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ON ATTAINING THE PRESCRIBED QUALITY OF A CONTROLLED FOURTH ORDER SYSTEM

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In this paper, we discuss a method of auxiliary controlled models and its application to solving some robust control problems for a system described by differential equations. As an illustration, a system of nonlinear differential equations of the fourth order is used. A solution algorithm, which is stable with respect to informational noise and computational errors, is presented. The algorithm is based on a combination of online state/input reconstruction and feedback control methods.

Keywords: auxiliary models, feedback control, online reconstruction.

1. Introduction: Problem statement

A dynamical model connecting the main economic and climatic indices was suggested by Nordhaus (1994). This model consists of various components:

(i) Constant parameters (Nordhaus, 1994, Tables 2.3 and 2.4, p. 21).

(ii) Functions that are considered (for simplicity of the analysis) exogenous with respect to the model and are given *a priori*.

(iii) Inner functions that are connected to one another and to exogenous parameters by means of some algebraic and differential equations (Nordhaus, 1994, Tables 2.3 and 2.2).

Consider the following functions: $\mu(t)$, the rate of emission reduction with respect to uncontrollable emissions; $E(t)$, the amount of emissions of Greenhouse Gases (GHGs) (CO_2 (carbonic acid gas) and chlorine-fluorine carbons only); $M_1(t) = (M(t) - 590)$, the excess of the mass of GHGs in the atmosphere with respect to the pre-industrial period; $T_0(t)$, the

average atmospheric temperature (on the Earth's surface); $T_1(t)$, the average deep-ocean temperature; $I(t)$, a gross investment; $K(t)$, a capital stock; $F(t)$, an atmospheric radiative forcing from GHGs; $O(t)$, a forcing of exogenous GHGs (i.e., of gases which are considered as uncontrollable; these are all GHGs except CO_2 (carbonic acid gas) and chlorine-fluorine carbons); $A(t)$, the level of technology; $\sigma(t)$, the ratio of GHGs emissions to global output; $L(t)$, the population at time t, also equal to labor inputs; $Q(t)$, the gross world product.

If we pass from the discrete-time model suggested by various authors to the "continuous" one, then the corresponding equations of the model Σ take the form

$$
\begin{aligned}
\dot{T}_0(t) &= c_1 T_0(t) + c_2 T_1(t) + c_3 F(t), \\
\dot{T}_1(t) &= c_4 (T_0(t) - T_1(t)), \\
\dot{M}_1(t) &= \beta E(t) - \delta_M M_1(t), \\
\dot{K}(t) &= -\delta_K K(t) + I(t), \quad t \in [0, \vartheta],
\end{aligned}
\tag{1}
$$

where t is time, ϑ is the terminal time instant,

$$
F(t) = 4.1 \cdot \log_2 \left(1 + \frac{M_1(t)}{590} \right) + O(t),
$$

$$E(t) = (1 - \mu(t))\sigma(t)Q(t),$$

$$Q(t) = \frac{(1 - b_1\mu(t)^{b_2})}{(1 + \theta_1 T_0(t)^{\theta_2})} A(t)K(t)^\gamma L(t)^{1-\gamma}.$$

Refer to the work of Nordhaus (1994) for more details on model description (Ch. 2) as well as on the physical meaning of coefficients in system (1) (Table 2.3, p. 20).

An initial state of Σ, $x(0) = \{T_0(0), T_1(0), M_1(0), K(0))\}$, is assumed to be known and given *a priori*. It is natural to set $T_0(0) > 0$, $T_1(0) > 0$, and $K(0) > 0$. Functions $\mu(\cdot)$ and $I(\cdot)$ are considered control inputs determining a strategy of global control of climate and economy. The numerical analysis of the model is performed by Nordhaus (1994). In that work, the direct problem is solved, namely, possible strategies (rules of forming $\mu(\cdot)$ and $I(\cdot)$) are specified, and the system's dynamics are computed. In what follows, functions μ and I, according to Nordhaus (1994), are treated as controls and are denoted by the symbol u, i.e., $u = \{\mu, I\}$. We transform the system (1) to the following form (setting small values of $b_1 = 0.0686$ and $\vartheta_1 = 0.00144$ to zero):

$$\dot{T}_0(t) = c_1 T_0(t) + c_2 T_1(t)$$
$$+ c_5 \log_2\left(1 + \frac{M_1(t)}{590}\right) + c_3 O(t),$$

$$\dot{T}_1(t) = c_4(T_0(t) - T_1(t)),$$
$$\dot{M}_1(t) = E_1(t)(1 - \mu(t)) - \delta_M M_1(t), \qquad (2)$$
$$\dot{K}(t) = -\delta_K K(t) + I(t), \quad t \in [0, \vartheta],$$

where $c_5 = 4.1\, c_3$,

$$E_1(t) = E_1(t, K) = \beta\sigma(t)A(t)K(t)^\gamma L(t)^{1-\gamma}.$$

In the sequel we consider the system Σ of the form (2). The symbol $x(\cdot) = x(\cdot; x(0), u(\cdot))$ stands for the solution of the system (2) with an initial state $x(0)$ and a control $u(\cdot) = \{\mu(\cdot), I(\cdot)\}$.

Our aim differs from that of Nordhaus (1994). We consider an "inverse" problem consisting in the following. Some system dynamics, i.e., a function $x_*(\cdot) = \{T_{0*}(\cdot), T_{1*}(\cdot), K_*(\cdot), M_{1*}(\cdot)\}$ generated by some unknown controls $\mu = \mu_*(\cdot)$ and $I = I_*(\cdot)$ are given. These controls may be program or feedback controls; the latter is formed, for example, by the rule $\mu_*(t) = \mu(t, x_*(t))$, $I_*(t) = I(t, x_*(t))$. Thus, the functions $x_*(\cdot) = \{T_{0*}(\cdot), T_{1*}(\cdot), K_*(\cdot), M_{1*}(\cdot)\}$ satisfy the system of equations

$$\dot{T}_{0*}(t) = c_1 T_{0*}(t) + c_2 T_{1*}(t)$$
$$+ c_5 \log_2\left(1 + \frac{M_{1*}(t)}{590}\right) + c_3 O_*(t),$$

$$\dot{T}_{1*}(t) = c_4(T_{0*}(t) - T_{1*}(t)), \qquad (3)$$
$$\dot{M}_{1*}(t) = E_{1*}(t, K_*)(1 - \mu_*(t)) - \delta_M M_{1*}(t),$$
$$\dot{K}_*(t) = -\delta_K K_*(t) + I_*(t), \quad t \in [0, \vartheta],$$

where, we emphasize it once again, the functions $\mu_*(\cdot)$ and $I_*(\cdot)$ are unknown. It is known only that they are subject to restrictions of the form

$$I_*(t) \in [I_-, I_+], \quad \mu_*(t) \in [f_-, f_+] \text{ for } t \in [0, \vartheta]. \quad (4)$$

Here

$$-\infty < f_- < f_+ < +\infty, \quad 0 \le I_- < I_+ < +\infty.$$

The initial state of the system (3), $x_*(0) = \{T_{0*}(0), T_{1*}(0), M_{1*}(0), K_*(0)\}$, is assumed to be $x(0)$.

The control problem under discussion may be formulated in the following way. At time instants frequent enough

$$\tau_i \in \Delta = \{\tau_i\}_{i=0}^m, \quad \tau_{i+1} = \tau_i + \delta, \quad \tau_0 = 0, \quad \tau_m = \vartheta,$$

values of $T_0(\tau_i)$, $T_1(\tau_i)$, and $K(\tau_i)$ are inaccurately measured. Results of measurements (vectors $\{\xi_{1i}^h, \xi_{2i}^h, \xi_{3i}^h\} \in \mathbb{R}^3$) satisfy the inequalities

$$\{(T_0(\tau_i) - \xi_{1i}^h)^2 + (T_1(\tau_i) - \xi_{2i}^h)^2$$
$$+ (K(\tau_i) - \xi_{3i}^h)^2\} \le h, \quad (5)$$

where $h \in (0, 1)$ is the level of informational noise. A number $\varepsilon > 0$ is given. It is required to construct an algorithm for forming a feedback control,

$$u = u_i^h = u(\tau_i; x_*(\cdot), \xi^h(\cdot)),$$

$$t \in [\tau_i, \tau_{i+1}), \quad i \in \{\phi, \dots, m-1\},$$

that ensures a prescribed quality of the controlled system (2). Namely, the algorithm is to implement the following condition. Whatever unknown possible Lebesgue measurable functions $\mu_*(\cdot)$ and $I_*(\cdot)$ with the properties (4) may be, the distance between $x^h(t)$ and $x_*(t)$ at all moments $t \in [0, \vartheta]$ should not exceed the value of ε provided the values of h and δ are sufficiently small.

Here $x^h(\cdot) = x(\cdot; x(0), u^h(\cdot)) = \{T_0^h(\cdot), T_1^h(\cdot), M_1^h(\cdot), K^h(\cdot)\}$ is the trajectory of Σ generated by the control $u^h(\cdot)$ which is formed according to the feedback principle, i.e., $x^h(\cdot)$ is the solution of the system (2) with the feedback controls $\mu(\cdot) = \mu^h(\cdot)$ and $I(\cdot) = I^h(\cdot)$.

This is an informal statement of the problem discussed in the present paper.

In what follows, the symbol $\mathcal{U}$ stands for the set of admissible controls, i.e., the set of Lebesgue measurable functions $u(\cdot) = \{\mu(\cdot), I(\cdot)\}$ such that $\mu(t) \in [f_-, f_+]$, $I(t) \in [I_-, I_+]$ for a.a. $t \in [0, \vartheta]$.

Note that the system (1) is "well-posed". Under this term, we mean the following. First, as is easily seen, in the domain $1 + M_1/590 > 0$, the Carathodory solution of the system (1) is uniquely defined for any $\{\mu(\cdot), I(\cdot)\} \in \mathcal{U}$ and any initial state. Second, the discrete-time model of

Nordhaus (1994, p. 19) can be obtained from (1) by the discretization of our system via the Euler method with an appropriate step.

One of the approaches to solving the problems of guaranteed control (they are also called positional differential games) for dynamical systems described by ordinary differential equations was developed by Krasovskii and Subbotin (1988). In the work cited above, the cases when the full phase state of a system is inaccurately measured at frequent enough time instants are considered. In the present work, using the approach described by Osipov and Kryazhimskii (1995), the problems of guaranteed control under the measurement of a "part" of a system's phase state (a "part" of coordinates) are investigated (see also Osipov and Kryazhimskii, 1995; Blizorukova and Maksimov, 2006).

To form a control u being a solution to our problem, along with the information on the "part" of coordinates of the solution of the system Σ (namely, on the values ξ_i^h satisfying the inequalities (5)), it is necessary to obtain some additional information on the coordinate M_1 which is missing. To get such a piece of information during the control process, it is reasonable, following the approach developed by Osipov and Kryazhimskii (1995), Maksimov (2010) as well as Osipov *et al.* (2011), to introduce an auxiliary controlled system M. This system is described by a differential equation (its form is specified below). The equation has an output $w^h(t)$ and an input $v^h(t)$. The input $v^h(\cdot)$ is some new auxiliary control; it should be formed by the feedback principle in such a way that $v^h(\cdot)$ "approximates" the unknown coordinate $M_1(\cdot)$ in the mean uniform metric. Thus, along with the block of forming the control in the real system (it is called a *controller*), we need to incorporate into the control contour one more block (it is called an *identifier*) allowing us to reconstruct the missing coordinate $M_1(\cdot)$ in a real time mode. The scheme of algorithms for solving the problem is given in Fig. 1.
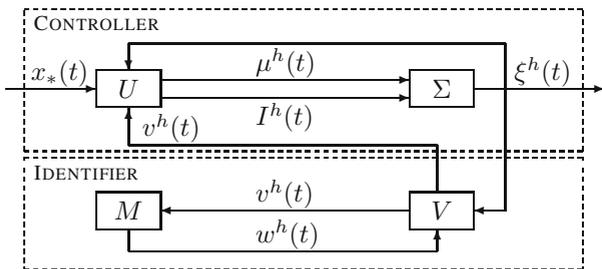

Fig. 1. Scheme of solution algorithms.

To start with an auxiliary dynamical system M (a model) is introduced. This model functioning on the time interval $[0, \vartheta]$ has an input $v^h(t)$ and an output $w^h(t)$. The model M with its control law V forms the identifier.

Before the algorithm starts, the value h and the partition Δ with the step δ, as well as the model M, are fixed. The process of synchronous feedback control of the systems Σ and M is organized on the interval $[0, \vartheta]$. This process is decomposed into $m-1$ identical steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, the following actions are fulfilled. First, at the time instant τ_i, according to the chosen rules U and V, the functions

$$v^h(t) = v_i^h \in V(\tau_i, \xi_i^h, w^h(\tau_i)), \quad t \in \delta_i, \qquad (6)$$

$$\begin{aligned} u^h(t) = u_i^h = \{\mu_i^h, I_i^h\} &\in U(\tau_i, v_i^h, \xi_i^h, x_*(\tau_i)) \\ &\subset [I_-, I_+] \times [f_-, f_+], \end{aligned} \qquad (7)$$

are calculated by measurements ξ_i^h and $w^h(\tau_i)$. Then (till the moment τ_{i+1}) the control $u = u^h(t)$, $\tau_i \leq t < \tau_{i+1}$, is fed onto the input of the system Σ and the control $v = v^h(t)$, $\tau_i \leq t < \tau_{i+1}$, onto the input of the model M. The values ξ_{i+1}^h and $w^h(\tau_{i+1})$ are the results of the work of the algorithm at the i-th step. The procedure stops at the moment ϑ.

Thus, the complexity of solving these problems is reduced to an appropriate choice of a model M as well as functions U and V. Consequently, a mathematical statement of the problem may be formulated as follows. In the sequel, a family of partitions

$$\Delta_h = \{\tau_{i,h}\}_{h=0}^{m_h}, \quad \tau_{i+1,h} = \tau_{i,h} + \delta(h),$$

$$\tau_{0,h} = 0, \quad \tau_{m_h,h} = \vartheta,$$

of the interval $[0, \vartheta]$ is assumed to be fixed.

Problem of robust control. It is required to specify differential equations of the model M in the form

$$\dot{w}^h(t) = f_1(\xi_i^h, w^h(\tau_i), v^h(t)), \qquad (8)$$

$$t \in \delta_{h,i} = [\tau_i, \tau_{i+1}), \quad \tau_i = \tau_{i,h},$$

$$w^h(0) = w_0^h, \quad w^h(t) \in \mathbb{R},$$

and the rule of choosing controls $v_i^h(t)$ in the model (8) and $u_i^h(t)$ in the system (2) at the moments τ_i being a mapping of the form (6), (7) such that the inequality

$$\max_{t \in [0, \vartheta]} \|x^h(t) - x_*(t)\| \leq \varepsilon \qquad (9)$$

holds for $h \in (0, h_*(\varepsilon))$ and $\delta = \delta(h) \in (0, \delta(h_*(\varepsilon))$.

Here, the symbol $\|\cdot\|$ denotes the Euclidean norm of a vector, $x^h(\cdot) = x(\cdot; x(0), u^h(\cdot))$ is the trajectory of Σ generated by the control $u^h(\cdot)$ that is formed according to the feedback principle (6), (7).

After writing down the model equations (8) and the rules for forming controls (6) and (7), the sequence of actions which should be performed to provide "closeness" of a trajectory of system (2) and a given trajectory

$x_*(\cdot)$ (i.e., the inequality (9), is similar to the sequence described above (see also the end of Section 3, where the corresponding algorithm is presented).

From the above scheme, it follows that one of the important steps of the algorithm is the reconstruction of the unmeasured coordinate M_1 in "real-time" mode. It should be noted that the problems of online reconstruction of states/inputs were considered from the position of the approach above by Keesman and Maksimov (2008), Blizorukova and Maksimov (2010) as well as Osipov *et al.* (2011) for some classes of systems described by ordinary differential equations of the second and third orders. The algorithms designed in these works allowed reconstructing the corresponding characteristics in the L_2-metric. The algorithm for reconstructing the coordinate M_1 described in Section 2 allows us to get an approximation of M_1 in the uniform metric (the metric of space C of continuous functions).

Let the symbol $X(\cdot)$ denote the set of solutions of the system (2), i.e., $X(\cdot) = \{x(\cdot) : x(\cdot) = x(\cdot; x(0), u(\cdot)) = \{T_0(\cdot), T_1(\cdot), M_1(\cdot), K(\cdot)\}, u(\cdot) \in \mathcal{U}\}$. We assume that the following condition is fulfilled.

Condition 1.

$$d_* = \inf\Big\{ \min_{t \in [0,\vartheta]} \Big(1 + \frac{M_1(t)}{590}\Big) :$$
$$x(\cdot) = \{T_0(\cdot), T_1(\cdot), M_1(\cdot), K(\cdot)\} \in X(\cdot)\Big\} > 1.$$

In addition, the functions $\sigma(t)$, $A(t)$, $L(t)$, and $O(t)$ are considered to be known and continuous.

2. Algorithm for reconstructing $M_1(\cdot)$

First, we specify the algorithm for reconstructing $M_1(\cdot)$, which will be applied for solving the problem in question. Namely, we describe the identifier (see Fig. 1), i.e., specify the equation of the model M and its control law V. To substantiate this algorithm, we use ideas of Osipov and Kryazhimskii (1995), Osipov *et al.* (2011), Blizorukova and Maksimov (2006) as well as Maksimov (2013).

Introduce the notation

$$T(t) = \{T_0(t), T_1(t)\},$$

$$f(t, T(t)) = c_1 T_0(t) + c_2 T_1(t) + c_3 O(t),$$

$$\tilde{u}(t) = \log_2\Big(1 + \frac{M_1(t)}{590}\Big).$$

Here $x(\cdot) = \{T_0(\cdot), T_1(\cdot), M_1(\cdot), K(\cdot)\}$ is an arbitrary element of the set $X(\cdot)$. In this case, the first equation of the system (2) is rewritten in the form

$$\dot{T}_0(t) = f(t, T(t)) + c_5 \tilde{u}(t).$$

Note that one can specify a number $M_* > 0$ such that the following inequalities are valid:

$$\|\dot{T}(t)\| \le M_* \quad \text{for a.a. } t \in [0, \vartheta], \tag{10}$$

$$|f(t, T(t)) - f(\tau_i, \xi_i^h)| \le M_*(\delta + h + \omega(\delta)) \tag{11}$$

for $t \in \delta_i = [\tau_i, \tau_{i+1})$. Here the symbol $|\cdot|$ stands for the absolute value of a number, $\tau_i = \tau_{i,h}$, $\omega(\delta)$ is the continuity modulus of the function $t \to O(t)$, $t \in [0, \vartheta]$, i.e.,

$$\omega(\delta) = \sup\{|O(t) - O(t - \delta)| : t \in [\delta, \vartheta]\}.$$

The inequality (11) is a consequence of (5) and (10).

We fix a family Δ_h of partitions of the interval $[0, \vartheta]$ and some auxiliary function $\alpha(h) : (0, 1) \to (0, 1)$. As the model M, we take a linear system described by a scalar differential equation of the form

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h) + c_5 v^h(t) \tag{12}$$

for a.a. $t \in \delta_i = [\tau_i, \tau_{i+1})$, $i \in \{0, \ldots, m-1\}$, $\tau_i = \tau_{i,h}$, $m = m_h$, with the initial condition

$$w^h(0) = T_0(0).$$

Let

$$v^h(t) = v_i^h$$
$$\in V(\tau_i, \xi_i^h, w^h(\tau_i))$$
$$= -\frac{1}{\alpha} c_5 [w^h(\tau_i) - \xi_{1i}^h] \tag{13}$$

for a. a. $t \in \delta_i$. The control $v^h(t)$ in Eqn. (12) is found from (13). Thus, the model control is specified by the feedback principle (see (6)). Consequently, Eqn. (12) takes the form

$$\dot{w}^h(t) = f(\tau_i, \xi_i^h) - \frac{1}{\alpha} c_5{}^2 [w^h(\tau_i) - \xi_{1i}^h] \tag{14}$$

for a.a. $t \in \delta_i$.

Let us describe the algorithm for reconstructing the unmeasured coordinate $M_1(\cdot)$ in real time mode. Before the algorithm starts, we fix a value $h \in (0, 1)$ and a partition Δ_h. The work of the algorithm is decomposed into $m - 1$ identical steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{i,h}$, the following actions are made. First, at the moment τ_i, the control $v^h(t)$ is calculated by (13). This control is fed to the input of the model (12) on the interval $[\tau_i, \tau_{i+1})$. Under the action of this control, the model passes from the state $w^h(\tau_i)$ to the state $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; \tau_i, w^h(\tau_i), v_i^h)$. The work of the algorithm stops at the moment ϑ.

Denote by $\Xi(x(\cdot), h)$ the set of admissible measurements, i.e., the set of all piece wise constant functions $\xi^h(\cdot) \to \mathbb{R}^3$, $\xi^h(t) = \xi_i^h$ for $t \in [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{i,h}$, satisfying the inequalities (5). Here $\xi_i^h = \{\xi_{1i}^h, \xi_{2i}^h, \xi_{3i}^h\}$.

Lemma 1. *Let the conditions*

$$\alpha(h) \to 0, \qquad \delta(h) \to 0, \qquad (15)$$

$$\delta(h)\alpha^{-1}(h) \to 0, \quad h\alpha^{-1}(h) \to 0 \ \text{as } h \to 0$$

be fulfilled. Then, uniformly in all $x(\cdot) \in X(\cdot)$, $h \in (0,1)$, $\xi^h(\cdot) \in \Xi(x(\cdot), h)$, $i \in \{0, \ldots, m_h - 1\}$, *the inequalities*

$$\int_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)| \, ds \le C\delta \qquad (16)$$

are valid. Here $C = \text{const} > 0$, $\delta = \delta(h)$, *and* $\tau_i = \tau_{i,h}$.

Proof. Taking into account (14), we deduce

$$\frac{d}{dt}[w^h(t) - T_0(t)]$$

$$= f(\tau_i, \xi_i^h) - \frac{1}{\alpha}c_5{}^2[w^h(\tau_i) - \xi_{1i}^h]$$

$$\quad - f(t, T(t)) - c_5\tilde{u}(t)$$

$$= -\frac{1}{\alpha}c_5{}^2[w^h(t) - T_0(t)] + \Psi_h^{(1)}(t)$$

for a. a. $t \in \delta_i$, and

$$w^h(0) = T_0(0),$$

where

$$\Psi_h^{(1)}(s) = \Psi_h(s) + \frac{1}{\alpha}c_5{}^2[w^h(s) - w^h(\tau_i)],$$

$$\Psi_h(s) = -\frac{1}{\alpha}c_5{}^2[T_0(s) - \xi_{1i}^h] + [f(\tau_i, \xi_i^h)$$

$$\quad - f(s, T(s))] - c_5\tilde{u}(s) \quad \text{for a.a. } s \in \delta_i.$$

Here, by virtue of (5), (10), (11) and (15), the family of functions $\Psi_h(\cdot)$ is bounded,

$$|\Psi_h(s)| \le M^{(1)} \quad \text{for a.a. } t \in [0, \vartheta], \qquad (17)$$

uniformly with respect to all $h \in (0,1)$. Further, we have

$$w^h(t) - T_0(t)$$

$$= \int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)} \Psi_h^{(1)}(s) \, ds, \quad t \in [0, \vartheta]. \quad (18)$$

Let

$$\mu(t) = \max_{0 \le \tau \le t} |w^h(\tau) - T_0(\tau)|,$$

$$f_h(t) = f(\tau_i, \xi_i^h) \quad \text{for } t \in \delta_i.$$

Then the following estimations are true:

$$\frac{1}{\alpha}c_5{}^2 \int_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)| \, ds$$

$$\le \frac{K_0}{\alpha} \int_{\tau_i}^{\tau_{i+1}} |f_h(s) - \frac{1}{\alpha}c_5{}^2[w^h(\tau_i) - \xi_{1i}^h]| \, ds$$

$$\le K_1\frac{\delta}{\alpha} + K_2\frac{\delta}{\alpha^2}(\mu(\tau_i) + h),$$

$$\mu(\tau_i) \le \mu(\tau_{i+1}). \qquad (19)$$

Note that

$$|\Psi_h^{(1)}(t)|$$

$$\le |\Psi_h(t)| + \frac{1}{\alpha}c_5{}^2 \int_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)| \, ds \quad \text{for } t \in \delta_i. \qquad (20)$$

Thus, taking into account (18)–(20), we obtain

$$\mu(t) \le K_3\left(\frac{\delta}{\alpha} + \frac{\delta}{\alpha^2}\mu(\tau_i) + \frac{\delta h}{\alpha^2}\right) \int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)} \, ds$$

$$+ \int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)}|\Psi_h(s)| \, ds, \quad t \in \delta_i. \qquad (21)$$

Using (17), we derive

$$\int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)}|\Psi_h(s)| \, ds \le K_4 \int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)} \, ds. \qquad (22)$$

It is easily seen that

$$\int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)} \, ds$$

$$= \frac{\alpha}{c_5{}^2} e^{-\frac{c_5{}^2}{\alpha}(t-s)}\Big|_0^t$$

$$= \frac{\alpha}{c_5{}^2}(1 - e^{-\frac{c_5{}^2}{\alpha}t}) \le K_5\alpha. \qquad (23)$$

Using (22) and (23), we obtain

$$\int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)}|\Psi_h(s)| \, ds \le K_6\alpha. \qquad (24)$$

Taking into account (21), (17), (24) and assuming $t = \tau_i$, we have

$$\left(1 - \frac{K_3K_5\delta}{\alpha}\right)\mu(\tau_i) \le K_7\left(\alpha + \delta + \frac{\delta h}{\alpha}\right).$$

Therefore, for sufficiently small h (for example, such that $1 - K_3 K_5 \delta / \alpha \geq 1/2$), we obtain

$$\mu(\tau_i) \leq K_8 \Big(\alpha + \delta + \frac{\delta h}{\alpha}\Big) \leq K_9 (\alpha + \delta) \qquad (25)$$

(see (15)). From (19) it follows that

$$\int_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)| \, \mathrm{d}s \leq K_{10} \Big\{\delta + \frac{\delta}{\alpha}(\mu(\tau_i) + h)\Big\}.$$

In addition, by virtue of (25), we derive

$$\delta + \frac{\delta}{\alpha}(\mu(\tau_i) + h) \leq \delta + K_8 \frac{\delta}{\alpha}\Big(\alpha + h + \delta + \frac{\delta h}{\alpha}\Big)$$
$$\leq K_{11}\delta.$$

Therefore,

$$\int_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)| \, \mathrm{d}s \leq K_{12}\delta.$$

The inequality (16) is established and the lemma is proved. ∎

Lemma 2. *Let the conditions (15) be fulfilled. Let $\delta^\gamma(h)\alpha^{-1}(h) \to +\infty$ (for some $\gamma \in (0,1)$) as $h \to 0$ and*

$$u_e^h(t) = \begin{cases} \tilde{u}(0), & t \in [0, \delta^\gamma), \\ v^h(t), & t \in [\delta^\gamma, \vartheta]. \end{cases}$$

Then the inequality

$$\sup_{t \in [0,\vartheta]} |u_e^h(t) - \tilde{u}(t)|$$
$$\leq d_1^0 \alpha(h) + d_2^0 (h + \delta(h))\alpha^{-1}(h) + d_3^0 \omega(\delta(h))$$
$$+ d_4^0 \alpha(h) \delta^{-\gamma}(h) + d_5^0 \delta^\gamma(h)$$

is valid. Here the constants d_j^0, $j \in 1, \ldots, 5$, do not depend on $h \in (0,1)$.

Proof. It is easily seen that the equality

$$\frac{1}{\alpha} c_5{}^2 [w^h(t) - T_0(t)]$$
$$= \int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s} e^{-\frac{1}{\alpha}c_5{}^2(t-s)}\Big)\Psi_h^{(1)}(s) \, \mathrm{d}s$$
$$= -\int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s} e^{-\frac{1}{\alpha}c_5{}^2(t-s)}\Big)c_5\tilde{u}(s) \, \mathrm{d}s \qquad (26)$$
$$+ \sum_{j=1}^3 \int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s} e^{-\frac{1}{\alpha}c_5{}^2(t-s)}\Big)\gamma_\delta^{(j)}(s) \, \mathrm{d}s$$

is true. Here

$$\gamma_\delta^{(1)}(s) = \frac{1}{\alpha} c_5{}^2 [w^h(s) - w^h(\tau_i)],$$
$$\gamma_\delta^{(2)}(s) = -\frac{1}{\alpha} c_5{}^2 [T_0(s) - \xi_{1i}^h],$$
$$\gamma_\delta^{(3)}(s) = f(\tau_i, \xi_i^h) - f(s, T(s))$$

for a.a. $s \in \delta_i$. Using (16) we conclude that

$$|\gamma_\delta^{(1)}(s)| \leq C_1 \frac{\delta}{\alpha}, \quad s \in [0, \vartheta]. \qquad (27)$$

From (5) and (10) it follows that

$$|\gamma_\delta^{(2)}(s)| \leq C_2 \frac{\delta + h}{\alpha}, \quad s \in [0, \vartheta]. \qquad (28)$$

Using (11) we have

$$|\gamma_\delta^{(3)}(s)| \leq M_*(\delta + h + \omega(\delta)), \quad s \in [0, \vartheta]. \qquad (29)$$

In this case, from (27)–(29), (23), (24) we deduce that

$$\Big|\sum_{j=1}^3 \int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s} e^{-\frac{1}{\alpha}c_5{}^2(t-s)}\Big)\gamma_\delta^{(j)}(s) \, \mathrm{d}s\Big|$$
$$\leq \varrho(h, \alpha, \delta) = C_3\Big(\delta + h + \omega(\delta) + \frac{\delta + h}{\alpha}\Big). \qquad (30)$$

Integrating by parts the first term on the right-hand side of (26), we obtain

$$-\int_0^t \Big(\frac{\mathrm{d}}{\mathrm{d}s} e^{-\frac{1}{\alpha}c_5{}^2(t-s)}\Big)c_5\tilde{u}(s) \, \mathrm{d}s$$
$$= e^{-\frac{1}{\alpha}c_5{}^2 t}c_5\tilde{u}(0) - c_5\tilde{u}(t) \qquad (31)$$
$$+ \int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)}c_5\dot{\tilde{u}}(s) \, \mathrm{d}s.$$

From (5), (10), (16) (see Lemma 1), we have that for a.a. $t \in \delta_i$

$$\Big|\frac{1}{\alpha}c_5{}^2\{[w^h(t) - T_0(t)] - [w^h(\tau_i) - \xi_{1i}^h]\}\Big| \qquad (32)$$
$$\leq \frac{C_4}{\alpha}\Big\{\int_{\tau_i}^{\tau_{i+1}} |\dot{w}^h(s)| \, \mathrm{d}s + h + \delta\Big\} \leq C_5 \frac{h + \delta}{\alpha}.$$

By virtue of the boundedness of $\dot{\tilde{u}}(\cdot)$ ($\dot{\tilde{u}}(\cdot) \in L_\infty([0,\vartheta];\mathbb{R})$ (see Condition 1), using (24), we derive the inequality

$$\Big|\int_0^t e^{-\frac{1}{\alpha}c_5{}^2(t-s)}c_5\dot{\tilde{u}}(s) \, \mathrm{d}s\Big| \leq C_6\alpha.$$

Therefore, using this inequality as well as (26), (30)–(32) and (15), we obtain for $t \in \delta_i$

$$
\begin{aligned}
\left| \frac{1}{\alpha} c_5{}^2 [w^h(\tau_i) - T_0(\tau_i)] - c_5 \tilde{u}(t) \right| \\
\leq \varrho(h, \delta, \alpha) + C_5 \frac{h + \delta}{\alpha} \\
+ C_6 \alpha + |e^{-\frac{1}{\alpha} c_5{}^2 t} c_5 \tilde{u}(0)|.
\end{aligned}
\tag{33}
$$

Note that

$$
e^{-c_* \frac{\delta^\gamma}{\alpha}} \leq \frac{\alpha}{c_* \delta^\gamma} \quad \text{for } c_* > 0.
$$

The statement of the lemma follows from (33), since, by virtue of the boundedness of $M_1(\cdot)$, the inequalities

$$
|\tilde{u}(t) - u_e^h(t)| \leq c \delta^\gamma \quad \text{for a.a. } t \in [0, \delta^\gamma),
$$

$$
|\tilde{u}(t) - u_e^h(t)| \leq d_1^0 \alpha + d_2^0 \frac{h + \delta}{\alpha} + d_3^0 \omega(\delta)
$$

$$
+ d_4^0 \frac{\alpha}{\delta^\gamma} \quad \text{for a.a. } t \in [\delta^\gamma, \vartheta]
$$

are valid. The lemma is proved. ∎

Introduce the notation

$$
\tilde{u}_*^h(t) = 590(2^{u_e^h(t)} - 1).
$$

The following theorem is true.

Theorem 1. *Under the conditions of Lemma 2, the inequality*

$$
\begin{aligned}
\sup_{t \in [0, \vartheta]} |\tilde{u}_*^h(t) - M_1(t)| &\leq \nu(h, \delta(h), \alpha(h)) \\
&= d_1 \alpha(h) + d_2(h + \delta(h)) \alpha^{-1}(h) + d_3 \omega(\delta(h)) \\
&\quad + d_4 \alpha(h) \delta^{-\gamma}(h) + d_5 \delta^\gamma(h)
\end{aligned}
$$

holds. Here the constants d_j, $j \in \{1, \ldots, 5\}$, do not depend on $h \in (0, 1)$.

The theorem follows from Lemma 2 and the inequality

$$
|\tilde{u}_*^h(t) - M_1(t)| \leq 590|2^{u_e^h(t)} - 2^{\tilde{u}(t)}|.
$$

3. Algorithm for the control problem

Let us turn to the description of the algorithm for solving the control problem in question. From the above, it is necessary to specify the model (8) and control laws U (7) and V (6), providing the inequality (9).

We fix a family Δ_h of partitions of the interval $[0, \vartheta]$ and some function $\alpha(h) : (0, 1) \to (0, 1)$. Let the family Δ_h and the function $\alpha(h)$ be such that the following condition holds

Condition 2. *We have*

$$
\alpha(h) \to 0, \quad \delta(h) \to 0, \quad \delta(h) \alpha^{-1}(h) \to 0,
$$

$$
h \alpha^{-1}(h) \to 0, \quad \alpha^{-1}(h) \delta^\gamma(h) \to +\infty \quad \text{as } h \to 0
$$

for some $\gamma \in (0, 1)$.

Let the model (8) be of form (12), i.e.,

$$
\dot{w}^h(t) = f(\tau_i, \xi_i^h) + c_5 v^h(t)
\tag{34}
$$

for a.a. $t \in \delta_i = [\tau_i, \tau_{i+1})$, $i \in \{0, \ldots, m - 1\}$, $\tau_i = \tau_{i,h}$, $m = m_h$, with the initial condition

$$
w^h(0) = T_0(0).
$$

Let rules U (7) and V (6) for forming the controls u_i^h and v_i^h be as follows:

$$
v_i^h = V(\tau_i, \xi_i^h, w^h(\tau_i)) = -\frac{1}{\alpha} c_5 [w^h(\tau_i) - \xi_{1i}^h], \tag{35}
$$

$$
u_i^h = \{\mu^h(\tau_i), I^h(\tau_i)\} = U(\tau_i, v_i^h, \xi_i^h, x_*(\tau_i))
$$

for $t \in \delta_i$. Here

$$
I^h(\tau_i) = \arg\min\{(\xi_{3i}^h - K_*(\tau_i))I : I \in [I_-, I_+]\}, \tag{36}
$$

$$
\begin{aligned}
\mu^h(\tau_i) = \arg\min\{&E_1(\tau_i, K_*) \\
&\times (M_{1*}(\tau_i) - \tilde{u}_*^h(\tau_i))\mu : \mu \in [f_-, f_+]\},
\end{aligned}
\tag{37}
$$

$$
\tilde{u}_*^h(\tau_i) = 590(2^{u_\varepsilon^h(\tau_i)} - 1),
$$

$$
u_\varepsilon^h(\tau_i) = \begin{cases} \log_2\left(1 + \frac{M_1(0)}{590}\right) & \text{if } \tau_i \leq \delta^\gamma(h), \\ v_i^h & \text{otherwise.} \end{cases}
$$

In what follows, we need the lemma given below.

Lemma 3. *(Maksimov, 2011) Let the function $\varepsilon(t)$ be nonpositive for $t \in T$ and, for all $i \in \{0, \ldots, m - 1\}$, satisfy the inequalities*

$$
\varepsilon(\tau_{i+1}) \leq \varepsilon(\tau_i)(1 + \beta\delta) + \int_{\tau_i}^{\tau_{i+1}} |\varphi(t)| \, \mathrm{d}t,
$$

where $\tau_i \in \Delta$, $\beta = \text{const} > 0$, and $\varphi(\cdot) \in L_1(T; \mathbb{R})$. Then

$$
\varepsilon(\tau_i) \leq \left(\varepsilon(t_0) + \int_{t_0}^{\tau_i} |\varphi(t)| \, \mathrm{d}t\right) \exp(\beta(\tau_i - t_0)).
$$

Introduce the following condition.

Condition 3. *The inequalities*

$$
0 < C^{(1)} < K_*(t) < C^{(2)} < +\infty \quad \text{for } t \in [0, \vartheta]
$$

are valid.

Theorem 2. *For any $\varepsilon > 0$, one can specify $h_*(\varepsilon) \in (0, 1)$ such that, for all $h \in (0, h_*(\varepsilon))$ and $\delta(h) \in (0, \delta(h_*(\varepsilon)))$, the inequality (9) holds, if the model M is given by Eqn. (34), the strategies V and U are taken in the form (6), (7), (35)–(37).*

Proof. First, estimate the variation in the value

$$\varepsilon_1(t) = |K_*(t) - K^h(t)|^2, \quad t \in [0, \vartheta].$$

It is easily seen that for $t \in \delta_i = [\tau_i, \tau_{i+1})$ the following inequality is true:

$$
\varepsilon_1(t) \leq \varepsilon_1(\tau_i) + \delta(h) \int_{\tau_i}^{t} |\dot{K}_*(\tau) - \dot{K}^h(\tau)|^2 \, \mathrm{d}\tau
$$
$$
+ \int_{\tau_i}^{t} \nu_i(\tau) \, \mathrm{d}\tau. \tag{38}
$$

Here

$$\nu_i(\tau) = 2(K_*(\tau_i) - K^h(\tau_i))(\dot{K}_*(\tau) - \dot{K}^h(\tau)), \quad \tau \in \delta_i.$$

Consider the value $\nu_i(t)$. We have for $t \in \delta_i$

$$\nu_i(t) = \sum_{j=1}^{4} \nu_{ji}(t), \quad t \in \delta_i, \tag{39}$$

where

$$
\begin{aligned}
\nu_{1i}(t) &= 2\beta_i^{(1)} \delta_K \left(K_*(\tau_i) - K_*(t)\right), \\
\nu_{2i}(t) &= 2\beta_i^{(1)} \delta_K \left(K^h(t) - K^h(\tau_i)\right), \\
\nu_{3i}(t) &= 2\delta_K \varepsilon_1(\tau_i), \\
\nu_{4i}(t) &= 2\beta_i^{(1)} \delta_K \left(I_*(t) - I^h(\tau_i)\right), \\
\beta_i^{(1)} &= K_*(\tau_i) - K^h(\tau_i).
\end{aligned}
$$

Estimate each term on the right-hand side of the equality (39). From (2) and (4) it follows that

$$\nu_{1i}(t) \leq C_1\delta, \quad \nu_{2i}(t) \leq C_1\delta, \quad t \in \delta_i. \tag{40}$$

Taking into account the inequality

$$|K^h(\tau_i) - \xi_{3i}^h| \leq h$$

(see (5)) and the rule for choosing the control $I^h(\cdot)$ (36), we obtain for a.a. $t \in \delta_i$

$$\nu_{4i}(t) \leq C_2 h, \quad C_2 = 2\delta_K \max\{|I_-|, |I_+|\}. \tag{41}$$

From (39)–(41) it follows that for a.a. $t \in \delta_i$

$$\nu_i(t) \leq 2C_1\delta + C_2 h + C_3\varepsilon_1(\tau_i). \tag{42}$$

Consequently, due to (38) and (42), we have for $i \in \{0, \ldots, m - 1\}$ the estimate

$$\varepsilon_1(\tau_{i+1}) \leq (1 + C_3\delta)\varepsilon_1(\tau_i) + C_4\delta(\delta + h). \tag{43}$$

From (43) and Lemma 3, we derive (for all $i \in \{0, \ldots, m - 1\}$) the inequalities

$$\varepsilon_1(\tau_{i+1}) \leq \varepsilon_1(0) + C_5(\delta + h). \tag{44}$$

In this case,

$$\varepsilon_1(t) \leq C_*(\delta + h). \tag{45}$$

Here $C_* > 0$ is a constant that does not depend on h, δ and can be written explicitly. Now estimate the variation

$$\varepsilon_2(t) = |M_{*1}(t) - M_1^h(t)|^2, \quad t \in [0, \vartheta].$$

It is easily seen that for $t \in \delta_i = [\tau_i, \tau_{i+1})$ the following inequality is true:

$$
\varepsilon_2(t) \leq \varepsilon_2(\tau_i) + \delta(h) \int_{\tau_i}^{t} |\dot{M}_{*1}(\tau) - M_1^h(\tau)|^2 \, \mathrm{d}\tau
$$
$$
+ \int_{\tau_i}^{t} \mu_i^{(1)}(\tau) \, \mathrm{d}\tau, \tag{46}
$$

$$\mu_i^{(1)}(t) = 2(M_{*1}(\tau_i) - M_1^h(\tau_i))(\dot{M}_{*1}(t) - \dot{M}_1^h(t))$$

for $t \in \delta_i$. Note that

$$
\begin{aligned}
\dot{M}_{*1}(t) &= E_1(t, K_*)(1 - \mu_*(t)) - \delta_M M_{*1}(t), \\
\dot{M}_1^h(t) &= E_1(t, K^h)(1 - \mu^h(t)) - \delta_M M_1^h(t),
\end{aligned} \tag{47}
$$

$$|\dot{K}^h(t)| \leq C_0 \tag{48}$$

for a.a. $t \in [0, \vartheta]$. Consider the value $\mu_i(t)$. By virtue of (47), we have for $t \in \delta_i$

$$\mu_i(t) = \sum_{j=1}^{6} \lambda_{ji}(t), \quad t \in \delta_i, \tag{49}$$

where

$$
\begin{aligned}
\lambda_{1i}(t) &= 2\beta_i^{(2)} \left(E_1(\tau_i, K_*) - E_1(t, K_*)\right) \mu_*(t), \\
\lambda_{2i}(t) &= 2\beta_i^{(2)} \left(E_1(t, K_*) - E_1(t, K^h)\right), \\
\lambda_{3i}(t) &= 2\beta_i^{(2)} \left(E_1(t, K^h) - E_1(\tau_i, K^h)\right) \mu^h(t), \\
\lambda_{4i}(t) &= 2\delta_M \beta_i^{(2)} \left(M_1^h(t) - M_{*1}(t)\right), \\
\lambda_{5i}(t) &= 2\beta_i^{(2)} E_1(\tau_i, K_*) \left(\mu^h(t) - \mu_*(t)\right), \\
\lambda_{6i}(t) &= 2\beta_i^{(2)} (E_1(\tau_i, K^h) - E_1(\tau_i, K_*))\mu^h(t), \\
\beta_i^{(2)} &= M_{*1}(\tau_i) - M_1^h(\tau_i).
\end{aligned}
$$

Now, we estimate each term on the right-hand side of the equality (49). From (2), (3) and (48), it follows that

$$
\begin{aligned}
\lambda_{1i}(t) &\leq C_6(\delta + \omega_\varrho(\delta)), \\
\lambda_{3i}(t) &\leq C_7(\delta + \omega_\varrho(\delta)), \quad t \in \delta_i,
\end{aligned} \tag{50}
$$

$$|E_1(t, K_*) - E_1(t, K^h)|$$
$$\leq \varrho(t)|K_*^\gamma(t) - (K^h)^\gamma(t)|, \quad t \in [0, \vartheta], \quad (51)$$

where

$$\varrho(t) = \beta\sigma(t)A(t)L^{1-\gamma}(t),$$

$\omega_\varrho(\delta)$ is the continuity modulus of $\varrho(t)$, i.e.,

$$\omega_\varrho(\delta) = \sup\{|\varrho(t) - \varrho(t+\delta)| : t, t+\delta \in [0, \vartheta]\}.$$

By virtue of Condition 3, we obtain

$$|K_*^\gamma(t) - (K^h)^\gamma(t)| \leq K_*^\gamma(t)|1 - (K^h(t)/K_*(t))^\gamma|$$
$$\leq (C^{(2)})^\gamma|1 - (1 + \psi^h(t))^\gamma|,$$

where

$$\psi^h(t) = \frac{K^h(t) - K_*(t)}{K_*(t)}.$$

Taking into account (45), we conclude that the estimate

$$|\psi^h(t)| \leq \frac{\varepsilon_1^{1/2}(t)}{C^{(1)}} \leq \frac{C_*^{1/2}(h+\delta)^{1/2}}{C^{(1)}}$$

is valid. Let us choose $\delta_* \in (0, 1)$ and $h_* \in (0, 1)$ in such a way that the inequality

$$\frac{C_*^{1/2}}{C^{(1)}}(h_*^{1/2} + \delta_*^{1/2}) < 1$$

is valid. Then, for $h \in (0, h_*)$ and $\delta \in (0, \delta_*)$ we have

$$\sup_{t \in [0, \vartheta]} |K_*^\gamma(t) - (K^h)^\gamma(t)| \leq C_8(h^{1/2} + \delta^{1/2}), \quad (52)$$

where the constant C_8 does not depend on h and δ. Using (51) and (52), we obtain for $\delta_* \in (0, 1)$ and $h_* \in (0, 1)$

$$\lambda_{2i}(t) \leq C_9(h^{1/2} + \delta^{1/2}),$$
$$\lambda_{6i}(t) \leq C_{10}(h^{1/2} + \delta^{1/2}), \quad t \in \delta_i. \quad (53)$$

Furthermore, we have for $t \in \delta_i$

$$\lambda_{4i}(t) \leq C_{11}\varepsilon_2(\tau_i) + C_{12}\delta, \quad (54)$$
$$\lambda_{5i}(t) \leq \lambda_i^{(1)}(t) + \lambda_i^{(2)}(t), \quad (55)$$

where

$$\lambda_i^{(1)}(t) = 2E_1(\tau_i, K_*)(M_{1*}(\tau_i) - \tilde{u}_*^h(\tau_i))$$
$$\times (\mu^h(t) - \mu_*(\tau_i))$$

$$\lambda_i^{(2)}(t) = 4|M_1^h(\tau_i) - \tilde{u}_*^h(\tau_i)||E_1(\tau_i, K_*)|f, \quad t \in \delta_i,$$

$$f = \max\{|f_-|, |f_+|\}.$$

Using the rule of forming the control $\mu^h(\cdot)$ (see (37)), we get

$$\lambda_i^{(1)}(t) \leq 0 \quad \text{for a.a. } t \in \delta_i. \quad (56)$$

In addition, from Theorem 1 we derive

$$\lambda_i^{(2)}(t) \leq C_{13}\nu(h, \delta(h), \alpha(h)) \quad \text{for } t \in \delta_i. \quad (57)$$

Taking into account (46), (49), (50), (53)–(57), we conclude for $t \in [\tau_i, \tau_{i+1}]$ that

$$\varepsilon_2(t) \leq \varepsilon_2(\tau_i)(1 + C_{14}\delta)$$
$$+ C_{15}(h^{1/2} + \delta^{1/2})\delta \quad (58)$$
$$+ C_{16}\nu(h, \delta(h), \alpha(h))\delta.$$

Using Lemma 3, we have for $i \in \{0, \ldots, m\}$

$$\varepsilon_2(\tau_i) \leq \varepsilon_2(0) + C_{17}(h^{1/2}$$
$$+ \delta^{1/2} + \nu(h, \delta(h), \alpha(h))).$$

Then we get for $t \in [0, \vartheta]$

$$\varepsilon_2(t) \leq C_{18}(h^{1/2} + \delta^{1/2} + \nu(h, \delta(h), \alpha(h))). \quad (59)$$

The validity of the theorem follows from (45) and (59).
∎

Note that proving Theorem 2 we actually prove the convergence of the algorithm and not (time) convergence of the estimates.

Concluding, we describe the algorithm for the problem under consideration. Thus, we have the system (2) with the control $u = \{\mu, I\}$ and the system (3) with the unknown control $u_* = \{\mu_*, I_*\}$. We choose a family $\Delta_h = \{\tau_{i,h}\}_{i=0}^{m_h}$ of partitions of the interval $[0, \vartheta]$ with a step $\delta(h) = \tau_{i+1,h} - \tau_{i,h}$ and a function $\alpha(h)$: $(0, 1) \rightarrow (0, 1)$ depending on the parameter h. The family Δ_h and the function $\alpha(h)$ satisfy Condition 2. Before the algorithm starts, we fix some value of measurement accuracy h, a partition $\Delta = \Delta_h$ and a number $\alpha = \alpha(h)$. The work of the algorithm is decomposed into $m - 1$, $m = m_h$, identical steps. At the i-th step carried out during the time interval $\delta_i = [\tau_i, \tau_{i+1})$, $\tau_i = \tau_{i,h}$, the following actions are made. First, at the moment τ_i, using the state $w^h(\tau_i)$ of the model (34), the result ξ_i^h (satisfying the inequality (5)) of calculating the state of the system (2), we determine three numbers, namely, v_i^h and $u_i^h = \{\mu^h(\tau_i), I^h(\tau_i)\}$, by the formulas (35)–(37). Then, during the time interval δ_i, the constant control $v^h(t) = v_i^h$ is fed to the input of the model (34) and the constant control $u^h(t) = u_i^h$ is fed to the system (2), respectively. After these operations, at the moment τ_{i+1} the model state is recalculated (instead of the number $w^h(\tau_i)$, the number $w^h(\tau_{i+1}) = w^h(\tau_{i+1}; w^h(\tau_i), v_i^h)$ is found; in addition, the vector ξ_{i+1}^h is determined). Analogous actions are performed till the moment $\tau_{m_h-1,h}$.

As follows from Theorem 2, if the fixed measurement accuracy h is sufficiently small, then the algorithm described above for forming the control $u(\cdot)$ in the system (2) provides the "tracking" (in uniform metric) of the solution $x_*(\cdot)$ of the system (3) by the

solution $x^h(\cdot)$ of the system (2). Thus, the algorithm solves the problem of robust control.

Remark 1. Let us briefly dwell on the reasons that made us introduce Conditions 1–3. The appearance of Condition 1 is caused by the two factors: first, the "physical" (Nordhaus, 1994, Chapt. 2) sense of function $F(t)$ containing as a summand the value $4.1 \log_2(1 + M_1(t)/590)$ and, second, the necessity of boundedness in $L_\infty([0, \vartheta]; \mathbb{R})$ of the function $\dot{\tilde{u}}(\cdot)$ (see the proof of Lemma 2). Condition 2 is necessary to prove Theorem 2. Only under this condition does the right-hand side of the inequality (59) tend to zero as $h \to 0$. Condition 3 (namely, constants $C^{(1)}$ and $C^{(2)}$ from this condition) is used to obtain inequality (52).

4. Results of computer modeling

The algorithm described above was tested on a computer. In Figs. 2–5, the results of computer modeling are presented for the following case:

$$
\begin{aligned}
& c_1 = c_2 = c_3 = 1, && \sigma = 1 + 0.5t, \\
& c_4 = 0.5, && O(t) = 5 \sin t, \\
& \delta_K = 0.65, && L(t) = 1, \\
& a_1 = 30, && a_2 = 60, \\
& \delta_M = 0.0833, && \mu_*(t) = 1 + 0.5t, \\
& \beta = 0.1, && I_*(t) = 1 + 0.15t^2, \\
& \gamma = 0.1, && A(t) = 2t^{1/2}.
\end{aligned}
$$

The parameters are as follows:

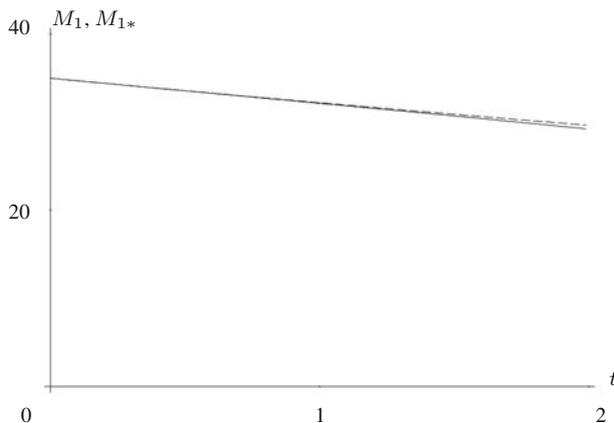$$ f_- = 1, \quad f_+ = 10, \quad I_- = 1, \quad I_+ = 2. $$

Fig. 2. Trajectories $M_1(t)$ (dashed line) and $M_{1*}(t)$ (solid line) in the case of $\delta = 0.0001$, $h = 0.001$.
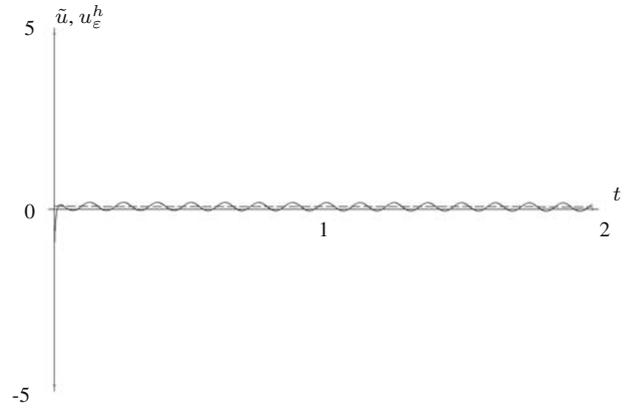

Fig. 3. Function $\tilde{u}(t)$ (solid line) and control $u_\varepsilon^h(t)$ (dashed line) in the case of $\delta = 0.0001$, $h = 0.001$.

The initial conditions for the system are the following:

$$
\begin{aligned}
& T_0(0) = 1, && T_1(0) = 0.5, \\
& M_1(0) = 35, && K(0) = 5.
\end{aligned}
$$

Equations (2) and (3) were solved by the Euler method with step δ. During the experiment, we assume

$$
\begin{aligned}
& \xi_{1i}^h = T_0(\tau_i) + h \sin(50t), \\
& \xi_{2i}^h = T_1(\tau_i) + h \sin(50t), \\
& \xi_{3i}^h = K(\tau_i) + h \sin(50t), \quad \alpha = 0.1.
\end{aligned}
$$

In Figs. 2 and 4, trajectories $M_1(t)$ (dashed line) and $M_{1*}(t)$ (solid line) are presented. In Figs. 3 and 5, control $u_\varepsilon^h(t)$ (dashed line) and function

$$ \tilde{u}(t) = \log_2\left(1 + \frac{M_1(t)}{590}\right) $$

(solid line) are presented. Figures 2 and 3 correspond to the case of $\delta = 0.0001$, $h = 0.001$; Figs. 4 and 5 correspond to the case of $\delta = 0.01$, $h = 0.5$.

5. Conclusions

A control problem for a fourth order system was considered. The problem consists in constructing an algorithm of stable formation of control characteristics providing a given quality of the process under conditions of measuring a part of the phase coordinates and the action of uncontrolled disturbances. On the basis of the feedback control method, a solving algorithm (the reconstruction-control controller) is designed. The algorithm consists of two blocks: the block of online reconstruction and the block of positional control.
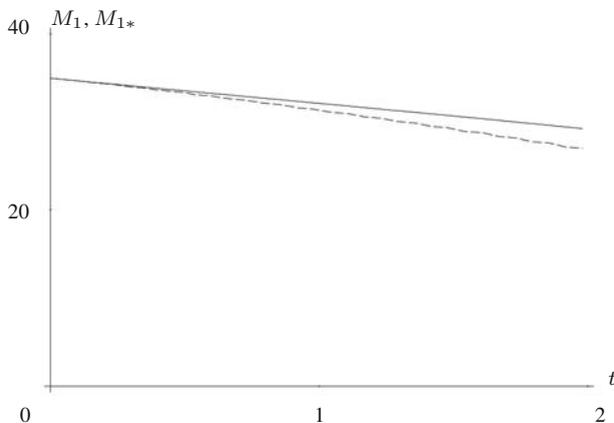
Acknowledgment

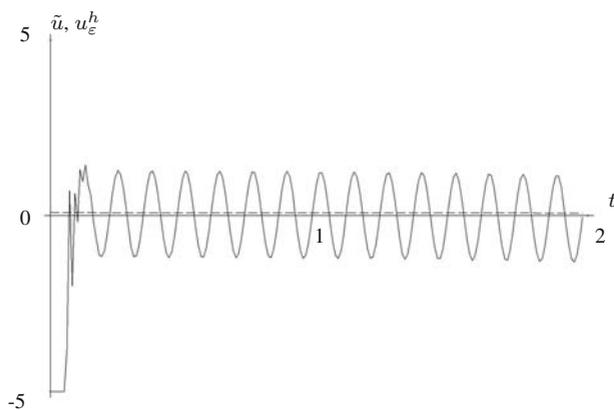Fig. 4. Trajectories $M_1(t)$ (dashed line) and $M_{1*}(t)$ (solid line) in the case of $\delta = 0.01$, $h = 0.5$.

Fig. 5. Function $\tilde{u}(t)$ (solid line) and control $u_\varepsilon^h(t)$ (dashed line) in the case of $\delta = 0.01$, $h = 0.5$.

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