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FAULT DIAGNOSIS IN NONLINEAR HYBRID SYSTEMS

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The problem of fault diagnosis in hybrid systems is investigated. It is assumed that the hybrid systems under consideration consist of a finite automaton, a set of nonlinear difference equations and the so-called mode activator that coordinates the action of the other two parts. To solve the fault diagnosis problem, hybrid residual generators based on both diagnostic observers and parity relations are used. It is shown that the hybrid nature of the system imposes some restrictions on the possibility of creating such generators. Sufficient solvability conditions of the fault diagnosis problem are found. Examples illustrate details of the solution.

Keywords: hybrid systems, finite automata, mode activator, fault diagnosis, nonparametric method.

1. Introduction

Hybrid systems (HSs), considered in this paper, are dynamic systems, whose behavior is determined by interaction of a number of discrete-time systems (DTSs) described by difference equations and a finite automaton (FA) with discrete-event dynamics. The DTS depends on parameters that take values from a set of real numbers, and the values are determined by the outputs of the FA. A switching occurs whenever states of the DTS reach some given domains, defined through a set of inequalities. This type of HS has been studied earlier (Cocquempot *et al.*, 2004; Gruyitch, 2007; Yang *et al.*, 2010; Leth and Wisniewski, 2014), also in the context of fault diagnosis (Shumsky and Zhirabok, 2012; Shumsky *et al.*, 2012).

There exist numerous descriptions of hybrid systems and switching rules, including time- and state-dependent switching schemes. Our choice is motivated mostly by three aspects. First, this description is general enough and accommodates some other descriptions based on state-dependent switching. Second, actual complex systems have two main parts: one is a control system that has a finite number of states, and the other is an operational system with discrete dynamics. Our model is well suited to describe such a class of actual systems. Third, the algebraic tools we will apply (algebra of partitions/functions) work both for DTSs and FAs. The algebra of partitions was developed for FAs by Hartmanis and Stearns (1966). A similar approach to DTSs, called the algebra of functions (Zhirabok and Shumsky, 2008), was inspired by the approach of Hartmanis and Stearns (1966) and mimics the latter. Since the two theories are interlinked, this helps us to build a bridge between the theories of FAs and DTSs. The partitions can be replaced by functions generating them, and analogous operations/operators are introduced. These aspects are advantages of the suggested approach over the existing methods.

Numerous methods have been elaborated for fault detection and isolation (FDI) in dynamic systems within the scope of the analytical redundancy concept. According to this concept, FDI is based on checking relations that exist among system inputs and outputs

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measured over a finite time window. The FDI process includes residual generation as a result of a mismatch between the system behavior and its reference model behavior, followed by decision making through evaluation of the residual. This paper concentrates only on the stage of residual generation.

There are two basic approaches to residual generation. One is based on closed-loop techniques (diagnostic observers, Kalman filters) (Alcorta and Frank, 1997; Shumsky and Zhirabok, 2006; Li *et al.*, 2016), while the other involves open-loop techniques (redundancy or parity relations) (Patton *et al.*, 2000; Blanke *et al.*, 2006). In this paper, both approaches are considered.

In the framework of the latter technique, there exists a promising method of fault diagnosis in technical systems known as a data-driven, or model-free, or nonparametric method (see, e.g., Ding, 2014). A notable feature of this method is that parameters of the system under consideration may be unknown.

Hybrid residual generators constructed in this paper have dynamic and finite automaton parts. The former coincides with its counterpart in the ordinary residual generators except for parameters, whose values are changed under time- or state-dependent switchings. The latter provides the main fault diagnosis properties such as sensitivity and insensitivity to faults, reliability, robustness with respect to disturbances, model errors, and measurement noise. These properties are the same as in the ordinary residual generators; they are well-investigated in numerous papers and books (see, e.g., Gertler, 1998; Blanke *et al.*, 2006; Ding, 2014; Witczak, 2014; Zhirabok *et al.*, 2017), and therefore are not considered in the present paper.

Note that the finite automaton part of hybrid systems imposes some restrictions on the possibility of creating the dynamic part of a residual generator when states of the FA part (hybrid system modes) are immediately unobservable. This aspect is not sufficiently studied in the literature. The present paper accents studying these restrictions; its main purpose is to find sufficient solvability conditions of the fault diagnosis problem for hybrid systems for the case when their modes are immediately unobservable. This is the main contribution and novelty of the present paper.

Various aspects of fault diagnosis in hybrid systems were studied in the literature: structural diagnosability (Pröll *et al.*, 2015), active diagnosis (Tabatabaeipour *et al.*, 2009), fault estimation (Laboudi *et al.*, 2015), diagnosis in linear hybrid systems (Farhat and Koenig, 2017; Zhao *et al.*, 2015), fault-tolerant control (Cocquempot *et al.*, 2004; Yang *et al.*, 2010), disturbance decoupling (Zattoni, 2018).

In this paper, both diagnostic observers and parity relations in nonparametric form are used for fault diagnosis in nonlinear HSs. In comparison with the earlier works (Shumsky and Zhirabok, 2012; Shumsky *et al.*, 2012), where HSs with continuous-time systems are studied, here we consider an HS with discrete-time systems that allows using a unified mathematical technique to analyze the DTS and the FA. Besides, additional possibilities to construct the diagnostic HS are used, which allows to us extend a class of systems for which the FDI problem can be solved.

The rest of the paper is organized as follows. In Section 2, the HS is described and the problem statement is formulated. Section 3 recalls some facts from the algebra of partitions/functions, important to prove the results of this paper. In Section 4, residual generators are constructed. Section 5 is devoted to the hybrid part of diagnostic system design. A practical example is considered in Section 6. Section 7 concludes the paper.

2. Basic models and problem statement

The HS is represented schematically in Fig. 1. In this figure, there are three basic elements: the FA, the DTS (actually a set of them, one for each mode), and the so-called mode activator (MA), which coordinates the actions of the FA and the DTS. The FA is described by the model $A = (I, S, O, \delta, \lambda)$, where I, S and O are finite sets of inputs, states, and outputs, while δ and λ are state transition and output functions, respectively, described by

$$s^+ = \delta(s, i), \qquad o = \lambda(s), \tag{1}$$

where $s^+ \in S$ is the new state after transition from the state $s \in S$, initiated by the input $i \in I$, and $o \in O$ is the output. It is assumed that both functions δ and λ are specified by appropriate tables. The automaton A is assumed to be minimal (irreducible), i.e., the number of its states cannot be decreased.

The DTS is described by a set of nonlinear difference equations of the form

$$\begin{aligned} x(t+1) &= f_o(x(t), u(t)), \quad o \in O, \\ y(t) &= h(x(t)), \end{aligned} \tag{2}$$

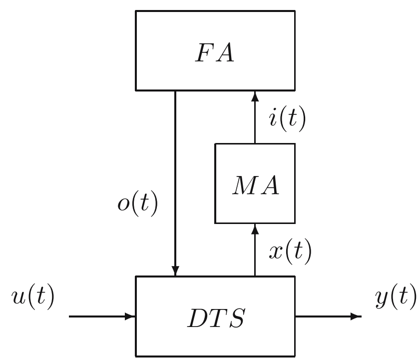

Fig. 1. Structure of a hybrid system.

where $x \in X \subseteq \mathbb{R}^n$, $u \in U \subseteq \mathbb{R}^m$, $y \in Y \subseteq \mathbb{R}^l$ are states, controls, and measured outputs, respectively. The function f_o in (2) depends on the system mode, activated by the output $o \in O$ of the FA.

It is assumed that the functions f_o and h contain parameters from the set $C = \{a_1, \ldots, a_p\}$, whose values depend on o. This dependence is described by the function specified by an appropriate table. It is assumed that some parameters from C do not depend on O they reflect faults: if there are no faults, $a_j = a_{j0}$, where a_{j0} is the nominal value of the parameter a_j, but if the j-th fault occurs, a_j becomes an unknown function of time.

The mode activator is described by the function β as

$$i(t) = \beta(x(t)). \tag{3}$$

The hybrid diagnostic system we are looking for has a similar hybrid structure and is shown in Fig. 2. The notation corresponds to that of the original HS in Fig. 1; just a letter "D" (from the word "diagnosis") is added. The description of the FAD subsystem is also similar, except that we use the subindex "*" everywhere:

$$s_*^+ = \delta_*(s_*, i_*), \quad o_* = \lambda_*(s_*). \tag{4}$$

The MAD is given by

$$i_*(t) = \beta_*(y(t), x_*(t)), \tag{5}$$

where β_* is a function specified below, $x_* \in \mathbb{R}^{n_*}$ is the state of the residual generator (RG) described in Subsections 4.1 and 4.2, $n_* \le n$.

Use of the state $x_*(t)$ in (5) may cause the mode difference when the mode of the HS under diagnosis and that of the RG may be asynchronous. However, if the RG is disturbance decoupled, this problem does not arise. If not, $x_*(t)$ should be removed from (5).

In Section 5.4 we show that use of $x_*(t)$ in (5) allows us to extend the class of hybrid systems for which the FDI problem can be solved. Nevertheless, the asynchronous problem exists, and it will be considered in future works.

Problem statement. Find an automaton FAD, described by (4), under the restrictions imposed by the MAD (5), and a bank of residual generators to solve the problem of fault isolation based on the structural residual vector and the matrix of syndromes (Gertler, 1998).

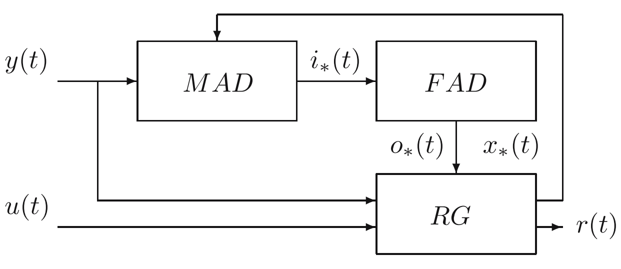

Fig. 2. Structure of a hybrid diagnostic system.

3. Algebra of partitions/functions

3.1. General definitions and properties. We briefly recall the definitions and concepts of the algebra of partitions/functions from the works of Hartmanis and Stearns (1966), Zhirabok and Shumsky (2008), Shumsky and Zhirabok (2010) as well as Kaldmäe *et al.* (2013) used to solve the problem. Note that the algebra of partitions operates with partitions of some set S and the algebra of functions operates with vector functions, defined on the set S. Let F_S be a set of vector functions with the domain S.

Definition 1. (*Relation of partial preorder*) Given $\alpha, \beta \in F_S$, the notation $\alpha \le \beta$ means that there exists a function γ such that $\beta(s) = \gamma(\alpha(s))$ for $\forall s \in S$.

Definition 2. (*Equivalence*) If $\alpha \le \beta$ and $\beta \le \alpha$, then α and β are called equivalent, which is denoted by $\alpha \cong \beta$.

Note that $\cong$ is an equivalence relation. It divides the set F_S into the equivalence classes containing equivalent functions. Denote by F_E the set of all these equivalence classes; then the relation $\le$ is a partial order on this set and the pair $(F_E, \le)$ is a lattice. There exist two special vector functions **0** and **1**, such that for every function α, $\mathbf{0} \le \alpha \le \mathbf{1}$. The function **0** is equivalent to the identity function, and **1** to a constant function.

Since the pair $(F_E, \le)$ is a lattice, two binary operations $\times$ and $\oplus$ exist defined by

$$\alpha \times \beta = \inf(\alpha, \beta), \quad \alpha \oplus \beta = \sup(\alpha, \beta).$$

In simple cases, the second expression may be used to compute $\alpha \oplus \beta$. The rule for the operation $\times$ is simple:

$$(\alpha \times \beta)(s) = \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix}.$$

Example 1. Let $F_S = \mathbb{R}^3$, $\alpha(s) = (s_1+s_2, s_3)^T$, $\beta(s) = (s_1 s_3, s_2 s_3)^T$. Then $(\alpha \times \beta)(s) \cong (s_1 + s_2, s_3, s_1 s_3)^T$ and $(\alpha \oplus \beta)(s) = s_3(s_1 + s_2)$. ♦

Definition 3. (*Binary relation* Δ) Given $\alpha, \beta \in F_X$, $(\alpha, \beta) \in \Delta$ we mean that there exists a function f_* such that $\beta(f(x, u)) = f_*(\alpha(x), u)$ for all $(x, u) \in X \times U$. Here f is a function f_o for any $o \in O$.

The binary relation Δ is used for the definition of the operators $\mathbf{m}$ and $\mathbf{M}$.

Definition 4. The operator $\mathbf{m}(\alpha)$ is a function in F_X that satisfies the following conditions:
(i) $(\alpha, \mathbf{m}(\alpha)) \in \Delta$,
(ii) if $(\alpha, \beta) \in \Delta$, then $\mathbf{m}(\alpha) \le \beta$.

Definition 5. The operator $\mathbf{M}(\beta)$ is a function in F_X that satisfies the following conditions:
(i) $(\mathbf{M}(\beta), \beta) \in \Delta$,
(ii) if $(\alpha, \beta) \in \Delta$, then $\alpha \le \mathbf{M}(\beta)$.

The main properties of the operations and operators are as follows (Zhirabok and Shumsky, 2008).

Lemma 1. *Let α and β be some functions. Then*
1. $\alpha \le \beta \Rightarrow \boldsymbol{M}(\alpha) \le \boldsymbol{M}(\beta)$; $\alpha \le \beta \Rightarrow \boldsymbol{m}(\alpha) \le \boldsymbol{m}(\beta)$;
2. $\boldsymbol{M}(\boldsymbol{m}(\alpha)) \ge \alpha$, $\boldsymbol{m}(\boldsymbol{M}(\beta)) \le \beta$;
3. $\alpha \le \boldsymbol{M}(\beta) \Rightarrow (\alpha, \beta) \in \Delta$; $\boldsymbol{m}(\alpha) \le \beta \Rightarrow (\alpha, \beta) \in \Delta$;
4. $\boldsymbol{M}(\alpha \times \beta) = \boldsymbol{M}(\alpha) \times \boldsymbol{M}(\beta)$.

To evaluate the operations and operators, webMathematica based software has been developed so that anyone can use it with only an internet browser (`http://webmathematica.cc.ioc.ee/mathematica/NLControl/main/index.html`).

The set of vector functions with the domain S and the set of partitions of S are closely linked. A vector function α induces a partition π_α as follows:

$$\forall s, s' \in S \quad \alpha(s) = \alpha(s') \Leftrightarrow s \equiv s'(\pi_\alpha). \tag{6}$$

When π_α and π_β are some partitions of S, one says that $\pi_\alpha \le \pi_\beta$ iff $s \equiv s'(\pi_\alpha) \Rightarrow s \equiv s'(\pi_\beta)$. The notation $s \equiv s'(\pi_\alpha)$ means that s and s' are in the same block of the partition π_α. If $\pi_\alpha \le \pi_\beta$ and $\pi_\beta \le \pi_\alpha$, then $\pi_\alpha = \pi_\beta$.

On the analogy of functions, one may define operations $\times$ and $\oplus$, a binary relation Δ, and operators **m** and **M**.

Example 2. Let $S = \{1, 2, 3, 4, 5, 6\}$, $\pi_\alpha = \{(1, 2), (3, 4), (5, 6)\}$, and $\pi_\beta = \{(1, 3), (2, 4), (5, 6)\}$. Then $\pi_\alpha \times \pi_\beta = \{(1), (2), (3), (4), (5, 6)\}$ and $\pi_\alpha \oplus \pi_\beta = \{(1, 2, 3, 4), (5, 6)\}$. Clearly, $\pi_\alpha \times \pi_\beta \le \pi_\alpha \le \pi_\alpha \oplus \pi_\beta$. ♦

Other examples of evaluating the operations $\times$ and $\oplus$ are given in Section 5.

We also introduce two additional operators $\mathbf{m}_I$ and $\mathbf{M}_I$ as follows. The operator $\mathbf{m}_I : F_I \to F_S$ yields a minimal function satisfying for $\chi \in F_I$ the condition

$$\chi(i) = \chi(i') \Rightarrow \mathbf{m}_I(\chi)(\delta(s, i)) = \mathbf{m}_I(\chi)(\delta(s, i')) \tag{7}$$

for all $i, i' \in I$ and $s \in S$. The operator $\mathbf{M}_I : F_S \to F_I$ yields a maximal function satisfying the condition

$$\mathbf{M}_I(\beta)(i) = \mathbf{M}_I(\beta)(i') \Rightarrow \beta(\delta(s, i)) = \beta(\delta(s, i')) \tag{8}$$

for all $i, i' \in I$ and $s \in S$.

The operator $\mathbf{m}_I$ is similar to **m** except that its domain is F_I. Analogously, the operator $\mathbf{M}_I$ is similar to **M** except that it defines the function with the domain I.

Definition 6. Given $\alpha \in F_X$, we say that α is (h, f)-invariant if $(\alpha \times h, \alpha) \in \Delta$, or $\alpha \times h \le \mathbf{M}(\alpha)$, or $\mathbf{m}(\alpha \times h) \le \alpha$.

In the case of smooth functions, Definition 6 is a generalization of the concept of the (h, f)-invariant distribution (or codistribution) (Isidori, 1995).

If $h = \mathbf{1}$ in Definition 6, we say that α is an f-invariant function. By analogy, we may talk about a δ-invariant function $\xi \in F_S$ defined for the FA (1).

Lemma 2. *If α and β are (h, f)-invariant, so is $\alpha \times \beta$.*

Proof. Let α and β be (h, f)-invariant. Then $\alpha \times h \le \mathbf{M}(\alpha)$ and $\beta \times h \le \mathbf{M}(\beta)$. Multiply the respective right- and left-hand sides of these inequalities: $\alpha \times h \times \beta \times h \cong \alpha \times \beta \times h \le \mathbf{M}(\alpha) \times \mathbf{M}(\beta)$. By Lemma 1, we obtain $(\alpha \times \beta) \times h \le \mathbf{M}(\alpha \times \beta)$. The last inequality means that $\alpha \times \beta$ is (h, f)-invariant. ■

3.2. Computation of the operators $\mathbf{m}$, $\mathbf{M}$, $\mathbf{m}_I$, and $\mathbf{M}_I$.

It is known that a function γ exists that satisfies the condition $(\alpha \times u) \oplus f \cong \gamma(f)$; define $\mathbf{m}(\alpha) \cong \gamma$ (see Zhirabok and Shumsky, 2008). Examples of how to compute γ are given by Kaldmäe *et al.* (2013).

For evaluation of the operator **M**, see the work of Zhirabok and Shumsky (2008), though this rule is not used in the paper.

In the case of partitions, the operators **m** and **M** can be evaluated by

$$\mathbf{m}(\pi) = \prod_{\{j\}} \sigma_j, \quad (\pi, \sigma_j) \in \Delta,$$
$$\mathbf{M}(\sigma) = \sum_{\{j\}} \pi_j, \quad (\pi_j, \sigma) \in \Delta,$$

respectively (Hartmanis and Stearns, 1966). Note that the symbol $\prod$ corresponds to applying $\times$, and the symbol $\sum$ corresponds to applying $\oplus$.

To calculate $\mathbf{m}_I$ for partitions, use the following rule. Denote by Ω_τ the set of all partitions σ such that

$$i \equiv i'(\tau) \Rightarrow \delta(s, i) \equiv \delta(s, i')(\sigma)$$

for all $i, i' \in I$ and $s \in S$. Then

$$\mathbf{m}_I(\tau) = \prod_{\sigma \in \Omega_\tau} \sigma.$$

To calculate $\mathbf{M}_I$, denote by Ω_σ the set of all partitions τ such that

$$i \equiv i'(\tau) \Rightarrow \delta(s, i) \equiv \delta(s, i')(\sigma)$$

for all $i, i' \in I$ and $s \in S$. Then

$$\mathbf{M}_I(\sigma) = \sum_{\tau \in \Omega_\sigma} \tau.$$

Examples for calculations involving the operator **m** are given in Example 3 and Section 5.

4. Residual generator design

To generate residuals, in this section we use both diagnostic observers and parity relations. First, consider an observer based generator. All the results of this section should be understood as given for a fixed (single) mode. This holds for the main relations of the section.

4.1. Observer based RG design. To solve the problem of fault isolation, introduce for the parameter a_j a vector function $\alpha^{(j)}$ with the maximal number of components such that

$$\frac{\partial}{\partial a_j}\alpha^{(j)}(f(x,u)) = 0.$$

Consider the system (2) for some $o \in O$ and introduce the coordinate transformation

$$x_*(t) = \varphi(x(t)), \quad y_*(t) = \phi(y(t)), \quad t \geq 0, \tag{9}$$

such that the transformed system can be used as a diagnostic observer and is described by

$$\begin{aligned} x_*(t+1) &= f_*(x_*(t), y(t), u(t)) + Kr(t), \\ y_*(t) &= h_*(x_*(t)), \end{aligned} \tag{10}$$

where f_* and h_* are some functions, K is the gain matrix, $r(t)$ is a residual generated as follows:

$$r(t) = \phi(y(t)) - y_*(t).$$

Since the problem of choosing the matrix K has been studied in the literature (see, e.g., Alcorta and Frank, 1997; Schreier *et al.*, 1997), it is not considered in this paper.

It is known (Zhirabok and Shumsky, 2008) that the model obtained via the state transformation does not depend on the parameter a_j if

$$\alpha^{(j)} \leq \varphi. \tag{11}$$

From (9) and (10) it follows that the functions φ and ϕ satisfy

$$\begin{aligned} \varphi(f(x,u)) &= f_*(\varphi(x), h(x), u), \\ \phi(h(x)) &= h_*(\varphi(x)). \end{aligned}$$

By the definitions of the relation Δ and the operation $\oplus$, we obtain from these equations $(\varphi \times h, \varphi) \in \Delta$ and $h \oplus \varphi \neq \mathbf{1}$. Thus, φ is an (h, f)-invariant function satisfying the condition $h \oplus \varphi \neq \mathbf{1}$.

Given $\gamma^0 := \alpha^{(j)}$, compute recursively for $c \geq 1$, using the formula

$$\gamma^{c+1} = \gamma^c \oplus \mathbf{m}(\gamma^c \times h), \tag{12}$$

the sequence of non-decreasing vector functions $\gamma^0 \leq \gamma^1 \leq \ldots$. There exists a finite c such that $\gamma^c \ncong \gamma^{c-1}$ but $\gamma^c \cong \gamma^{c+d}$ for all $d \geq 1$. Define $\varphi := \gamma^c$.

Theorem 1. (Zhirabok and Shumsky, 2008) *The formula (12) yields a minimal (h, f)-invariant vector function φ satisfying the condition $\alpha^{(j)} \leq \varphi$.*

The minimality of the function φ gives the best choice to satisfy the condition $h \oplus \varphi \neq \mathbf{1}$. If $h \oplus \varphi = \mathbf{1}$, the model (10) independent of the parameter a_j does not exist.

If $h \oplus \varphi \neq \mathbf{1}$, we obtain the functions ϕ and h_* from the functional equation $\phi(h) = h \oplus \varphi = h_*(\varphi)$. The function f_* can be constructed as follows. Write down

$$x_*(t+1) = \varphi(x(t+1)) = \varphi(f(x(t), u(t))$$

and on the right-hand side of the last expression replace the variable x by x_* and y; this is possible since φ is an (h, f)-invariant function.

4.2. RG design based on parity relations. Consider the system (2) for some $o \in O$ and introduce the coordinate transformation

$$\begin{aligned} x_{*j}(t) &= \varphi^{(j)}(x(t)), \quad j = 1, \ldots, k, \\ y_*(t) &= \phi(y(t)), \quad t \geq 0, \end{aligned} \tag{13}$$

such that the transformed system is feedback free:

$$\begin{aligned} x_{*1}(t+1) &= f_*^1(y(t), u(t)), \\ x_{*j}(t+1) &= f_*^j(x_{*j-1}(t), y(t), u(t)), \\ &\qquad j = 2, \ldots, k, \\ y_*(t) &= h_*(x_{*k}(t)), \end{aligned} \tag{14}$$

where f_*^j, $j = 1, \ldots, k$, and h_* are some functions, k is the dimension of the transformed system. In general, the feedback free system is of the form $x_{*j}(t+1) = f_*^j(x_{*j-1}(t), \ldots, x_{*1}(t), y(t), u(t))$; we consider (14) for simplicity.

Note that (14) is a special case of (10) and $\varphi = \varphi^{(1)} \times \cdots \times \varphi^{(k)}$ is an (h, f)-invariant function. Perform in (14) several temporal shifts and substitutions:

$$\begin{aligned} x_{*2}(t+2) &= f_*^2(f_*^1(y(t), u(t)), y(t+1), u(t+1)) \\ &= f_*^{21}(y(t+1), y(t), u(t+1), u(t)), \\ &\;\;\vdots \\ x_{*k}(t+k) &= F_*(y(t+k-1), \ldots, y(t), \\ &\qquad u(t+k-1)), \ldots, u(t)) \end{aligned}$$

for some functions f_*^{21} and F_*. Taking into account (14), we get

$$\begin{aligned} y_*(t+k) &= F_{**}(y(t+k-1), \ldots, y(t), \\ &\qquad u(t+k-1)), \ldots, u(t)), \end{aligned} \tag{15}$$

where $F_{**} = h_*(F_*)$.

Assume for simplicity that the functions f_o and h in (2) are described by polynomials. Assuming that the functions $\varphi^{(1)}, \dots, \varphi^{(k)}$, and ϕ are polynomials too, we conclude that the functions $f_*^1, \dots, f_*^k$, and h_* are polynomials as well. As a result, the right-hand side of (15) can be written in the form

$$\begin{aligned} y_*(t+k) &= F_{**}(y(t+k-1), \dots, y(t), \\ &\qquad u(t+k-1), \dots, u(t)) \\ &= (\Gamma_1(a)\ \Gamma_2(a)\ \dots\ \Gamma_q(a)) \\ &\quad \times \begin{pmatrix} P_1^*(t+k-1, \dots, t) \\ P_2^*(t+k-1, \dots, t) \\ \vdots \\ P_q^*(t+k-1, \dots, t) \end{pmatrix}, \end{aligned}$$

where $(\Gamma_1(a)\ \dots\ \Gamma_q(a)) =: \Gamma(a)$ is a row of algebraic expressions which are functions of parameters from the set $C = \{a_1, \dots, a_p\}$; $P_j^*(t+k-1, \dots, t) := P_j(y(t+k-1), u(t+k-1), \dots, y(t), u(t))$; $P_j(y(t+k-1), u(t+k-1), \dots, y(t), u(t))$, $j = 1, \dots, q$, is the polynomial formed by the functions f_o and h at instants of time from t till $t+k-1$.

Write down the expression for y_* for T instants of time:

$$\begin{aligned} Y_T(t) &= (y_*(t)\ y_*(t-1)\ \dots\ y_*(t-T+1)) \\ &= \Gamma(a) P_T(t), \end{aligned} \tag{16}$$

where

$$P_T(t) = \begin{pmatrix} P_1^*(t-1, \dots, t-k) & \dots & \dots & P_1^*(t-T, \dots, t-k-T+1) \\ P_2^*(t-1, \dots, t-k) & \dots & \dots & P_2^*(t-T, \dots, t-k-T+1) \\ \dots & \dots & \dots & \\ P_q^*(t-1, \dots, t-k) & \dots & \dots & P_q^*(t-T, \dots, t-k-T+1) \end{pmatrix}.$$

4.3. Residual generation. To generate the residual for decision making about faults, use the method based on the matrix $P_T(t)$ kernel. In the works of Ding (2014) and Zhirabok *et al.* (2017), the size of the temporal window T is minimal such that $\mathrm{rank}(P_T(t)) = \mathrm{rank}(P_{T-1}(t))$. This means that the last column of the matrix $P_T(t)$ linearly depends on other columns, i.e., vector $v(T)$ exists such that $P_T(t)v(T) = 0$ and $P_T(t)$ has a nonempty kernel. From (16) it follows that $Y_T(t)v(T) = 0$, so the rule

$$r_T(t) = Y_T(t)v(T), \quad v(T) \in ker(P_T(t))$$

is robust since it is independent of the values of the system parameters.

It is well known that calculation of the ranks of matrices formed on the basis of experimental data is an ill-conditioned problem. To overcome this difficulty and reduce computational complexity, one may take T such that the number of columns of the matrix $P_T(t)$ is bigger than that of its rows, i.e., $T \geq q+1$; here the equality $\mathrm{rank}(P_T(t)) = \mathrm{rank}(P_{T-1}(t))$ is valid. In some specific cases, the value of T may be reduced.

To isolate faults, we construct RGs based on a bank of observers or parity relations based. Each RG has to be invariant with respect to some group of faults and sensitive to other faults. To make a decision, the matrix of symptoms is used (Gertler, 1998), where rows correspond to residuals, and columns to faults.

4.4. Condition of parity relation design. It is known (Zhirabok and Shumsky, 2008) that the model obtained via the state transformation $\varphi = \varphi^{(1)} \times \varphi^{(2)} \times \dots \times \varphi^{(k)}$ does not depend on the parameter a_j if (11) holds.

Theorem 2. *The system (2) can be transformed into the model (14) independent of the parameter a_j if and only if*

$$\alpha^{(j)} \oplus \boldsymbol{m}(h) \neq \boldsymbol{1}, \tag{17}$$

$$\begin{aligned} (\alpha^{(j)} \oplus h) \oplus \boldsymbol{m}(h \times (\alpha^{(j)} \oplus \boldsymbol{m}(h \times \dots \\ \times (\alpha^{(j)} \oplus \boldsymbol{m}(h)) \dots))) \neq \boldsymbol{1} \end{aligned} \tag{18}$$

(the operator **m** *is used k times).*

Proof.
(*Necessity*). Assume that the system (1) can be transformed into the model (14) independent of a_j. This means that there exist functions $\varphi^{(1)}, \dots, \varphi^{(k)}$, and ϕ such that $\alpha^{(j)} \leq \varphi = \varphi^{(1)} \times \dots \times \varphi^{(k)}$ and $y_* = \phi(y)$. The last condition yields $\alpha^{(j)} \leq \varphi^{(c)}$ for all c. Consider the first equation in (14) and replace $x_{*1}(t+1)$ with $\varphi^{(1)}(x(t+1))$ and $y(t)$ with $h(x(t))$: $\varphi^{(1)}(x(t+1)) = \varphi^{(1)}(f(x(t), y(t))) = f_*^1(h(x(t)), u(t))$. By the definitions of relation Δ and operator **m**, this implies $(h, \varphi^{(1)}) \in \Delta$ and $\varphi^{(1)} \geq \mathbf{m}(h)$. By analogy, the second equation in (14) yields $(h \times \varphi^{(c-1)}, \varphi^{(c)}) \in \Delta$, $c = 2, \dots, k$. The property of the operator **m** (Lemma 1) and the inequality $\alpha^{(j)} \leq \varphi^{(c)}$ yield $\varphi^{(c)} \geq \alpha^{(j)} \oplus \mathbf{m}(h \times \varphi^{(c-1)})$, $c = 2, \dots, k$, and $\varphi^{(1)} \geq \alpha^{(j)} \oplus \mathbf{m}(h)$, respectively. The last inequality implies (17).

Consider for simplicity $k = 3$. Then

$$\begin{aligned} \varphi^{(2)} &\geq \alpha^{(j)} \oplus \mathbf{m}(h \times \varphi^{(1)}) \\ &\geq \alpha^{(j)} \oplus \mathbf{m}(h \times (\alpha^{(j)} \oplus \mathbf{m}(h))), \\ \varphi^{(3)} &\geq \alpha^{(i)} \oplus \mathbf{m}(h \times \varphi^{(2)}) \\ &\geq \alpha^{(j)} \oplus \mathbf{m}(h \times (\alpha^{(j)} \oplus \mathbf{m}(h \times (\alpha^{(j)} \oplus \mathbf{m}(h))))). \end{aligned} \tag{19}$$

Since $y_* = h_*(x_{*k})$ and $y_* = \phi(y)$, we have $(\phi(h))(x) = (h_*(\varphi^{(k)}))(x)$. This means that $h \oplus \varphi^{(k)} \neq \mathbf{1}$, and taking into account (19), we obtain (18).

(*Sufficiency*). From (18) it follows that there exist functions γ_1, γ_2, and $\varphi^{(k)}$ such that

$$\gamma_1(\mathbf{m}(h \times (\alpha^{(j)} \oplus \mathbf{m}(h \times \cdots \times (\alpha^{(j)} \oplus \mathbf{m}(h)) \dots))) = \gamma_2(\alpha^{(j)} \oplus h) = \varphi^{(k)}.$$

The relation above yields $\varphi^{(k)} \geq \alpha^{(j)}$, $\varphi^{(k)} \geq h$, and

$$\mathbf{m}(h \times (\alpha^{(j)} \oplus \mathbf{m}(h \times \cdots \times (\alpha^{(j)} \oplus \mathbf{m}(h)) \dots))) \leq \varphi^{(k)}.$$

From the last inequality and the properties of operators $\mathbf{m}$ and $\mathbf{M}$ it follows that

$$h \times (\alpha^{(j)} \oplus \mathbf{m}(h \times \cdots \times (\alpha^{(j)} \oplus \mathbf{m}(h)) \dots)) \leq \mathbf{M}(\varphi^{(k)}).$$

Choose $\varphi^{(k-1)}$ as a maximal function satisfying the conditions $h \times \varphi^{(k-1)} \leq \mathbf{M}(\varphi^{(k)})$ and

$$\alpha^{(j)} \oplus \mathbf{m}(h \times \cdots \times (\alpha^{(j)} \oplus \mathbf{m}(h)) \dots) \leq \varphi^{(k-1)}$$

(the operator $\mathbf{m}$ is used $k-1$ times); then the inclusion $(h \times \varphi^{(k-1)}, \varphi^{(k)}) \in \Delta$ and the inequalities $\varphi^{(k-1)} \geq \mathbf{m}(h \times \cdots \times (\alpha^{(i)} \oplus \mathbf{m}(h)) \dots)$ and $\varphi^{(k-1)} \geq \alpha^{(j)}$ are true. By analogy, functions $\varphi^{(k-2)}$, ..., $\varphi^{(2)}$ can be chosen such that the inclusion $(h \times \varphi^{(c-1)}, \varphi^{(c)}) \in \Delta$ and the inequality $\varphi^{(c)} \geq \alpha^{(j)}$ are true, $c = k-2, \dots, 2$. Using the operator $\mathbf{M}$ $k-1$ times, from (18) we obtain the nontrivial (due to (17)) function $\varphi^{(1)} = \alpha^{(j)} \oplus \mathbf{m}(h)$. Then $\varphi^{(1)} \geq \mathbf{m}(h)$ and $\varphi^{(1)} \geq \alpha^{(j)}$ hold, i.e., the inclusion $(h, \varphi^{(1)}) \in \Delta$ is true. Since $\varphi = \varphi^{(1)} \times \cdots \times \varphi^{(k)}$ and $\varphi^{(c)} \geq \alpha^{(i)}$ for all c, we get $\varphi \geq \alpha^{(j)}$, i.e., the condition (11) is valid. From the above and the definition of relation Δ it follows that (14) are true for some functions $f_*^{(1)}$, ..., $f_*^{(k)}$. The inequality $\varphi^{(k)} \geq h$ means that a function ϕ exists such that $\varphi^{(k)} = \phi(h)$; then $y_* = \phi(h(x)) = \varphi^{(k)}(x) = x_{*1}$. ■

Theorem 2 gives an exhaustive solution of the problem considered. Its main difficulty is the use of a complex mathematical tool (the algebra of functions) to construct the RG. To overcome this shortcoming, one may use the so-called logic-dynamic approach (Zhirabok *et al.*, 2017) which allows solving the problem for nonlinear systems with methods of linear algebra.

Example 3. Consider the system described by

$$x^+ = f_o(x, u) = \begin{pmatrix} a_1x_2 - a_2x_1 \\ x_1|x_1| \\ a_1x_2 - a_2x_3 \\ x_3 - x_4 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix} u,$$

$$y_1 = x_1, \quad y_2 = x_4.$$

Here, the notation x^+ means $x(t+1)$.

Check the possibility to obtain the relation of the form (14) free of the coefficient a_1. From the definition of the function $\alpha^{(j)}$ it follows that $\alpha^{(1)}(x) = (x_1 - x_3, x_2, x_4)^T$. To compute $\mathbf{m}(h)$, use the rule from Section 3: since $(\alpha \times u) \oplus f = (f_{o2}, f_{o1} - f_{o3} - a_2f_{o4})^T$, where f_{oi} is the i-th component of the function f_o, we have $\mathbf{m}(h)(x) = (x_2, x_1 - x_3 - a_2x_4)^T$. Since $\alpha^{(1)} \leq \mathbf{m}(h)$, the condition (17) is valid. Next, $(\alpha^{(1)} \oplus \mathbf{m}(h))(x) = \mathbf{m}(h)(x) = (x_2, x_1 - x_3 - a_2x_4)^T$ and $h \times (\alpha^{(1)} \oplus \mathbf{m}(h))(x) = \mathbf{m}(h)(x) = (x_1, x_2, x_3, x_4)^T = \mathbf{0}$. Since $\mathbf{m}(\mathbf{0}) = \mathbf{0}$, the condition (18) is true with $k = 2$, i.e., (14) can be constructed without the coefficient a_1.

Set $\varphi^{(2)}(x) := (\alpha^{(1)} \oplus h)(x) = x_4$, $\varphi^{(1)}(x) := x_1 - x_3 - a_2x_4 \geq (\alpha^{(1)} \oplus \mathbf{m}(h))(x)$, $x_{*1} := x_1 - x_3 - a_2x_4$, $x_{*2} := x_4$; then (14) takes the form

$$\begin{aligned} x_{*1}(t+1) &= -a_2y_1(t) + a_2y_2(t) + u_1(t), \\ x_{*2}(t+1) &= -x_{*1}(t) + y_1(t) - (a_2+1)y_2(t), \\ y_*(t) &= y_2(t) = x_{*2}(t). \end{aligned}$$

Make temporal shifts and substitutions and obtain the input-output description:

$$y_2(t+2) = y_1(t+1) - (a_2+1)y_2(t+1) + a_2y_1(t) - a_2y_2(t) - u_1(t). \tag{20}$$

Based on (20), we obtain

$$\begin{aligned} y_2(t) &= y_1(t-1) - (a_2+1)y_2(t-1) \\ &\quad + a_2y_1(t-2) - a_2y_2(t-2) - u_1(t-2) \\ &= (1 \ \ a_2+1 \ \ a_2 \ \ -a_2 \ \ -1) \begin{pmatrix} y_1(t-1) \\ y_2(t-1) \\ y_1(t-2) \\ y_2(t-2) \\ u_1(t-2) \end{pmatrix}. \end{aligned}$$

Clearly, $T = 6$. As a result, $Y_6(t)$, $\Gamma(a)$, and $P_6(t)$ in (16) are as follows:

$$\begin{aligned} Y_6(t) &= (y_2(t) \ \dots \ y_2(t-5)), \\ \Gamma(a) &= (1 \ \ a_2+1 \ \ a_2 \ \ -a_2 \ \ -1), \\ P_6(t) &= \begin{pmatrix} y_1(t-1) & \dots & y_1(t-6) \\ y_2(t-1) & \dots & y_2(t-6) \\ y_1(t-2) & \dots & y_1(t-7) \\ y_2(t-2) & \dots & y_2(t-7) \\ u_1(t-2) & \dots & u_1(t-7) \end{pmatrix}. \end{aligned}$$

♦

By construction, the parity relation based RG is founded on the feedback free model (14). This condition is rather restrictive. Therefore the observer based RG is preferable to attain the purpose of fault diagnosis.

5. MAD and FAD design

Solvability of the FDI problem for each mode, in general, is not enough to solve the problem for the HS, except in the rare occasions when Eqns. (10) or (14) do not depend on parameters at all (are same for each mode), and therefore A_* and MAD are missing. If this is not the case, then one has to coordinate the work of the FA and the FAD as well as that of the MA and the MAD. This is possible only under certain conditions, specified in Theorems 3 and 4 below.

In general, the function φ depends on the system mode and therefore, in the following, we use the notation φ_o instead.

5.1. Main relations. The FA, described by (1) and the FAD, described by (4), are well-coordinated functions $\theta : S \to S_*$, $\rho : I \to I_*$, and $\eta : O \to O_*$ exist such that

$$\delta_*(\theta(s), \rho(i)) = \theta(\delta(s, i)), \tag{21}$$

$$\lambda_*(\theta(s)) = \eta(\lambda(s)). \tag{22}$$

Theorem 3. *The relations (21) and (22) hold if the following conditions are satisfied:*

(i) θ is a δ-invariant function satisfying the condition $\theta \leq \eta(\lambda)$;

(ii) $\boldsymbol{m}_I(\rho) \leq \theta$.

Proof. Since $\theta \leq \lambda' = \eta(\lambda)$, a function λ_* exists such that (22) holds. From (ii), the definition of relation $\leq$, and (7) it follows that

$$\rho(i) = \rho(i') \Rightarrow \theta(\delta(s, i)) = \theta(\delta(s, i'))$$

for all $i, i' \in I$ and $s \in S$. Since θ is δ-invariant, we have

$$\theta(s) = \theta(s') \Rightarrow \theta(\delta(s, i)) = \theta(\delta(s', i))$$

for all $i \in I$ and $s, s' \in S$. Clearly, the last relation is true for $i' \in I$ such that $\rho(i) = \rho(i')$. Therefore

$$[\rho(i) = \rho(i') \wedge \theta(s) = \theta(s')] \\ \Rightarrow \theta(\delta(s, i)) = \theta(\delta(s', i')).$$

The latter means that there exists a function δ_* such that (21) holds. ■

5.2. MAD design. From (5) it can be shown that $i_* = (\beta_*(h \times \varphi_o))(x)$. Since $i_* = \rho(i)$ and $i = \beta(x)$, we have

$$\beta_*(h \times \varphi_o) = \rho(\beta). \tag{23}$$

Theorem 4. *The FDI problem is solvable for the HS if the following conditions are satisfied:*

(i) the FDI problem is solvable for every mode of the HS;

(ii) $(h \times \varphi_o) \oplus \beta \neq \boldsymbol{1}$ for all $o \in O$;

(iii) $\boldsymbol{m}_I(\rho) \leq \theta$ for some δ-invariant function θ satisfying the condition $\theta \leq \lambda' = \eta(\lambda)$.

Proof. When (ii) is true, then there exists nonconstant functions ρ and β_* such that $\rho(\beta) = \beta_*(h \times \varphi_o)$. The function ρ defines the inputs i_* for the FAD as $i_* = \rho(i)$, $i \in I$, the function β_* defines the MAD (5). When (iii) is true, from Theorem 3 it follows that (21) and (22) are true, i.e., the FAD can be constructed. ■

The functions ρ and β_* can be found as follows. Let the functions h, β, and φ_o define partitions π_h, π_β, and π_{φ_o} on the set X, respectively, on the analogy of (6). Find the partition $\pi_* = \pi_\beta \oplus (\pi_h \times \pi_{\varphi_o})$. Note that the number of its blocks is finite due to the finite number of blocks of the partition π_β.

Assume that the partition π_β contains several blocks, and each corresponds to some input i according to (3). The partition $\pi_* = \pi_\beta \oplus (\pi_h \times \pi_{\varphi_o})$ contains bigger blocks and each consist of some blocks of π_β. Let some block B_{π_*} contain blocks $B_{\pi_\beta,1}, \dots, B_{\pi_\beta,d}$ and $\beta(B_{\pi_\beta,1}) = i_1, \dots, \beta(B_{\pi_\beta,d}) = i_d$. Then $\beta_*(B_{\pi_*}) = i_* = \rho(i_1) = \dots = \rho(i_d)$.

Example 4. Consider the system from Example 3 and let the function β from (3) be as follows:

$$i = \beta(x) = \begin{cases} i_1 & \text{if } |x_1| \leq 1 \text{ and } x_2 < 0, \\ i_2 & \text{if } |x_1| \leq 1 \text{ and } x_2 \geq 0, \\ i_3 & \text{if } |x_1| > 1 \text{ and } x_2 < 0, \\ i_4 & \text{if } |x_1| > 1 \text{ and } x_2 \geq 0, \end{cases}$$

$$h = (x_1 \ \ x_4)^T.$$

The partition π_β has four blocks corresponding to the inputs i_1 to i_4: the block $B_{\pi_\beta,1}$ contains states satisfying the conditions $|x_1| \leq 1$ and $x_2 < 0$, the block $B_{\pi_\beta,2}$ the conditions $|x_1| \leq 1$ and $x_2 \geq 0$, etc. Since $\varphi_o(x) = (x_1 - x_3 - a_2 x_4, x_4)^T$, we have $(h \times \varphi_o)(x) = (x_1, x_3, x_4)^T$ and $((h \times \varphi_o) \oplus \beta)(x) = x_1$. Therefore, the condition (ii) of Theorem 4 is satisfied for all $o \in O$.

The partition π_* corresponding to the function $((h \times \varphi_o) \oplus \beta)(x) = x_1$ has two blocks $B_{\pi_*,1}$ and $B_{\pi_*,2}$. The first contains states satisfying the condition $|x_1| = |y_1| \leq 1$, while for the second $|x_1| = |y_1| > 1$. Accordingly, block $B_{\pi_*,1}$ contains blocks $B_{\pi_\beta,1}$ and $B_{\pi_\beta,2}$, block $B_{\pi_*,2}$ contains $B_{\pi_\beta,3}$ and $B_{\pi_\beta,4}$. Since $\beta(B_{\pi_\beta,1}) = i_1$ and $\beta(B_{\pi_\beta,2}) = i_2$, we have $\beta_*(B_{\pi_*,1}) = i_{*1} = \rho(i_1) = \rho(i_2)$; by analogy, $\beta_*(B_{\pi_*,2}) = i_{*2} = \rho(i_3) = \rho(i_3)$. As a result, we write

$$i_* = \beta_*(y) = \begin{cases} i_{*1} & \text{if } |y_1| \le 1, \\ i_{*2} & \text{if } |y_1| > 1, \end{cases}$$

$$\rho(i) = \begin{cases} i_{*1} & \text{if } i \in \{i_1, i_2\}, \\ i_{*2} & \text{if } i \in \{i_3, i_4\}, \end{cases}$$

$$\pi_\rho = \{(i_1, i_2), (i_3, i_4)\}.$$

♦

Note that systems with the output y of a low dimension have some limits on the use of the suggested approach. Assume that the function φ_o for some parameter a_j and mode $o \in O$ is such that $h \oplus \varphi_o = \mathbf{1}$. In this case, the RG invariant with respect to the parameter a_j and the appropriate MAD do not exist. Hence, the problem of fault isolation may have only a partial solution. The problem can be solved by using additional sensors.

5.3. FAD design. Note that if ρ is an identity function, then the FAD coincides with the FA since it is irreducible by assumption. We assume that ρ is not an identity function "compressing" the set of inputs I. Therefore, the set of states S can be "compressed" as well by the function θ and the number of the states of automaton A can be reduced.

Note that when the function ρ is not the identity, we need to reduce the automaton A (if it is possible) since $\theta \ge \mathbf{m}_I(\rho) \ne \mathbf{0}$. Such a reduction may lead to a decrease in the system detectability. Algorithm 1 below constructs the FAD A_*.

Example 5. The description of the FA is given in Table 1, and the values of system parameters in Example 3, defined by modes, are given in Table 2.

Table 1. Description of the FA.

s	s^+				o
	i_1	i_2	i_3	i_4	
s_1	s_1	s_2	s_3	s_4	o_1
s_2	s_1	s_2	s_6	s_6	o_2
s_3	s_2	s_2	s_3	s_4	o_3
s_4	s_5	s_5	s_4	s_3	o_3
s_5	s_5	s_5	s_3	s_4	o_2
s_6	s_2	s_1	s_6	s_6	o_3

From Table 1 it follows that

$$\pi_\lambda = \{(s_1), (s_2, s_5), (s_3, s_4, s_6)\},$$

while the function ρ and the partition π_ρ are found in Example 4.

Algorithm 1. Construction of the FAD A_*.

Step 1. At first find the functions θ and η. Introduce the partitions π_θ and π_η, respectively corresponding to the functions θ and η on the analogy of (6). Write a recursion

$$\pi_{\theta^{j+1}} = \pi_{\theta^j} \oplus \mathbf{m}(\pi_{\theta^j}), \quad j = 0, 1, \dots, \tag{24}$$

where $\pi_{\theta^0} = \mathbf{m}_I(\pi_\rho)$. From (24) it follows that $\pi_{\theta^0} \le \pi_{\theta^1} \le \dots$. It is known (Hartmanis and Stearns, 1966) that an integer c exists such that $\pi_{\theta^c} = \pi_{\theta^{c+1}}$. Then $\pi_\theta = \pi_{\theta^c}$ is a minimal partition corresponding to the function θ. If $\pi_\theta = \mathbf{1}$, the FDI problem cannot be solved; assume that $\pi_\theta \ne \mathbf{1}$.

Step 2. Denote by S_* the set of blocks of the partition π_θ. To construct the function δ_* of the automaton A_*, replace the states in the table of A by the states from the set S_* according to the following rule: if the state s is in the block of the partition π_θ corresponding to s_*, then s is replaced by s_*. Analogously, replace the inputs of the automaton A by the inputs from the set I_*.

Step 3. To find the functions λ_* and η, introduce the partition π_λ corresponding to the function λ and compute the partition $\pi_\Lambda = \pi_\theta \oplus \pi_\lambda$; assume that $\pi_\Lambda \ne \mathbf{1}$, otherwise the FDI problem cannot be solved. Consider the first block of the partition π_Λ and the states s_1 and s_2 from this block. Let $\lambda(s_1) = o_1$ and $\lambda(s_2) = o_2$. Then set $\eta(o_1) = \eta(o_2) := o_{*1}$. Other blocks of the partition π_Λ are considered analogously.

Step 4. The function λ_* is constructed as follows: if the state s_* corresponds to the block B_{π_θ} of the partition π_θ, then $\lambda_*(s_*) = \eta(\lambda(s))$, $s \in B_{\pi_\theta}$.

Since $\rho(i_1) = \rho(i_2)$ and $\rho(i_3) = \rho(i_4)$, the states $s_1 = \delta(s_1, i_1)$ and $s_2 = \delta(s_1, i_2)$ form some block of the partition π_{θ^0}; the same is true for the states $s_3 = \delta(s_1, i_3)$ and $s_4 = \delta(s_1, i_4)$. As a result, $\pi_{\theta^0} = \{(s_1, s_2), (s_3, s_4), (s_5), (s_6)\}$.

From (24) it follows that $\mathbf{m}(\pi_{\theta^0}) = \{(s_1), (s_2, s_5), (s_3, s_4, s_6)\}$ and $\pi_{\theta^1} = \{(s_1, s_2, s_5), (s_3, s_4, s_6)\}$. It can be shown that $\pi_{\theta^2} = \pi_{\theta^1}$. Then $\pi_\theta := \pi_{\theta^1}$, $S_* := \{s_{*1}, s_{*2}\}$, and

$$\theta(s) = \begin{cases} s_{*1} & \text{if } s \in \{s_1, s_2, s_5\}, \\ s_{*2} & \text{if } s \in \{s_3, s_4, s_6\}. \end{cases}$$

Compute the partition $\pi_\Lambda = \pi_\theta \oplus \pi_\lambda = \{(s_1, s_2, s_5), (s_3, s_4, s_6)\} \oplus \{(s_1), (s_2, s_5), (s_3, s_4, s_6)\} = \{(s_1, s_2, s_5), (s_3, s_4, s_6)\}$. Since the outputs o_1 and o_2 correspond to the states s_1, s_2, and s_5, the output o_3 correspond to the states s_2, s_4, and s_6, then

$$\eta(o) = \begin{cases} o_{*1} & \text{if } o \in \{o_1, o_2\}, \\ o_{*2} & \text{if } o = o_3, \end{cases}$$
$$\pi_\eta = \{(o_1, o_2), (o_3)\}.$$

Table 2. Values of system parameters.

Parameter	Mode		
	o_1	o_2	o_3
a_1	1	2	3
a_2	1	1	2

To construct the function λ_*, consider the state s_{*1} corresponding to the block (s_1, s_2, s_5). Since $\lambda(s_1) = o_1$, $\lambda(s_2) = \lambda(s_5) = o_2$, and $\eta(o_1) = \eta(o_2) = o_{*1}$, we have $\lambda_*(s_{*1}) = o_{*1}$. The state s_{*1} yields $\lambda_*(s_{*2}) = o_{*2}$ by analogy. As a result, the automaton A_* is described by Table 3.

Table 3. Description of A_*.

s_*	s_*^+		o_*
	i_{*1}	i_{*2}	
s_{*1}	s_{*1}	s_{*2}	o_{*1}
s_{*2}	s_{*1}	s_{*2}	o_{*2}

In addition to (23), there exists a relation between the forms (10) or (14) and the FAD. Introduce in O the set of partitions $\pi_1, \dots, \pi_p$: $o \equiv o'(\pi_j) \Leftrightarrow$ the coefficient a_j has the same values in the modes o and o', $j = 1, \dots, p$.

Theorem 5. *If the model (10) or (14) contains the coefficient a_j, then*

$$\pi_\eta \le \pi_j. \tag{25}$$

Proof. Let (10) or (14) contain the coefficient a_j, and modes o and o' be in some block of the partition π_η. The last means that the values of the coefficient a_j are the same in the modes o and o', i.e., these modes are in some block of the partition π_j. From the definition of the relation $\le$ it follows that $\pi_\eta \le \pi_j$. ■

Based on Table 2, we obtain $\pi_1 = \{(o_1), (o_2), (o_3)\}$ and $\pi_2 = \{(o_1, o_2), (o_3)\}$. Since $\pi_\eta = \{(o_1, o_2), (o_3)\}$, (25) is satisfied for the parameter a_2, and (14) should be free from the parameter a_1. From Example 3 it follows that (20) does not contain the parameter a_1. The values of parameter a_2 in that FAD, defined by modes, are given by

$$a_2 = \begin{cases} 1 & \text{if } o_* = o_{*1}, \\ 2 & \text{if } o_* = o_{*2}. \end{cases}$$

♦

5.4. Analysis of the solvability of the FDI problem.

Consider the condition $(h \times \varphi_o) \oplus \beta \ne \mathbf{1}$ from Theorem 4. If $h \oplus \beta \ne \mathbf{1}$, then we can construct the FAD without using the function φ_o. But it is strongly recommended to use this function since it provides a better possibility to solve the FDI problem. If $h \oplus \beta = \mathbf{1}$, use of the function φ_o is obligatory. Consider this in detail.

From the property of operations $\times$ and $\oplus$ it follows that (Zhirabok and Shumsky, 2008)

$$(h \times \varphi_o) \oplus \beta \le (h \oplus \beta) \times (\varphi_o \oplus \beta).$$

If $h \oplus \beta = \mathbf{1}$, then the only possibility to solve the FDI problem is to find a function φ_o such that $\varphi_o \oplus \beta \ne \mathbf{1}$ for all $o \in O$. Algorithm 2 below checks the possibility to solve the FDI problem under the condition $h \oplus \beta = \mathbf{1}$.

Algorithm 2. Checking the possibility of solving the FDI problem.

Step 1. Set $j = 1$, find an (h, f)-invariant function φ_o such that $a^{(j)} \le \varphi_o$, and construct the RG in the form (10) or (14) (if possible). Let the model (10) or (14) contain the parameters $a_{c_1}, a_{c_2}, \dots, a_{c_d}$.

Step 2. Check the condition $\varphi_o \oplus \beta \ne \mathbf{1}$ for all $o \in O$. If it does not hold, set $j := j + 1$ and go to Step 1.

Step 3. Find the functions β_* and ρ from (23) and the partition π_η. Check the condition (25) for $j = c_1$, $j = c_2$, $\dots$, $j = c_d$. If all of them hold, then the FAD for the j-th RG can be constructed, otherwise it cannot. If $j = p$, stop; otherwise set $j := j + 1$ and go to Step 1.

When the problem is not solved for some j and j' as well as appropriate functions φ_o and φ_o', we can construct an RG in the form (10) or (14) for the function $\varphi_o'' = \varphi_o \times \varphi_o'$ since it is (h, f)-invariant due to Lemma 2. In this case we have a better possibility of solving the FDI problem since $\pi_\eta'' \le \pi_\eta$ and $\pi_\eta'' \le \pi_\eta'$, where π_η'' corresponds to the function φ_o'', which provides a better chance to satisfy the condition (25).

If $(h \times \varphi_o) \le \beta$, then $(h \times \varphi_o) \oplus \beta \cong \beta$, and based on (23) we can set $\rho := \mathbf{0}$. In this case $\mathbf{m}_I(\rho) = \mathbf{0}$, and there is no need to reduce the automaton A, but we can try to reduce it using (25) as follows.

Let the model (10) or (14) contain the parameters $a_{j_1}, \dots, a_{j_c}$; then, by Theorem 5, $\pi_\eta \le \pi_{j_1}, \dots, \pi_\eta \le \pi_{j_c}$ hold, which implies that $\pi_\eta \le \pi_{j_1} \times \cdots \times \pi_{j_c}$. If $\pi_{j_1} \times \cdots \times \pi_{j_c} \ne \mathbf{0}$, we can set $\pi_\eta := \pi_{j_1} \times \cdots \times \pi_{j_c}$ and find the functions η and $\lambda' = \eta(\lambda)$.

The procedure of reducing the automaton A with the output function λ', i.e., finding the functions δ_* and λ_*, is as follows. Find the sequence of functions

$$\xi^0 := \lambda', \quad \xi^j = \lambda' \times \mathbf{M}(\lambda') \times \cdots \times \mathbf{M}^j(\lambda'),$$
$$j = 0, 1, \dots,$$

where $\mathbf{M}^0(\lambda') = \lambda'$. When $\xi^c \cong \xi^{c+1}$ for some c, set $\theta := \xi^c$; the function θ is maximal δ-invariant by construction (Zhirabok and Shumsky, 2008).

Since the functions $\xi^0 = \lambda'$ and $\mathbf{M}^j(\lambda')$ are specified by tables, for computations we use the partitions that correspond to these functions. Let $\pi_{\lambda'}$ be the partition on the set S induced by λ' according to (6). Find the sequence of partitions, corresponding to the functions ξ^j:

$$\pi_\theta^0 := \pi_{\lambda'}, \quad \pi_\theta^j = \pi_{\lambda'} \times \mathbf{M}(\pi_{\lambda'}) \times \cdots \times \mathbf{M}^j(\pi_{\lambda'}), \qquad j = 0, 1, \ldots \tag{26}$$

When $\pi_\theta^c = \pi_\theta^{c+1}$ for some c, then the partition $\pi_\theta = \pi_\theta^c$ has a substitution property (Hartmanis and Stearns, 1966) that corresponds to δ-invariance; if $\pi_\theta \neq \mathbf{0}$, i.e., if at least one block of the partition contains more than a single state, the automaton A with the output function λ' is reducible.

In addition to reducing the set of states of the automaton A, we may reduce the set of its inputs using the operator $\mathbf{M}_I$ as follows. Calculate the partition $\pi_\rho = \mathbf{M}_I(\pi_\theta)$. Observe that Theorem 3 requires $\mathbf{m}_I(\rho) \leq \theta$, or $\rho \leq \mathbf{M}_I(\theta)$ to be satisfied. Note that the choice $\rho = \mathbf{M}_I(\theta)$ guarantees the minimal number of inputs in A_*. If $\rho \neq \mathbf{0}$, denote by I_* the set of blocks of the partition π_ρ, corresponding to the function ρ, and construct the automaton A_* by Algorithm 1.

6. Example

Consider the control system

$$\begin{aligned} x_1^+ &= a_1\frac{u_1}{\vartheta_1} - a_4\mathrm{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} + x_1, \\ x_2^+ &= a_3\frac{u_2}{\vartheta_2} + a_4\mathrm{sign}(x_1 - x_2)\sqrt{|x_1 - x_2|} \\ &\quad - a_2a_5\mathrm{sign}(x_2 - x_3)\sqrt{|x_2 - x_3|} + x_2, \\ x_3^+ &= a_2a_5\mathrm{sign}(x_2 - x_3)\sqrt{|x_2 - x_3|} \\ &\quad - a_6\sqrt{x_3 - \vartheta_7} + x_3, \\ y_1 &= x_1, \quad y_2 = x_2, \end{aligned} \tag{27}$$

where $a_4 = \vartheta_4\sqrt{2\vartheta_8}/\vartheta_1$, $a_5 = \vartheta_5\sqrt{2\vartheta_8}/\vartheta_2$, and $a_6 = \vartheta_6\sqrt{2\vartheta_8}/\vartheta_3$.

Equations (27) constitute a modified sampled-data model of the well-known example of a three tank system (Patton, 1994), see Fig. 3. The system consists of three consecutively linked tanks with the areas of cross-section ϑ_1, ϑ_2, and ϑ_3. The tanks are linked by pipes with the areas of cross-section ϑ_4 and ϑ_5. The liquid flows into the first and the second tanks and flows out of the third one through the pipe with the area of cross-section ϑ_6 located at height ϑ_7; ϑ_8 is the gravitational constant. The levels of liquid in the tanks are x_1, x_2, and x_3, respectively.

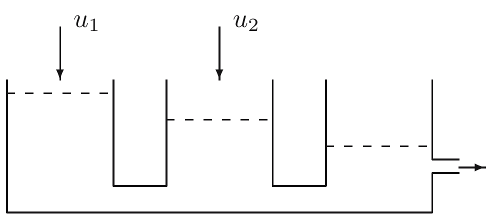

Fig. 3. Three-tank system.

The FA is given by Table 4 and the MA by the function

$$i = \beta(x) = \begin{cases} i_1 & \text{if } x_2 - x_3 \geq a, \\ i_2 & \text{if } -1 \leq x_2 - x_3 < a, \\ i_3 & \text{if } x_2 - x_3 < -1 \end{cases} \tag{28}$$

for some constant $a > 0$. The parameters a_1, a_2, and a_3 depend on the output of the FA as described in Table 5. The parameters a_4, a_5, and a_6 do not depend on the output; they reflect faults in the system which appear as leakages in the first, second, and third tank, respectively.

Table 4. Description of the FA.

s	s^+			o
	i_1	i_2	i_3	
s_1	s_1	s_2	s_4	o_1
s_2	s_1	s_2	s_3	o_2
s_3	s_1	s_2	s_3	o_3
s_4	s_1	s_2	s_4	o_4

Table 5. Values of system parameters.

Parameter	Mode			
	o_1	o_2	o_3	o_4
a_1	0.1	0.1	0.1	0.2
a_2	0.1	0.07	0.05	0.03
a_3	0.1	0.2	0.2	0.2

Clearly, $h \oplus \beta = \mathbf{1}$; therefore, we use Algorithm 2. Evaluate the function $\alpha^{(4)}(x) = (x_1 + x_2, x_3)^T$. Check the possibility of constructing an RG. Set $\gamma^0 := (x_1 + x_2, x_3)^T$; since $\gamma^0 \times h = \mathbf{0}$ and $\mathbf{m}(\mathbf{0}) = \mathbf{0}$, from (12) it follows that $\varphi(x) = \gamma^0(x) = (x_1 + x_2, x_3)^T$. Since $\varphi \times h = \mathbf{0} \leq \beta$ and $(\varphi \oplus h)(x) = x_1 + x_2$, we can construct an observer based RG and set $\rho := \mathbf{0}$.

Set $x_{*1} := \varphi_1(x) = x_1 + x_2$, $x_{*2} := \varphi_2(x) = x_3$. The observer is described by

$$\begin{aligned} x_{*1}^+ &= a_1\frac{u_1}{\vartheta_1} + a_3\frac{u_2}{\vartheta_2} \\ &\quad - a_2a_5\mathrm{sign}(y_2 - x_{*2})\sqrt{|y_2 - x_{*2}|} + y_1 + y_2, \\ x_{*2}^+ &= a_2a_5\mathrm{sign}(y_2 - x_{*2})\sqrt{|y_2 - x_{*2}|} \end{aligned}$$

$$- a_6\sqrt{x_{*2} - \vartheta_7} + x_{*2},$$
$$y_* = x_{*1}, \quad r_1 = y_1 + y_2 - y_*. \tag{29}$$

and the gain K is omitted for simplicity. Since the observer contains all parameters a_1 to a_3, the FAD coincides with the FA. The function β_* is defined as follows:

$$\beta_*(x_*, y) = \begin{cases} i_1 & \text{if } y_2 - x_{*2} \geq a, \\ i_2 & \text{if } -1 \leq y_2 - x_{*2} < a, \\ i_3 & \text{if } y_2 - x_{*2} < -1. \end{cases}$$

Since $\mathbf{m}(h) = x_1$ and $\alpha^1(x) \oplus \mathbf{m}(h) = \mathbf{1}$, the condition (17) does not hold. Therefore no RG based on parity relations exists.

From (29) it follows that, since $a_2 \neq 0$, the sensitivity of the residual r_1 to deviation in a_5 may change but does not equal zero; simulation confirms this fact. Besides, the sensitivity to a_6 does not depend on the mode.

Compute the function $\alpha^{(5)}(x) = (x_1, x_2 + x_3)^T$. It can be shown that $\varphi(x) = (x_1, x_2 + x_3)^T$, and the observer is described by the equations with $x_{*1} := x_1$, $x_{*2} := x_2 + x_3$:

$$x_{*1}^+ = a_1\frac{u_1}{\vartheta_1} - a_4\text{sign}(y_1 - y_2)\sqrt{|y_1 - y_2|} + y_1,$$
$$x_{*2}^+ = a_3\frac{u_2}{\vartheta_2} + a_4\text{sign}(y_1 - y_2)\sqrt{|y_1 - y_2|}$$
$$- a_6\sqrt{x_{*2} - y_2 - \vartheta_7} + x_{*2},$$
$$y_* = x_{*1}, \quad r_2 = y_1 - y_*. \tag{30}$$

Clearly, the sensitivity of the residual r_2 to a deviation in a_4 does not depend on the mode.

These equations contain the parameters a_1 and a_3, therefore we can set $\pi_\eta := \pi_1 \times \pi_3 = \{(o_1, o_2, o_3), (o_4)\} \times \{(o_1), (o_2, o_3, o_4)\} = \{(o_1), (o_2, o_3), (o_4)\}$, therefore $\pi_{\lambda'} = \{(s_1), (s_2, s_3), (s_4)\}$. The formula (26) yields $\pi_\theta = \{(s_1), (s_2, s_3), (s_4)\}$. The FAD is given in Table 6.

Table 6. Description of the FAD.

s_*	s_*^+			o_*
	i_1	i_2	i_3	
s_{1*}	s_{1*}	s_{2*}	s_{3*}	o_{1*}
s_{2*}	s_{1*}	s_{2*}	s_{2*}	o_{2*}
s_{3*}	s_{1*}	s_{2*}	s_{3*}	o_{3*}

The function β_* is defined by

$$\beta_*(x_*, y) = \begin{cases} i_1 & \text{if } 2y_2 - x_{*2} \geq a, \\ i_2 & \text{if } -1 \leq 2y_2 - x_{*2} < a, \\ i_3 & \text{if } 2y_2 - x_{*2} < -1. \end{cases}$$

Table 7. Values of the parameters a_1 and a_3.

Parameter	Mode		
	o_{1*}	o_{2*}	o_{3*}
a_1	0.1	0.1	0.2
a_3	0.1	0.2	0.2

The parameters a_1 and a_3 in the observer depend on the output of the FAD as given in Table 7.

One may check that the conditions of Theorem 2 hold and hence $\varphi(x) = x_1$. But $(\varphi \times h) \oplus \beta = \mathbf{1}$ and no RG based on parity relations exists.

Compute the function $\alpha^{(6)}(x) = (x_1, x_2)^T$. In this case, Theorems 1 and 2 produce the same result $\varphi(x) = x_1$. Since $(\varphi \times h) \oplus \beta = \mathbf{1}$, neither observer based nor parity relations based RG exists.

Clearly, the matrix of syndromes S_D is given by

$$S_D = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix},$$

where the rows correspond to the residuals r_1 and r_2, while the columns to the parameters a_4, a_5, and a_6.

For simulation, set $a_{40} := 0.2$, $a_{50} := 1$, $a_{60} := 0.1$, $\vartheta_{70} := 0.1$, $a := 0.3$, $u_1 = u_2 := 0.1$. Figures 4 and 5 show the behavior of the difference $x_2(t) - x_3(t)$ and the state s, respectively, where $s = i$ corresponds to the state s_i, $i = 1, \ldots, 4$. Multiple switchings in the interval $t = 50 \div 100$ can be explained by the proximity of $x_2(t) - x_3(t)$ to $a = 0.3$ here; two switchings after $t = 120$ are due to changing the parameter a_6.

Figure 6 shows the behaviour of the residual r_1 for the observer (29) when the parameter a_6 abruptly changes at $t = 120$ from $a_{60} = 0.1$ to $a_6 = 0.2$ while a_4 abruptly changes at $t = 30$ from $a_{40} = 0.2$ to $a_4 = 0.1$ onward. Figure 7 shows the behavior of the residual r_2 for the observer (30) when the parameter a_4 abruptly changes at $t = 65$ from $a_{40} = 0.2$ to $a_4 = 0.05$ onward. The behavior of r_1 shows its sensitivity to the parameter a_6 and insensitivity to a_4, which corresponds to the model (29). Simulation shows that the change in a_4 to $a_4 = 0.01$ yields another picture of multiple switchings.

7. Conclusion

The paper addressed the fault diagnosis problem in hybrid systems. A solution to this problem was proposed in the form of a bank of hybrid residual generators. Such generators are based on both closed-loop (diagnostic observers) and open-loop techniques (parity relations). The latter was is used for applying a nonparametric method of diagnosis. It was shown that the fault diagnosis problem can be solved under some (sufficient) solvability conditions.

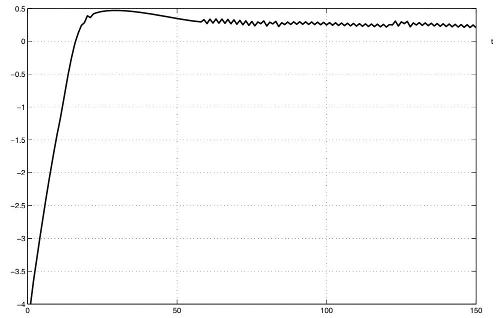

Fig. 4. Evolution of the difference $x_2 - x_3$.

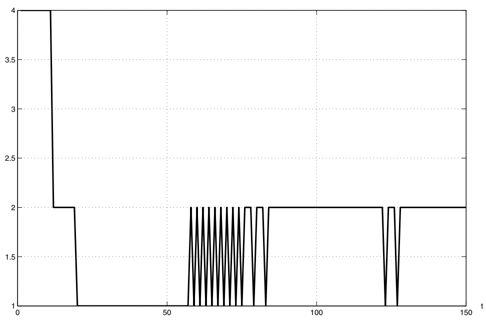

Fig. 5. Evolution of the state s.

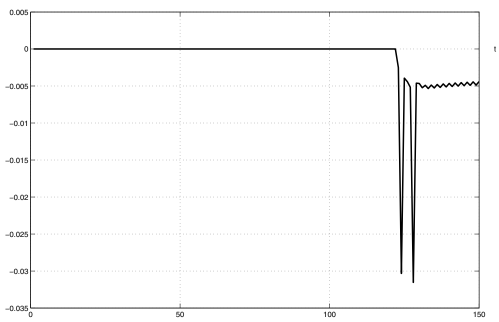

Fig. 6. Evolution of the residual r_1 for the observer (29).

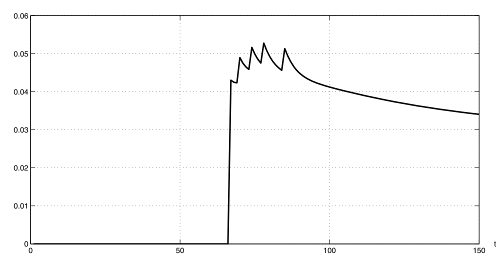

Fig. 7. Evolution of the residual r_2 for the observer (30).

The future plan regarding research covers studying the asynchronous problem and improving the detection efficiency based on weighted residual signals (Li *et al.*, 2016).

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