaker the possibility to find a common positive definite matrix solution, even if LMI techniques are applied. This approach requires to find a common positive definite matrix for r subsystems, which makes it very conservative and hence forces us, in the next section, to define non-quadratic stability conditions using a fuzzy Lyapunov function.

2.3. Non-Quadratic Stability Conditions. Due to their property of conservatism reduction, in this section we define a fuzzy Lyapunov function and then consider stability conditions. The candidate Lyapunov function satisfies the following conditions:

V is C^1,

$V(0) \neq 0$ and $V(x(t)) > 0$ for $x(t) \neq 0$,

$\|x(t)\| \longrightarrow \infty \Rightarrow V(x(t)) \longrightarrow \infty$.

This fuzzy Lyapunov function is defined (Khalil, 1996; Tanaka *et al.*, 2003) for studying the stability and stabilization of the Takagi-Sugeno fuzzy system (2).

Definition 2. (Tanaka and Sugeno, 1992) Equation

$$V(x(t)) = \sum_{i=1}^{r} h_i(z(t))\, x^T(t) P_i x(t), \tag{11}$$

where P_i is a positive definite matrix, is said to define a *fuzzy Lyapunov function* for the Takagi-Sugeno fuzzy system if the time derivative of $V(x(t))$ is always negative at $x(t) \neq 0$.

3. New Stabilization Approach

In this section, and based on the fuzzy Lyapunov function, we propose an approach that gives less conservative stability conditions.

The key assumptions are as follows:

Assumption 1. The time derivative of the premise membership function is upper bounded such that $|\dot{h}_i(z(t))| \leq \phi_i$ for $i = 1, \ldots, r$, where ϕ_i, $i = 1, \ldots, r$, are given positive constants.

Assumption 2. The local quadratic Lyapunov functions $x^T(t) P_i x(t)$, $i = 1, \ldots, r$ are proportionally related such that $P_j = \alpha_{ij} P_i$ for $i, j = 1, \ldots, r$, where $\alpha_{ij} \neq 1$ and $\alpha_{ij} > 0$ for $i \neq j$, and $\alpha_{ij} = 1$ for $i = j$.

Theorem 2. *Under Assumptions 1 and 2, the fuzzy system (6) can be stabilized via the PDC fuzzy controller (3) if there exist ϕ_ρ, α_{ij} for $i, j, \rho = 1, \ldots, r$, positive definite matrices $P_1, P_2, \ldots, P_r$ and matrices $F_1, F_2, \ldots, F_r$ such that*

$$P_i > 0, \quad i = 1, 2, \ldots, r, \tag{12}$$

$$\sum_{\rho=1}^{r} \phi_\rho P_\rho + \left(G_{jj}^T P_i + P_i G_{jj}\right) < 0, \quad i, j = 1, 2, \ldots, r, \tag{13}$$

$$\left\{\frac{G_{jk} + G_{kj}}{2}\right\}^T P_i + P_i \left\{\frac{G_{jk} + G_{kj}}{2}\right\} < 0,$$
$$\forall\, i, j, k \in \{1, 2, \ldots, r\} \text{ such that } j < k, \tag{14}$$

where $G_{jk} = A_j - B_j F_k$ and $G_{jj} = A_j - B_j F_j$.

Proof. The candidate Lyapunov function is defined by

$$V(x(t)) = \sum_{i=1}^{r} h_i(z(t))\, x^T(t) P_i x(t). \tag{15}$$

The time derivative of $V(x(t))$ is calculated as

$$\begin{aligned}\dot{V}(x(t)) = {} & \dot{x}^T(t) \left(\sum_{i=1}^{r} h_i(z(t)) P_i\right) x(t) \\ & + x^T(t) \left(\sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) P_\rho\right) x(t) \\ & + x^T(t) \left(\sum_{i=1}^{r} h_i(z(t)) P_i\right) \dot{x}(t). \end{aligned} \tag{16}$$

By substituting (7) into $\dot{V}(x(t))$, we obtain

$$\begin{aligned}& \dot{V}(x(t)) \\ & = x^T(t) \Bigg[\sum_{j=1}^{r}\sum_{j=1}^{r}\sum_{i=1}^{r} h_j(z(t))\, h_j(z(t))\, h_i(z(t))\, G_{jj}^T P_i \\ & \quad + \sum_{j=1}^{r}\sum_{j<k}\sum_{i=1}^{r} h_j(z(t))\, h_k(z(t))\, h_i(z(t)) \\ & \quad \times \left(\frac{G_{jk} + G_{kj}}{2}\right)^T P_i \\ & \quad + \sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) P_\rho \\ & \quad + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\, h_j(z(t))\, P_i G_{jj} \\ & \quad + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j<k} h_i(z(t))\, h_j(z(t))\, h_k(z(t))\, P_i \\ & \quad \times \left(\frac{G_{jk} + G_{kj}}{2}\right)\Bigg] x(t), \end{aligned} \tag{17}$$

and, finally, one can write

$$\begin{aligned}& \dot{V}(x(t)) \\ & = x^T(t) \Bigg[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r} h_i(z(t))\, h_j(z(t))\, h_j(z(t)) \\ & \quad \times \left(G_{jj}^T P_i + P_i G_{jj}\right) + \sum_{\rho=1}^{r} \dot{h}_\rho(z(t)) P_\rho \\ & \quad + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j<k} h_i(z(t))\, h_j(z(t))\, h_k(z(t)) \\ & \quad \times \left(\left(\frac{G_{jk}+G_{kj}}{2}\right)^T P_i + P_i \left(\frac{G_{jk}+G_{kj}}{2}\right)\right)\Bigg] x(t). \end{aligned} \tag{18}$$

Under the assumption $|\dot{h}_\rho(z(t))| \le \phi_\rho$, (18) can be rewritten as follows:

$$\begin{aligned}
&\dot{V}(x(t)) \\
&\le x^T(t)\Bigg[\sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j=1}^{r} h_i(z(t))\,h_j(z(t))\,h_j(z(t)) \\
&\quad \times \left(G_{jj}^T P_i + P_i G_{jj}\right) + \sum_{\rho=1}^{r}\phi_\rho P_\rho \\
&\quad + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j<k} h_i(z(t))\,h_j(z(t))\,h_k(z(t)) \\
&\quad \times\left(\left(\frac{G_{jk}+G_{kj}}{2}\right)^T P_i + P_i\left(\frac{G_{jk}+G_{kj}}{2}\right)\right)\Bigg] x(t). \quad (19)
\end{aligned}$$

If Eqns. (12)–(14) hold, the time derivative of the fuzzy Lyapunov function is negative. Consequently, we have

$$\begin{aligned}
&\dot{V}(x(t)) \\
&\le x^T(t)\Bigg[\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\,h_j^2(z(t)) \\
&\quad \times\left(\left(G_{jj}^T P_i + P_i G_{jj}\right) + \sum_{\rho=1}^{r}\phi_\rho P_\rho\right) \\
&\quad + \sum_{i=1}^{r}\sum_{j=1}^{r}\sum_{j<k} h_i(z(t))\,h_j(z(t))\,h_k(z(t)) \\
&\quad \times\left(\left(\frac{G_{jk}+G_{kj}}{2}\right)^T P_i + P_i\left(\frac{G_{jk}+G_{kj}}{2}\right)\right)\Bigg] x(t) < 0,
\end{aligned}$$

and the closed loop fuzzy system (6) is stable. ∎

3.1. Constraint on the Time Derivative of the Premise Membership Function.

The conditions of Theorem 2 were derived by including an assumption on the time derivative of the premise membership function

$$|\dot{h}_\rho(z(t))| \le \phi_\rho \quad \text{for } \rho = 1,\dots,r, \qquad (20)$$

so we need to select ϕ_ρ to satisfy the constraint.

In this subsection, the constraint imposed on the time derivative of the premise membership functions and hence on the derivative of the state variable i.e., the speed variable, is transformed into LMIs of Theorem 3 solved simultaneously with those of Theorem 2 to stabilize the Takagi-Sugeno fuzzy systems. The new LMIs, which support Assumption 1, allow us to increase the performance by limiting the displacement rate in the polytope, implying a facility to find the Lyapunov functions and thus a faster stabilization.

Theorem 3. *Assume that $x(0)$ and $z(0)$ are known. The assumption (20) holds if there exist positive definite matrices $P_1, P_2, \dots, P_r$ and matrices $F_1, F_2, \dots, F_r$ satisfying*

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & P_i^{-1} \end{bmatrix} \ge 0 \quad \text{for } i = 1,\dots,r \qquad (21)$$

$$\begin{bmatrix} \phi_\rho P_i & W_{ij\rho\ell}^T \\ W_{ij\rho\ell} & \phi_\rho I \end{bmatrix} \ge 0, \ \ \forall i,j,\rho \in \{1,2,\dots,r\}, \ \forall \ell, \qquad (22)$$

where $W_{ij\rho\ell} = \xi_{\rho\ell}(A_i - B_i F_j)$. The selection of $\xi_{\rho\ell}$ is performed from $\dot{h}_i(z(t))$ by using a simple procedure given in (Tanaka et al., 2001b). However, it is to be noted that the conditions of this theorem depend on initial states, so the initial conditions should be known and for different initial states we need to solve the LMIs again.

Proof. From (20) and for $x(t) = z(t)$ we have

$$|\dot{h}_\rho(z(t))| = \left|\frac{\partial h_\rho(z(t))}{\partial x(t)}\dot{x}(t)\right| \le \phi_\rho. \qquad (23)$$

We also assume that

$$\frac{\partial h_\rho(z(t))}{\partial x(t)} = \sum_{\ell=1}^{s} \upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}, \qquad (24)$$

where $\upsilon_{\rho\ell}(z(t)) \ge 0$ and $\sum_{i\ell=1}^{s}\upsilon_{\rho\ell}(z(t)) = 1$.

Using (24) we obtain LMIs that satisfy the assumption (23).

From (23) we have

$$\left(\frac{\partial h_\rho(z(t))}{\partial x(t)}\dot{x}(t)\right)^T\left(\frac{\partial h_\rho(z(t))}{\partial x(t)}\dot{x}(t)\right) \le \phi_\rho^2. \qquad (25)$$

Substituting (6) in (25), we obtain

$$\begin{aligned}
&\Bigg[\Bigg(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell} \\
&\quad\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))h_j(z(t))\,[A_i - B_i F_j]\,x(t)\right\}\Bigg)^T \\
&\quad\times\Bigg(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell} \\
&\quad\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r} h_i(z(t))\,h_j(z(t))\,[A_i - B_i F_j]\,x(t)\right\}\Bigg)\Bigg] \\
&\le \phi_\rho^2. \qquad (26)
\end{aligned}$$

Dividing by ϕ_ρ^2, we obtain

$$\frac{1}{\phi_\rho^2}x^T(t)\left[\left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}\right.\right.$$
$$\times\left.\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z(t))\,h_j(z(t))\,[A_i-B_iF_j]^T\right\}\right)$$
$$\times\left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}\right.$$
$$\left.\left.\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z(t))\,h_j(z(t))\,[A_i-B_iF_j]\right\}\right)\right]x(t)$$
$$\le 1. \tag{27}$$

We assume that for the fuzzy Lyapunov function (11) the inequality (28) holds (Bernal and Hušek, 2005; Tanaka and Wang, 2001):

$$V(x(t)) \le V(x(0)) \le 1, \quad t \ge 0, \tag{28}$$

i.e.,

$$\sum_{i=1}^{r}h_i(z(t))\,x^T(t)P_ix(t) \le \sum_{i=1}^{r}h_i(z(0))\,x^T(0)\,P_ix(0) \le 1. \tag{29}$$

Then we have

$$1-\sum_{i=1}^{r}h_i(z(0))\,x^T(0)\,P_ix(0) \ge 0, \tag{30}$$

and

$$1-x^T(0)\left(\sum_{i=1}^{r}h_i(z(0))\,P_i\right)x(0) \ge 0, \tag{31}$$

which is expressed via LMIs using the Schur complement as follows:

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & \left(\sum_{i=1}^{r}h_i(z(0))\,P_i\right)^{-1}\end{bmatrix} \ge 0. \tag{32}$$

This is implied by

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & P_i^{-1}\end{bmatrix} \ge 0 \quad \text{for } i=1,\dots,r,$$

which leads to the LMI condition (21).

On the other hand, by considering (27) and (29), we deduce that (23) holds if

$$\frac{1}{\phi_\rho^2}\left[\left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}\right.\right.$$
$$\times\left.\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z(t))\,h_j(z(t))\,[A_i-B_iF_j]^T\right\}\right)$$
$$\times\left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}\right.$$
$$\left.\left.\times\left\{\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z(t))\,h_j(z(t))\,[A_i-B_iF_j]\right\}\right)\right]$$
$$-\sum_{i=1}^{r}h_i(z(t))\,P_i \le 0, \tag{33}$$

which is equivalent to

$$\begin{bmatrix} \phi_\rho\sum_{i=1}^{r}h_i(z(t))\,P_i & \left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}Q^T\right) \\ \left(\sum_{\ell=1}^{s}\upsilon_{\rho\ell}(z(t))\,\xi_{\rho\ell}Q\right) & \phi_\rho I\end{bmatrix} \ge 0, \tag{34}$$

where

$$Q=\sum_{i=1}^{r}\sum_{j=1}^{r}h_i(z(t))\,h_j(z(t))\,[A_i-B_iF_j].$$

This leads to the LMI condition (22):

$$\begin{bmatrix} \phi_\rho P_i & W_{ij\rho\ell} \\ W_{ij\rho\ell} & \phi_\rho I\end{bmatrix} \ge 0, \quad \forall\, i,j,\rho \in \{1,2,\dots,r\} \quad \forall\,\ell,$$

where $W_{ij\rho\ell}=\xi_{\rho\ell}(A_i-B_iF_j)$. ■

3.2. Stable Fuzzy Controller Design. In this part we are interested in non-quadratic stabilization of T-S fuzzy models by using PDC laws. The fuzzy controller design is supposed to determine the local feedback gains F_i for the closed-loop Takagi-Sugeno fuzzy system (6). We define $X_i = P_i^{-1}$, $F_i = M_iX_i^{-1}$, $X_i = \alpha_{ij}X_j$ for $i,j=1,\dots,r$, where $\alpha_{ij} \ne 1$ and $\alpha_{ij} > 0$ for $i \ne j$, and $\alpha_{ij}=1$ for $i=j$. By giving $\phi_\rho > 0$ and α_{ij} for $i,j,\rho=1,\dots,r$, we obtain the following LMIs conditions that constitute a stable fuzzy controller design problem:

$$X_i > 0, \quad i=1,2,\dots,r, \tag{35}$$

$$\sum_{\rho=1}^{r}\phi_\rho X_\rho + X_iA_j^T - \alpha_{ij}M_j^TB_j^T + A_jX_i - \alpha_{ij}B_jM_j < 0, \quad i,j=1,2,\dots,r, \tag{36}$$

$$X_iA_j^T - \alpha_{ik}M_k^TB_j^T + X_iA_k^T - \alpha_{ij}M_j^TB_k^T + A_jX_i - \alpha_{ik}B_jM_k + A_kX_i - \alpha_{ij}B_kM_j < 0.$$

for each setting of $i, j, k \in \{1, 2, \dots, r\}$ such that $j < k$,

$$\begin{bmatrix} 1 & x^T(0) \\ x(0) & X_i \end{bmatrix} \geq 0 \quad \text{for } i = 1, \dots, r, \tag{37}$$

$$\begin{bmatrix} \phi_\rho X_i & W_{ij\rho\ell}^T \\ W_{ij\rho\ell} & \phi_\rho I \end{bmatrix} \geq 0, \quad \forall\, i, j, \rho \in \{1, 2, \dots, r\} \quad \forall\, \ell, \tag{38}$$

where $W_{ij\rho\ell} = \xi_{\rho\ell}(A_iX_i - \alpha_{ij}B_iM_j)$.

It is to be noted that from $X_i = \alpha_{ij}X_j$ we have $X_j = (1/\alpha_{ij})X_i = \alpha_{ji}X_i$, so that $\alpha_{ij} = 1/\alpha_{ji}$ $\forall,\ i, j \in \{1, 2, \dots, r\}$, and hence, according to our proposal and for given i and j, the relation $\alpha_{ij}\alpha_{ji} = 1$ is used. The coefficients α_{ij} and ϕ_ρ for $i, j, \rho = 1, 2, \dots, r$ and $i \neq j$, can be chosen heuristically according to the application considered. In particular, the ϕ_ρ's are chosen in such a way so as to obtain a fast switching among IF–THEN rules in order to keep the speed of response for a closed-loop system (Tanaka *et al.*, 2001b).

4. Design Examples

This part presents four different examples that illustrate the effectiveness of the new non-quadratic stabilization conditions that we propose in this paper.

Example 1. Consider the following fuzzy system (Tanaka *et al.*, 2001c) that shows the effectiveness of our approach knowing that it admits also a quadratic stabilization:

$$\dot{x}(t) = \sum_{i=1}^{r} h_i(z(t))\,(A_ix(t) + B_iu(t)), \tag{39}$$

$$h_1(x_1(t)) = \frac{1+\sin x_1(t)}{2}, \quad h_2(x_1(t)) = \frac{1-\sin x_1(t)}{2},$$

$$A_1 = \begin{bmatrix} -5 & -4 \\ -1 & -2 \end{bmatrix}, \quad A_2 = \begin{bmatrix} -2 & -4 \\ 20 & -2 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ 10 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 3 \end{bmatrix}.$$

For this fuzzy system, which admits a quadratic stabilization and where it is assumed that $|x_1(t)| \leq \pi/2$ and $|x_2(t)| \leq \pi/2$. For $\alpha_{12} = 0.2$, $\alpha_{21} = 1/\alpha_{12}$, $\phi_1 = \phi_2 = 0.5$ and $\xi_{11} = 0$, $\xi_{12} = 0.5$, $\xi_{21} = -0.5$, $\xi_{22} = 0$, we obtain

$$F_1 = \begin{bmatrix} 0.0262 & 0.1232 \end{bmatrix}, \quad P_1 = \begin{bmatrix} 8.2039 & 1.0367 \\ 1.0367 & 3.0338 \end{bmatrix} > 0,$$

$$F_2 = \begin{bmatrix} -3.4925 & 1.9967 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 30.5563 & -6.3970 \\ -6.3970 & 4.7558 \end{bmatrix} > 0.$$

which depend on the initial conditions and satisfy the LMIs given in Theorems 2 and 3 simultaneously.

The new PDC fuzzy controller design condition has feasible solutions for different initial conditions and hence stabilizes the system. Figure 1 shows the evolution of the states and the control input for the initial condition $x(0) = [\,1 \quad 1\,]^T$. As can be seen, the conservatism reduction leads to very interesting results regarding fast convergence in the stabilization of this Takagi Sugeno fuzzy system. ♦

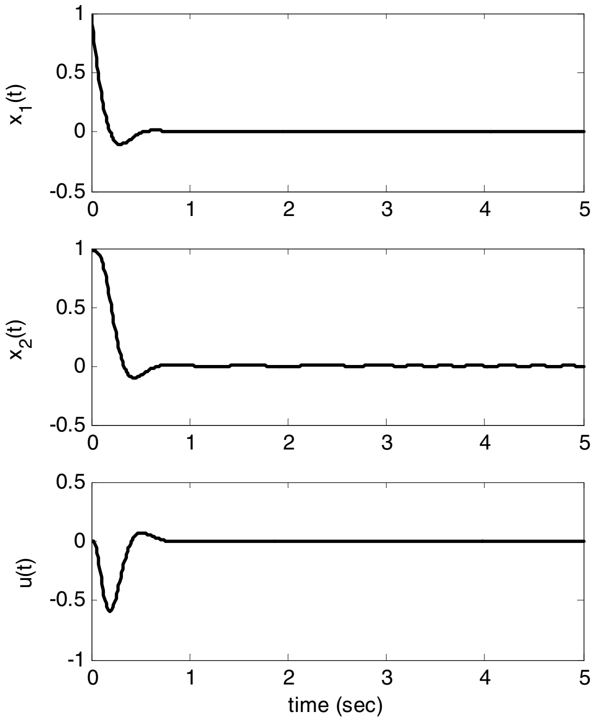


Fig. 1. Simulation results for Example 1.

Example 2. This is another example which does admit a single Lyapunov function (Morère, 2001). The utility of the proposed conditions is shown by the obtained results. We have

$$h_1(x_1(t)) = \frac{1}{\pi}\left[\frac{\pi}{2} - \tan^{-1}(x_1(t))\right],$$

$$h_2(x_1(t)) = \frac{1}{\pi}\left[\frac{\pi}{2} + \tan^{-1}(x_1(t))\right],$$

$$A_1 = \begin{bmatrix} 0.1000 & -1.0000 \\ -0.2500 & 1.0000 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 1.0000 & 0.5000 \\ 0.7500 & 2.0000 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -0.6500 \\ -0.2000 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} -1.0000 \\ -0.0500 \end{bmatrix}.$$

The design of a state-feedback controller using a global Lyapunov function is not possible since the LMI problem (8) is infeasible. However, if we consider local Lyapunov functions, the LMI problem (35)–(39) is feasible.

Our approach gives feasible solutions for different initial conditions, and hence stabilizes the system.

For $\alpha_{12} = 1.5$, $\alpha_{21} = 1/\alpha_{12}$, $\phi_1 = \phi_2 = 5$, and $\xi_{11} = 0.25$, $\xi_{12} = 0.75$, $\xi_{21} = 0.25$, $\xi_{22} = 0.75$, we obtain

$$P_1 = \begin{bmatrix} 12.1789 & -104.4753 \\ -104.4753 & 997.4141 \end{bmatrix} > 0,$$

$$P_2 = \begin{bmatrix} 12.3823 & -103.6178 \\ -103.6178 & 989.7426 \end{bmatrix} > 0,$$

$$F_1 = \begin{bmatrix} 14.1362 & -211.3544 \end{bmatrix},$$

$$F_2 = \begin{bmatrix} -0.3676 & -72.8607 \end{bmatrix}.$$

Figure 2 shows the evolution of the system states and control for the initial values $x(0) = [\ 0.1 \quad 0.1\ ]^T$.

♦

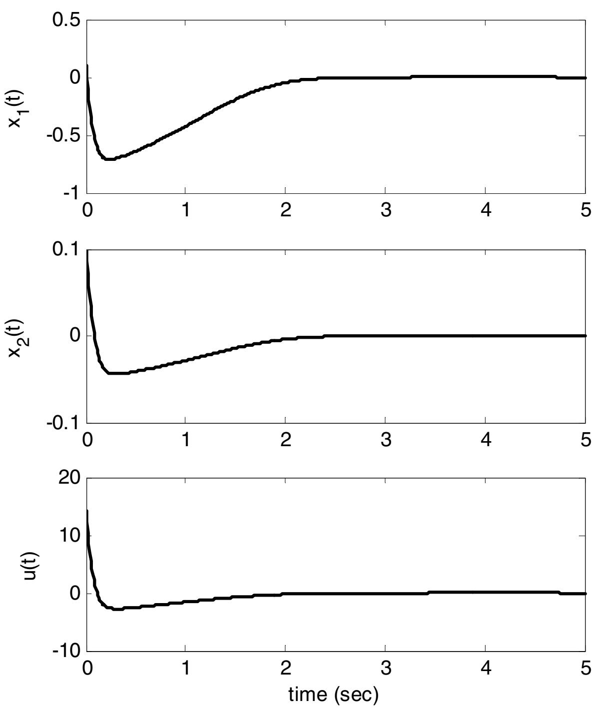


Fig. 2. Simulation results for Example 2.

Example 3. (An inverted pendulum on a cart) Consider now the problem of balancing and swinging-up an inverted pendulum on a cart using the proposed approach. The equations of motion are (Tanaka and Wang, 2001):

$$\begin{aligned} x_1(t) &= x_2(t), \\ \dot{x}_2(t) &= \frac{1}{4/3l - aml\cos^2(x_1(t))} \\ &\quad \times \Big[g\sin(x_1(t)) - (1/2)amlx_2^2(t)\sin(2x_1(t)) \\ &\quad - a\cos(x_1(t))\,u(t) \Big], \end{aligned} \tag{40}$$

where $x_1(t)$ denotes the angle (in radians) of the pendulum from the vertical axis and $x_2(t)$ is the angular velocity, $g = 9.8\,\mathrm{m/s^2}$ is the gravity constant, m is the mass of the pendulum, M is the mass of the car, $2l$ is the length of the pendulum, u [N] is the force applied to the cart and $a = 1/(m+M)$. For the simulations, the values of the parameters are $m = 2.0\,\mathrm{kg}$, $M = 8.0\,\mathrm{kg}$, $2l = 1.0\,\mathrm{m}$.

The control objective for this example is to balance the inverted pendulum for the approximate range $x_1 \in (-\pi/2, \pi/2)$ by using our fuzzy controller. The system (41) is modelled by the following two fuzzy rules:

Rule 1: IF $x_1(t)$ is about 0
THEN $\dot{x}(t) = A_1x(t) + B_1u(t)$,

Rule 2: IF $x_1(t)$ is about $\pm\,\pi/2$ $(|x_1| < \pi/2)$
THEN $\dot{x}(t) = A_2x(t) + B_2u(t)$,

where

$$A_1 = \begin{bmatrix} 0 & 1 \\ \dfrac{2g}{4l/3 - aml} & 0 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ \dfrac{2g}{\pi(4l/3 - aml\beta^2)} & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ -\dfrac{a}{4l/3 - aml} \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 \\ -\dfrac{a\beta}{4l/3 - aml\beta^2} \end{bmatrix}$$

and $\beta = \cos(88°)$.

The PDC control laws are as follows:

Rule 1 : IF $x_1(t)$ is about 0 THEN $u(t) = -F_1x(t)$,

Rule 2 : IF $x_1(t)$ is about $\pm\,\pi/2$ $(|x_1| < \pi/2)$
THEN $u(t) = -F_2x(t)$.

Hence the control law that grantees the stability of the fuzzy control system is given by

$$u(t) = -h_1(x_1(t))\,F_1x(t) - h_2(x_1(t))\,F_2x(t), \tag{41}$$

where h_1 and h_2 are the membership values of Rules 1 and 2, respectively.

Applying our approach, the objective of balancing and stabilizing the pendulum is reached with success for different initial conditions of $x_1 \in (-\pi/2, \pi/2)$ and $x_2 = 0$. For $\alpha_{12} = 1.3$, $\alpha_{21} = 1/\alpha_{12}$, $\phi_1 = \phi_2 = \pi/1.5$ and $\xi_{11} = -2/\pi$, $\xi_{12} = 2/\pi$, $\xi_{21} = -2/\pi$, $\xi_{22} = 2/\pi$, we obtain the following P_1, P_2, F_1 and F_2 for each initial condition:

For $x(0) = [\ \pi/6 \quad 0\ ]^T$ we have

$$P_1 = \begin{bmatrix} 57.7603 & 23.2068 \\ 23.2068 & 10.3697 \end{bmatrix} > 0,$$
$$P_2 = \begin{bmatrix} 58.1998 & 17.5082 \\ 17.5082 & 6.1428 \end{bmatrix} > 0,$$
$$F_1 = \begin{bmatrix} -630.7446 & -164.6591 \end{bmatrix},$$
$$F_2 = 10^{-3} \begin{bmatrix} -1.2396 & -0.2958 \end{bmatrix}.$$

For $x(0) = [\ \pi/4 \quad 0\ ]^T$ we have

$$P_1 = \begin{bmatrix} 32.0668 & 13.1229 \\ 13.1229 & 6.4541 \end{bmatrix} > 0,$$
$$P_2 = \begin{bmatrix} 39.1987 & 11.4436 \\ 11.4436 & 4.1936 \end{bmatrix} > 0,$$
$$F_1 = \begin{bmatrix} -530.6214 & -127.4777 \end{bmatrix},$$
$$F_2 = 10^{3} \begin{bmatrix} -1.0859 & -0.2427 \end{bmatrix}.$$

For $x(0) = [\ \pi/3 \quad 0\ ]^T$, we get

$$P_1 = \begin{bmatrix} 27.3149 & 10.8202 \\ 10.8202 & 5.6005 \end{bmatrix} > 0,$$
$$P_2 = \begin{bmatrix} 51.3551 & 14.8473 \\ 14.8473 & 5.3225 \end{bmatrix} > 0,$$
$$F_1 = \begin{bmatrix} -502.4650 & -115.6213 \end{bmatrix},$$
$$F_2 = 10^{3} \begin{bmatrix} -1.3235 & -0.3102 \end{bmatrix}.$$

Figure 3 shows the evolution of the inverted pendulum position, velocity and control force for different initial conditions. ♦

Example 4. (A two-link robot) To show the effectiveness of our approach, we apply it to a more complicated system, i.e., a two-link robot manipulator. The dynamic equation of the two-link robot system is as follows:

$$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + G(q) = \tau, \tag{42}$$

where

$$M(q) = \begin{bmatrix} (m_1+m_2)l_1^2 & m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) \\ m_2 l_1 l_2 (s_1 s_2 + c_1 c_2) & m_2 l_2^2 \end{bmatrix},$$

$$C(q,\dot{q}) = m_2 l_1 l_2 (c_1 s_2 + s_1 c_2) \begin{bmatrix} 0 & -\dot{q}_2 \\ -\dot{q}_1 & 0 \end{bmatrix},$$

$$G(q) = \begin{bmatrix} -(m_1+m_2) l_1 g s_1 \\ -m_2 l_2 g s_2 \end{bmatrix},$$

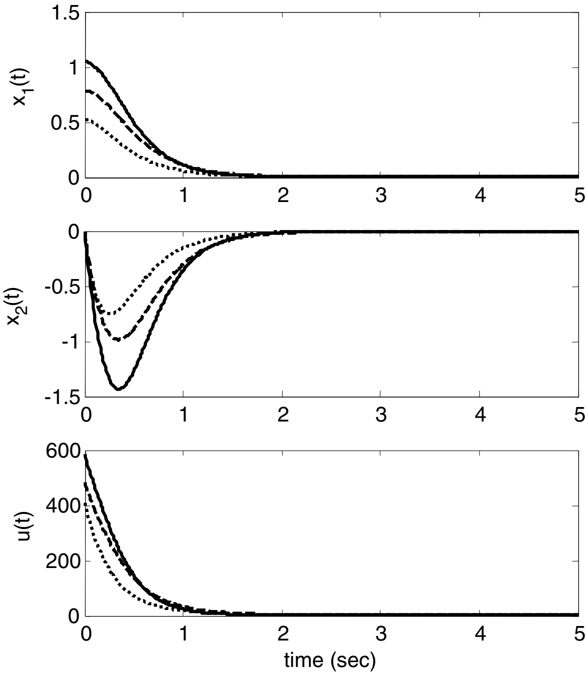


Fig. 3. Simulation results for Example 3.

and $q = [q_1, q_2]^T$, q_1 and q_2 being generalized coordinates, $M(q)$ is the inertia matrix, $C(q,\dot{q})$ includes Coriolis, centripetal forces, and $G(q)$ is the gravitational force. The different parameters are: link mass m_1, m_2 [kg], link length l_1, l_2 [m], angular position q_1, q_2 [rad], applied torques $\tau = [\ \tau_1 \quad \tau_2\ ]^T (N-m)$, and acceleration due to gravity $g = 9.8\,(\mathrm{m/s^2})$. We also introduce the compact notation $s_1 = \sin(q_1)$, $s_2 = \sin(q_2)$, $c_1 = \cos(q_1)$, and $c_2 = \cos(q_2)$. Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$ and $x_4 = \dot{q}_2$. The state space representation is given by

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= f_1(x) + g_{11}(x)\tau_1 + g_{12}\tau_2, \\ \dot{x}_3 &= x_4, \\ \dot{x}_4 &= f_2(x) + g_{21}(x)\tau_1 + g_{22}\tau_2, \\ y_1 &= x_1, \\ y_2 &= x_3. \end{aligned} \tag{43}$$

See (Tsen *et al.*, 2001) for more details concerning $f_1(x)$, $f_2(x)$, $g_{11}(x)$, g_{12}, $g_{21}(x)$ and g_{22}.

The objective is fuzzy stabilization of the two-link robot using our approach. The link masses are $m_1 = 1$ [kg], $m_2 = 1$ [kg], the link lengths are $l_1 = 1$ [m], $l_2 = 1$ [m] and angular positions q_1, q_2 are constrained within $[-(\pi/2), (\pi/2)]$. The T-S fuzzy model is given by the following nine rules whose membership functions are of triangular forms (Tsen *et al.*, 2001):

Rule 1: IF $x_1(t)$ is about $-\pi/2$ and $x_3(t)$ is about $\pi/2$
THEN $\dot{x}(t) = A_1 x(t) + B_1 u(t)$, $\quad y = C_1 x(t)$,

Rule 2: IF $x_1(t)$ is about $-\pi/2$ and $x_3(t)$ is about 0
THEN $\dot{x}(t) = A_2x(t) + B_2u(t)$, $y = C_2x(t)$,

Rule 3: IF $x_1(t)$ is about $-\pi/2$ and $x_3(t)$ is about $-\pi/2$
THEN $\dot{x}(t) = A_3x(t) + B_3u(t)$, $y = C_3x(t)$,

Rule 4: IF $x_1(t)$ is about 0 and $x_3(t)$ is about $-\pi/2$
THEN $\dot{x}(t) = A_4x(t) + B_4u(t)$, $y = C_4x(t)$,

Rule 5: IF $x_1(t)$ is about 0 and $x_3(t)$ is about 0
THEN $\dot{x}(t) = A_5x(t) + B_5u(t)$, $y = C_5x(t)$,

Rule 6: IF $x_1(t)$ is about 0 and $x_3(t)$ is about $\pi/2$
THEN $\dot{x}(t) = A_6x(t) + B_6u(t)$, $y = C_6x(t)$,

Rule 7: IF $x_1(t)$ is about $\pi/2$ and $x_3(t)$ is about $-\pi/2$
THEN $\dot{x}(t) = A_7x(t) + B_7u(t)$, $y = C_7x(t)$,

Rule 8: IF $x_1(t)$ is about $\pi/2$ and $x_3(t)$ is about 0
THEN $\dot{x}(t) = A_8x(t) + B_8u(t)$, $y = C_8x(t)$,

Rule 9: IF $x_1(t)$ is about $\pi/2$ and $x_3(t)$ is about $\pi/2$
THEN $\dot{x}(t) = A_9x(t) + B_9u(t)$, $y = C_9x(t)$,

where $x = [x_1, x_2, x_3, x_4]^T$, $u = [\tau_1, \tau_2]^T$, and the local models matrices given by

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 5.927 & -0.001 & -0.315 & -8.4\times10^{-6} \\ 0 & 0 & 0 & 1 \\ -6.859 & 0.002 & 3.155 & 6.2\times10^{-6} \end{bmatrix},$$

$$A_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.0428 & -0.0011 & 0.1791 & -0.0002 \\ 0 & 0 & 0 & 1 \\ 3.5436 & 0.0313 & 2.5611 & 1.14\times10^{-5} \end{bmatrix},$$

$$A_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2728 & 0.0030 & 0.4339 & -0.0001 \\ 0 & 0 & 0 & 1 \\ 9.1041 & 0.0158 & -1.0574 & -3.2\times10^{-5} \end{bmatrix},$$

$$A_4 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.4535 & 0.0017 & 1.2427 & 0.0002 \\ 0 & 0 & 0 & 1 \\ -3.1873 & -0.0306 & 5.1911 & -1.8\times10^{-6} \end{bmatrix},$$

$$A_5 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 11.1336 & 0.0 & -1.8145 & 0.0 \\ 0 & 0 & 0 & 1 \\ -9.0918 & 0.0 & 9.1638 & 0.0 \end{bmatrix},$$

$$A_6 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1702 & -0.0010 & 1.6870 & -0.0002 \\ 0 & 0 & 0 & 1 \\ -2.3559 & 0.0314 & 4.5298 & 1.1\times10^{-5} \end{bmatrix},$$

$$A_7 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.1206 & -0.0041 & 0.6205 & 0.0001 \\ 0 & 0 & 0 & 1 \\ 8.8794 & -0.0193 & -1.0119 & 4.4\times10^{-5} \end{bmatrix},$$

$$A_8 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3.6421 & 0.0018 & 0.0721 & 0.0002 \\ 0 & 0 & 0 & 1 \\ 2.4290 & -0.0305 & 2.9832 & 1.9\times10^{-5} \end{bmatrix},$$

$$A_9 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 6.2933 & -0.0009 & -0.2188 & -1.2\times10^{-5} \\ 0 & 0 & 0 & 1 \\ -7.4649 & 0.0024 & 3.2693 & 9.2\times10^{-6} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_3 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad B_4 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_5 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, \quad B_6 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_7 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \\ 0 & 0 \\ 1 & 2 \end{bmatrix}, \quad B_8 = \begin{bmatrix} 0 & 0 \\ 0.5 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_9 = \begin{bmatrix} 0 & 0 \\ 1 & -1 \\ 0 & 0 \\ -1 & 2 \end{bmatrix}, \quad C_i = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

For $\alpha_{12} = 1.2$, $\alpha_{13} = 0.66$, $\alpha_{14} = 0.9$, $\alpha_{15} = 0.8$, $\alpha_{16} = 0.7$, $\alpha_{17} = 1.5$, $\alpha_{19} = 1.1$, $\alpha_{23} = 1.6$, $\alpha_{24} = 1.18$, $\alpha_{25} = 1.05$, $\alpha_{26} = 1.9$, $\alpha_{27} = 1.99$, $\alpha_{28} = 0.99$, $\alpha_{29} = 0.77$, $\alpha_{34} = 0.88$, $\alpha_{35} = 1.33$, $\alpha_{36} = 1.4$, $\alpha_{37} = 1.7$, $\alpha_{38} = 1.66$, $\alpha_{39} = 1.56$, $\alpha_{45} = 1.48$, $\alpha_{46} = 1.39$, $\alpha_{47} = 1.69$, $\alpha_{48} = 1.11$, $\alpha_{49} = 1.88$, $\alpha_{56} = 2.1$, $\alpha_{57} = 2.2$, $\alpha_{58} = 1.61$, $\alpha_{37} = 1.7$, $\alpha_{38} = 1.66$, $\alpha_{39} = 1.56$, $\alpha_{45} = 1.48$, $\alpha_{46} = 1.39$, $\alpha_{47} = 1.69$, $\alpha_{48} = 1.11$, $\alpha_{49} = 1.88$, $\alpha_{56} = 2.1$, $\alpha_{57} = 2.2$, $\alpha_{58} = 1.61$, $\alpha_{59} = 1.23$, $\alpha_{67} = 2.11$, $\alpha_{68} = 2.2$, $\alpha_{69} = 1.52$, $\alpha_{78} = 0.78$, $\alpha_{78} = 0.98$, $\alpha_{79} = 0.82$,$\phi_1 = 1$, $\phi_2 = 1.5$, $\phi_3 = 1.2$, $\phi_4 = 2$, $\phi_5 = 1.6$, $\phi_6 = 1.8$, $\phi_7 = 2.5$, $\phi_8 = 2.2$, $\phi_9 = 2.4$, $\xi_{11} = 0$, $\xi_{12} = 2/\pi$, $\xi_{21} = 0$, $\xi_{22} = 2/\pi$, $\xi_{31} = -4/\pi$,

$\xi_{32} = 0$, $\xi_{41} = -2/\pi$, $\xi_{42} = 0$, $\xi_{51} = 0$, $\xi_{52} = 4/\pi$, $\xi_{61} = 2/\pi$, $\xi_{62} = 4/\pi$, $\xi_{71} = -2/\pi$, $\xi_{72} = 0$, $\xi_{81} = 2/\pi$, $\xi_{82} = 4/\pi$, $\xi_{91} = 0$, $\xi_{92} = 4/\pi$, we obtain the following P_i and F_i for $i = 1, \ldots, 9$:

$$P_1 = \begin{bmatrix} 0.0044 & 0.0005 & -0.0002 & -0.0000 \\ 0.0005 & 0.0001 & 0.0000 & 0.0000 \\ -0.0002 & 0.0000 & 0.0180 & 0.0011 \\ -0.0000 & 0.0000 & 0.0011 & 0.0001 \end{bmatrix} > 0,$$

$$P_2 = \begin{bmatrix} 0.0063 & 0.0007 & -0.0003 & -0.0000 \\ 0.0007 & 0.0001 & 0.0000 & 0.0000 \\ -0.0003 & 0.0000 & 0.0238 & 0.0014 \\ -0.0000 & 0.0000 & 0.0014 & 0.0001 \end{bmatrix} > 0,$$

$$P_3 = \begin{bmatrix} 0.0052 & 0.0006 & -0.0003 & -0.0000 \\ 0.0006 & 0.0001 & 0.0000 & 0.0000 \\ -0.0003 & 0.0000 & 0.0206 & 0.0012 \\ -0.0000 & 0.0000 & 0.0012 & 0.0001 \end{bmatrix} > 0,$$

$$P_4 = \begin{bmatrix} 0.0081 & 0.0008 & -0.0004 & -0.0001 \\ 0.0008 & 0.0001 & 0.0000 & -0.0000 \\ -0.0004 & 0.0000 & 0.0286 & 0.0014 \\ -0.0001 & -0.0000 & 0.0014 & 0.0001 \end{bmatrix} > 0,$$

$$P_5 = \begin{bmatrix} 0.0066 & 0.0007 & -0.0004 & -0.0000 \\ 0.0007 & 0.0001 & 0.0001 & 0.0000 \\ -0.0004 & 0.0001 & 0.0249 & 0.0014 \\ -0.0000 & 0.0000 & 0.0014 & 0.0001 \end{bmatrix} > 0,$$

$$P_6 = \begin{bmatrix} 0.0073 & 0.0007 & -0.0004 & -0.0001 \\ 0.0007 & 0.0001 & 0.0001 & -0.0000 \\ -0.0004 & 0.0001 & 0.0268 & 0.0015 \\ -0.0001 & -0.0000 & 0.0015 & 0.0001 \end{bmatrix} > 0,$$

$$P_7 = \begin{bmatrix} 0.0098 & 0.0008 & -0.0005 & -0.0002 \\ 0.0008 & 0.0001 & -0.0001 & -0.0000 \\ -0.0005 & -0.0001 & 0.0321 & 0.0018 \\ -0.0002 & -0.0000 & 0.0018 & 0.0001 \end{bmatrix} > 0,$$

$$P_8 = \begin{bmatrix} 0.0089 & 0.0007 & -0.0003 & -0.0001 \\ 0.0007 & 0.0001 & -0.0001 & -0.0000 \\ -0.0003 & -0.0001 & 0.0303 & 0.0016 \\ -0.0001 & -0.0000 & 0.0016 & 0.0001 \end{bmatrix} > 0,$$

$$P_9 = \begin{bmatrix} 0.0095 & 0.0006 & -0.0004 & 0.0001 \\ -0.0006 & 0.0001 & 0.0001 & 0.0000 \\ -0.0004 & 0.0001 & 0.0316 & 0.0002 \\ 0.0001 & 0.0000 & 0.0002 & 0.0001 \end{bmatrix} > 0,$$

$$F_1 = 10^4 \begin{bmatrix} 0.1972 & 0.0304 & 1.5274 & 0.0935 \\ -0.1281 & -0.0118 & 1.0172 & 0.0685 \end{bmatrix},$$

$$F_2 = 10^3 \begin{bmatrix} 7.1742 & 0.7969 & -0.8884 & -0.0548 \\ 0.9590 & 0.1227 & 7.8326 & 0.5126 \end{bmatrix},$$

$$F_3 = 10^3 \begin{bmatrix} 6.0170 & 0.6897 & -3.3359 & -0.2237 \\ -2.7700 & -0.3101 & 3.3993 & 0.2093 \end{bmatrix},$$

$$F_4 = 10^3 \begin{bmatrix} 8.4037 & 0.8734 & 0.3998 & -0.0322 \\ 0.2103 & 0.0442 & 7.2717 & 0.3699 \end{bmatrix},$$

$$F_5 = 10^3 \begin{bmatrix} 9.4557 & 1.0430 & 1.7476 & 0.0815 \\ 5.8559 & 0.6682 & 5.5596 & 0.3062 \end{bmatrix},$$

$$F_6 = 10^3 \begin{bmatrix} 6.9498 & 0.7022 & 1.4199 & 0.0196 \\ 0.8046 & 0.1075 & 6.1771 & 0.3639 \end{bmatrix},$$

$$F_7 = 10^3 \begin{bmatrix} 5.1527 & 0.4600 & 1.4825 & -0.0170 \\ -3.5642 & -0.3247 & 2.7930 & 0.2575 \end{bmatrix},$$

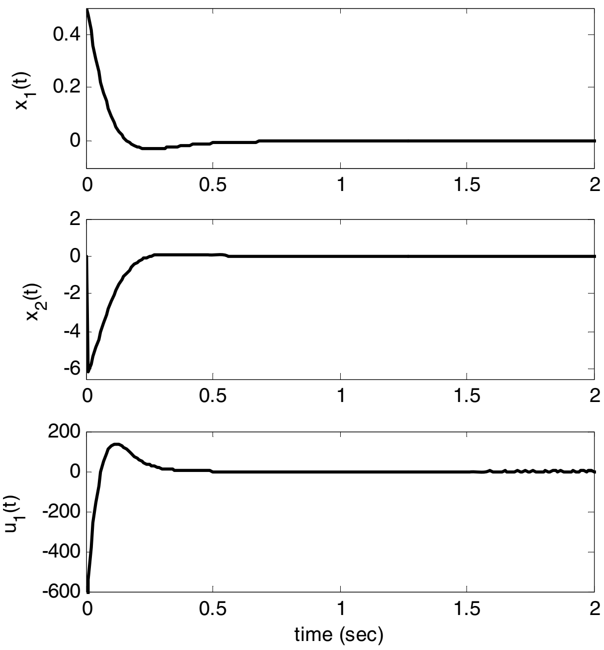
$$F_8 = 10^3 \begin{bmatrix} -3.5642 & -0.3247 & 2.7930 & 0.2575 \\ -0.0917 & -0.0166 & 4.9180 & 0.2621 \end{bmatrix},$$

$$F_9 = 10^3 \begin{bmatrix} 4.7285 & -0.1862 & 1.7978 & 0.2830 \\ 0.4183 & 0.2383 & -5.1097 & 0.5124 \end{bmatrix}.$$

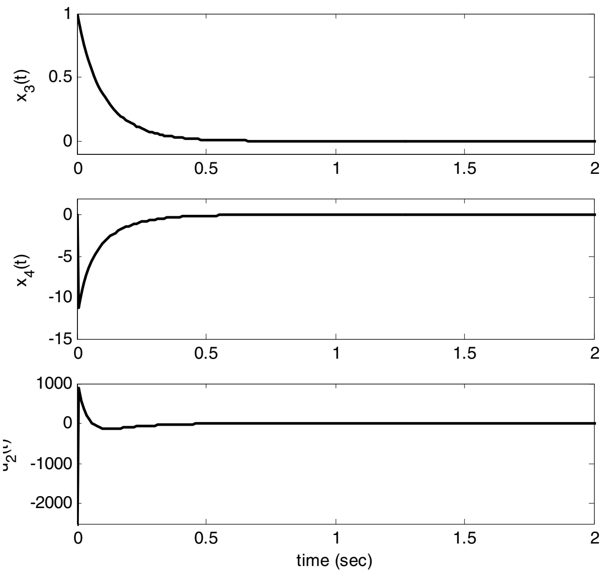
The control objective for this example is to stabilize the two-link robot using the proposed approach. Satisfactory and less conservative results are obtained, showing the effectiveness of our approach. Figure 4 shows the evolution of the states and control torques for the initial values $x(0) = [\ \pi/3 \quad 0 \quad \pi/6 \quad 0\ ]^T$. ♦

5. Conclusion

This paper presents a new fuzzy Lyapunov approach to the stabilization of Takagi-Sugeno fuzzy models, based on a fuzzy Lyapunov function, which is defined by fuzzily mixing quadratic Lyapunov functions. Conditions are derived in a logical way while taking full advantage of the fuzzy Lyapunov function and making two assumptions that concern a proportional relation between multiple quadratic Lyapunov functions and an upper bound to the time derivative of the premise membership function for which the respective LMIs that support it in Theorem 3 are solved with those of Theorem 2 to stabilize the system. Hence, the PDC procedure of constructing local feedback gains is simple and can be solved effectively by optimization solvers. Our approach leads to less conservative results and very good effects are obtained for various examples, even for those that do not admit a single Lyapunov func-



(a) Link 1



(b) Link 2

Fig. 4. Simulation results for Example 4.

tion, thus illustrating the effeciency of the proposed stabilization approach.

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