Int. J. Appl. Math. Comput. Sci., 2020, Vol. 30, No. 2, 267–279
DOI: 10.34768/amcs-2020-0021

STABILIZATION ANALYSIS OF IMPULSIVE STATE–DEPENDENT NEURAL NETWORKS WITH NONLINEAR DISTURBANCE: A QUANTIZATION APPROACH

YAXIAN HONG [a], HONGHUA BIN [a,*], ZHENKUN HUANG [a,*]

[a] School of Science
Jimei University
Xiamen 361021, Fujian Province, China
e-mail: {hhbin,hzk974226}@jmu.edu.cn

In this paper, the problem of feedback stabilization for a class of impulsive state-dependent neural networks (ISDNNs) with nonlinear disturbance inputs via quantized input signals is discussed. By constructing quasi-invariant sets and attracting sets for ISDNNs, we design a quantized controller with adjustable parameters. In combination with a suitable ISS-Lyapunov functional and a hybrid quantized control strategy, we propose novel criteria on input-to-state stability and global asymptotical stability for ISDNNs. Our results complement the existing ones. Numerical simulations are reported to substantiate the theoretical results and effectiveness of the proposed strategy.

Keywords: state-dependent neural networks, quantized input, stabilization.

1. Introduction

The memristor was called the fourth circuit component by Chua (1971). It can be used as a synapse in an artificial neural network or as a transistor in a new generation of computers. Owing to the appealing physical characteristics of the memory device, memristive neural networks (MNNs), a class of state-dependent neural networks (SDNNs), have attracted close attention in recent years (Duan *et al.*, 2017; Guo *et al.*, 2018; Huang *et al.*, 2019; 2018b; Wang *et al.*, 2016; Yang *et al.*, 2016; 2019; Zhang and Shen, 2015; Zhu *et al.*, 2018). Guo *et al.* (2018) analyzed multiple stable equilibrium points using reasonable assumptions on the decomposition of index sets and switching thresholds, and obtained a new theoretical result on recursive neural network switchings with stable radial basis functions and state-dependent switchings. Zhang and Shen (2015) obtained sufficient conditions to guarantee the global exponential stability of a class of delayed neural networks with state-dependent switching. The criteria of Lagrange stability for Takagi–Sugeno (T–S) with time-varying delays are obtained by constructing a scale-limited generalized Halanay inequality (Huang *et al.*, 2018b).

*Corresponding author

In the realization of electronic networks, due to the influence of frequency changes, the switching phenomenon, voltage mutation or sudden noise, the state of SDNNs may be disturbed by a sudden change and instantaneous disturbance in some cases. Impulsive systems are widely used in engineering, physics and science, which can be used to describe disturbances. Therefore, it is necessary to study the dynamical behavior of SDNNs with impulsive effects. For instance, the exponential stability of the SDNN model with variable delays and impulsive control is studied by choosing a suitable function (Duan *et al.*, 2017). A generalized model of SDNNs with an impulse time window is discussed by Yang *et al.* (2016), and the relationship between the exponential convergence rate and impulsive parameters is also pointed out. Zhu *et al.* (2018) studied systems with event-based impulsive control and applied it to SDNNs.

Many of the results mentioned above ignore the impact of data quantification, assuming that all data transmissions can be performed with unlimited precision, but this is not realistic in real-world networks. As usual, quantization may change the dynamical behavior of the system, and it will result in instability, oscillation or chaos. A quantizer can be regarded as a device that converts a real-valued signal into a piecewise constant one taking on a finite set of values. There have been two main control methods developed in recent years. One is the static quantizer, the other is the flexible quantizer (Chang *et al.*, 2013; Fan *et al.*, 2016; Sun *et al.*, 2019; Wan *et al.*, 2017; Wu *et al.*, 2018; Hong *et al.*, 2019; Zhang *et al.*, 2019). The former divides the domain of the quantized signal and has a fixed quantization level.

Chang *et al.* (2013) quantified all the error sampling measurements by a logarithmic quantizer, and studied the synchronization of complex networks. In order to estimate the unknown disturbance, Sun *et al.* (2019) designed a novel fixed-time disturbance observer and constructed a fixed-time controller to ensure the convergence of the system state. Zhang *et al.* (2019) gave some sufficient conditions for finite-time and fixed-time synchronization of discontinuous complex networks under the control of the static logarithmic quantizer. On the other hand, the flexible quantizer can be zooming-in or zooming-out according to the system state.

Fan *et al.* (2016) used a quantizer with adjustable time-varying parameters to study the synchronization of chaotic systems. Wan *et al.* (2017) proposed a novel quantized output control strategy to make discrete closed-loop systems with two quantized signals asymptotically stable or satisfy a specified H_∞ performance. The advantage of this method is that it can dynamically expand the quantization level to increase the region of attraction and attenuate the steady-state limit cycle.

Wu *et al.* (2018) combined quantitative control and event triggering control with the bounded consistency problem of multi-agent systems with external disturbances. Qian *et al.* (2012) studied a hybrid impulsive control system and gave a simple criterion for system stability. Hao *et al.* (2011) assumed that the network has two time-varying additive delays and limited capacity and proposed a continuous time Takagi–Sugeno (T–S) fuzzy system correlation stabilization method for pulse effects to ensure the asymptotic stability of the closed-loop system. Although some papers have considered a hybrid pulse system under quantized control, they often only quantify the disturbance input, but do not take the quantification of impulse terms into account.

Inspired by the aforementioned discussion, our purpose is to study how to obtain better performance of ISDNNs with nonlinear disturbance inputs considering the quantization effect. Our contributions are highlighted as follows:

1. It is for the first time that a flexible quantizer is discussed in ISDNNs. By proposing a quantitative control scheme, we propose novel criteria on input-to-state stability (ISS) and global asymptotical stability for ISDNNs.

2. The control strategy is different from those used by Wang *et al.* (2016); we adopt quantized control for external inputs and impulses, which is divided into two stages: zooming-in and zooming-out. It is superior to error-free control (Duan *et al.*, 2017; Guo *et al.*, 2018; Wang *et al.*, 2016; Zhu *et al.*, 2018).

3. By introducing a quasi-invariant set and an attracting basin, it can be obtained that the corresponding trajectory is inside an ellipsoid in a certain time interval. Through quantized control, the difficulty of channel blockage and limited channel can be solved.

The rest of the paper is organized as follows. In Section 2, the model of ISDNNs with quantized control is given. Some necessary assumptions, definitions, and lemmas are introduced. In Section 3, we first prove ISS and introduce quasi-invariant sets and attracting sets for ISDNNs with an input quantizer. Due to a quantized input-feedback control, we obtain the global asymptotic stability of ISDNNs. In Section 4, the numerical results show that the quantization controller is effective.

Notation. As usual, let $\mathbb{R}_+ = [0, \infty)$ and I be an identity matrix. $\mathbb{R}^n$ denotes the n-dimensional space with Euclidean vector norm $||\cdot||$ and $\mathbb{R}^{m\times n}$ is the set of $m \times n$ real matrices. $\mathrm{co}\{\check{a}, \hat{a}\}$ denotes the closure of the convex hull generated by real numbers $\check{a}$ and $\hat{a}$. $\lambda_{\min}(\cdot)$ and $\lambda_{\max}(\cdot)$ denote the minimum and maximum eigenvalues of the corresponding matrix, respectively. $\mathcal{C}[\mathbb{R}, \mathbb{R}] : \mathbb{R} \to \mathbb{R}$ is a continuous function. $P > 0$ denotes a positive definite matrix P. By $\mathfrak{R}(z) = \{x \in \mathbb{R}^n | x^T P x \le z\}$ we denote the corresponding ellipsoid.

2. Preliminaries and the model

In this section, we give our model description and recall some useful lemmas.

Define $\mathscr{U}$ as the set of measurable locally essentially bounded functions $u : \mathbb{R}^n \to \mathbb{R}^n$ and $\mathscr{V}$ as the set of bounded functions $J_k : \mathbb{R}^n \to \mathbb{R}^n$. A function $\gamma : \mathbb{R}_+ \to \mathbb{R}_+$ is said to be a class of $\mathscr{K}$ if it is continuous, strictly increasing and $\gamma(0) = 0$; $\gamma \in \mathscr{K}_\infty$ if $\gamma \in \mathscr{K}$ and also $\gamma(s) \to \infty$ as $s \to \infty$; $\xi \in \mathscr{KL}$ if $\xi : \mathbb{R}_+ \times \mathbb{R}_+ \to R_+, \xi(\cdot, t) \in \mathscr{K}$ for each fixed $t, \xi(s, t)$ decreases to zero as $t \to \infty$ for each fixed s.

Consider the following SDNNs with a nonlinear disturbance inputs model (Wang *et al.*, 2016):

$$\dot{x}_i(t) = -d_i(x_i(t))x_i(t) + \sum_{j=1}^{n} a_{ij}(x_i(t))f_j(x_j(t)) + u_i(x_i(t)), \quad t \ge 0, \tag{1}$$

where $i \in \mathcal{N} := \{1, 2, 3, \dots, n\}$, $x_i(t)$ is the voltage of the capacitor $\mathbb{C}_i$, n stands for the number of neurons, $f_j :$

$\mathbb{R} \longrightarrow \mathbb{R}$ is a neural activation function, $u_i(x_i(t))$ is the nonlinear disturbance inputs, $d_i(\cdot)$ and $a_{ij}(\cdot)$ represent the parallel-memristor corresponding to the capacitor $\mathbb{C}_i$ and the memductances of the memristor between the function $f_j(x_j(t))$ and $(x_i(t))$, respectively, and

$$d_i(x_i(t)) = \begin{cases} {d_i}^M, & |x_i(t)| \le \mathcal{T}_i, \\ {d_i}^S, & |x_i(t)| > \mathcal{T}_i, \end{cases}$$

$$a_{ij}(x_i(t)) = \begin{cases} a_{ij}^M, & |x_i(t)| \le \mathcal{T}_i, \\ a_{ij}^S, & |x_i(t)| > \mathcal{T}_i, \end{cases}$$

where $\mathcal{T}_i > 0$ is a switching jump, while ${d_i}^M > 0, {d_i}^S > 0, a_{ij}^M, a_{ij}^S$ $(i, j \in \mathcal{N})$ are constants which satisfy ${d_i}^M \neq {d_i}^S$ and $a_{ij}^M \neq a_{ij}^S$.

To improve the resistance disturbance capacity of the SDNNs with nonlinear disturbance inputs (1), we can add impulsive controllers to the nodes. Then, the SDNNs of hybrid switching with impulses can be described as follows:

$$\begin{cases} X\dot{x}_i(t) = -d_i(x_i(t))x_i(t) + \displaystyle\sum_{j=1}^{n} a_{ij}(x_i(t))f_j(x_j(t)) \\ \qquad\qquad +u_i(x_i(t)), \quad t \ge 0, \quad t \neq t_k, \\ x_i(t_k^+) = J_{ik}(x_i\left(t_k^-\right)), \quad t = t_k, \end{cases}$$

where $J_{ik}(x_i) \in \mathcal{C}[\mathbb{R}, \mathbb{R}], x^+(t_k)$ is the reset of x at t_k and $t_k (k \in \mathcal{N})$ are the impulsive moments with $0 < t_k < t_{k+1}, \lim_{k\to\infty} t_k = \infty$. Assume that

$$0 < \theta \le t_k - t_{k-1} \le \rho, \tag{2}$$

for all $k \in \mathcal{N}$ and some numbers θ, ρ. As usual, if not explicitly stated, signals are assumed to be right-continuous and to have left limits at all times.

In the control system, since u_i is a continuous input signal and J_{ik} is a discrete one, we call (u_i, J_{ik}) the hybrid input of an impulsive system. In the above mentioned works, hybrid input is taken directly for the stability of impulsive systems. Due to limited information feedback, however, it is interesting that only quantized input is available to stabilize the system.

Let $z \in \mathbb{R}^n$ be the variable being quantized, and $q(\cdot) : \mathbb{R}^n \to U$ be a quantizer that is a piecewise constant function, where U is a finite subset of $\mathbb{R}^n$. Usually, we assume that $q(z) = 0$ for z in some neighborhood of the equilibrium point, which satisfies

- If $\|z\| \le M$, then $\|q(z) - z\| \le \Delta$.
- If $\|z\| > M$, then $\|q(z)\| > M - \Delta$.

In general, we can view M and Δ as the quantization range and quantization error, respectively.

In the following control strategy, a flexible quantizer with one adjustable parameter μ is adopted in the form of

$$q_\mu(z) = \mu q\Big(\frac{z}{\mu}\Big),$$

where $\mu > 0$. We can get it quite naturally with the quantization range of μM and the quantization error of $\mu\Delta$. We can refer to μ as a “zoom” variable, if we increase μ that is equivalent to zooming out, which is substantially getting a new quantizer with or larger quantization range and a larger quantization error, if we decrease μ that is equivalent to zoom in, which is substantially getting a quantizer with smaller quantization rage and a smaller quantization error.

Definition 1. (*Aubin and Cellina, 1984*) Consider the following ordinary differential equation:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = h(x), \quad h(0) = h_0 \in \mathbb{R}^n, \tag{3}$$

where $h(x) : \mathbb{R}^n \to \mathbb{R}^n$ is not necessarily continuous. The Filippov set-valued map of $h(x)$ is defined as follows:

$$\mathbb{F}(x) = \bigcap_{\delta>0} \bigcap_{\mu(N)=0} K[h(B(x,\delta)\backslash N)],$$

where $K(E)$ is the closure of the convex hull of set $E, B(x, \delta) = \{y : ||y - x|| \le \delta\}$ and $\mu(N)$ is the Lebesgue measure of the set $N \subset \mathbb{R}^N$.

An absolutely continuous vector valued function $x(t)$ defined on [0, T] is called a solution in the sense of the Filippov of $\mathrm{d}x/\mathrm{d}t = h(x)$ if for almost all t, $\mathrm{d}x/\mathrm{d}t \in \mathbb{F}(x)$ and if, it satisfies the initial condition $h(0) = h_0$.

According to Aubin and Cellina (1984), (3) has at least one Filippov solution on $\mathbb{R}^n$, i.e., there exists a measurable function $\chi \in \mathbb{F}(x(t))$ such that

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \chi.$$

For $i, j \in \mathcal{N}$, write

$$\begin{aligned} \overline{d}_i &:= \max\{{d_i}^M, {d_i}^S\}, \\ \underline{d}_i &:= \min\{{d_i}^M, {d_i}^S\}, \\ \overline{a}_{ij} &:= \max\{a_{ij}^M, a_{ij}^S\}, \\ \underline{a}_{ij} &:= \min\{a_{ij}^M, a_{ij}^S\}, \\ \tilde{a}_{ij} &:= \max\{|a_{ij}^M|, |a_{ij}^S|\}. \end{aligned}$$

Based on Definition 1 and the theory of differential inclusion, an ISDNN with nonlinear disturbance inputs model can be described by the following differential inclusion:

$$\begin{cases} \dot{x}_i(t) \in -\mathrm{co}\{d_i{}^M, d_i{}^S\}x_i(t) + \sum_{j=1}^{n} \mathrm{co}\{a_{ij}{}^M, a_{ij}{}^S\} \\ \qquad \times f_j(x_j(t)) + u_i(x_i(t)), \quad t \neq t_k, \\ x_i(t_k^+) = J_{ik}(x_i(t_k^-)), \quad t = t_k. \end{cases} \tag{4}$$

There exist measurable functions $\breve{d}_i(t) \in \mathrm{co}\{d_i{}^M, d_i{}^S\}$, $\breve{a}_{ij}(t) \in co\{a_{ij}{}^M, a_{ij}{}^S\}$ such that

$$\begin{cases} \dot{x}_i(t) = -\breve{d}_i(t)x_i(t) + \sum_{j=1}^{n} \breve{a}_{ij}(t) f_j(x_j(t)) \\ \qquad + u_i(x_i(t)), \quad t \neq t_k, \\ x_i(t_k^+) = J_{ik}(x_i(t_k^-)), t = t_k. \end{cases} \tag{5}$$

Set

$$D^M = \mathrm{diag}\{d_1^M, d_1^M, \dots, d_n^M\},$$
$$D^S = \mathrm{diag}\{d_1^S, d_2^S, \dots, d_n^S\},$$
$$\underline{D} = \mathrm{diag}\{\underline{d}_1, \underline{d}_2, \dots, \underline{d}_n\},$$
$$\breve{D}(t) = \mathrm{diag}\{\breve{d}_1(t), \breve{d}_2(t), \dots, \breve{d}_n(t)\},$$
$$A^M = (a_{ij}^M)_{n\times n},$$
$$A^S = (a_{ij}^S)_{n\times n}, \quad \breve{A}(t) = (\breve{a}_{ij}(t))_{n\times n},$$
$$\tilde{A} = (\tilde{a}_{ij})_{n\times n},$$
$$x(t) = (x_1(t), x_2(t), \dots, x_n(t))^T,$$
$$f(x(t)) = (f_1(x(t)), f_2(x(t)), \dots, f_n(x(t)))^T,$$
$$u(x(t)) = (u_1(x(t)), u_2(x(t)), \dots, u_n(x(t)))^T,$$
$$J_k(x(t)) = (J_{1k}(x(t)), J_{2k}(x(t)), \dots, J_{nk}(x(t)))^T.$$

Similarly to (4) and (5), we have

$$\begin{cases} \dot{x}(t) \in -\mathrm{co}\{D^M, D^S\}x(t) + \mathrm{co}\{A^M, A^S\} f(x(t)) \\ \qquad +u(x(t)), \quad t \neq t_k, \\ x(t_k^+) = J_k(x(t_k^-)), \quad t = t_k, \end{cases}$$

or, equivalently, there exist measurable functions $\breve{D}(t) \in \mathrm{co}\{D^M, D^S\}$, $\breve{A}(t) \in co\{A^M, A^S\}$ such that

$$\begin{cases} \dot{x}(t) = -\breve{D}(t)x(t) + \breve{A}(t) f(x(t)) \\ \qquad + u(x(t)), \quad t \neq t_k, \\ x(t_k^+) = J_k(x(t_k^-)), \quad t = t_k. \end{cases} \tag{6}$$

Based on (6), if the input feedback is quantized, then the closed-loop ISDNN with quantized measurements is as follows:

$$\begin{cases} \dot{x}(t) = -\breve{D}(t)x(t) + \breve{A}(t) f(x(t)) \\ \qquad +q_\mu(u(x(t))), \quad t \geq 0, \quad t \neq t_k, \\ x(t_k^+) = q_\upsilon(J_k(x(t_k^-))), \quad t = t_k, \end{cases} \tag{7}$$

where functions $u \in \mathscr{U}, J_k \in \mathscr{V}$ are continuous in $\mathbb{R}^n, (u, J_k)$ is the hybrid input, and μ and ν are zoom variables. We assume that there are some $\mathscr{K}_\infty$ class upper bounded functions $\bar{u}, \bar{J}_k$ such that

$$\|u(x)\| \leq \bar{u}(\|x\|), \quad \|J_k(x)\| \leq \bar{J}_k(\|x\|), \tag{8}$$

for all $x \in \mathbb{R}^n$. Actually, for the existence of the upper bounded functions, we can take

$$\bar{u}(s) = \max_{||x|| \leq s} ||u(x)||,$$

$$\bar{J}_k(s) = \max_{||x|| \leq s} ||J_k(x)||.$$

Rewrite the ISDNN system with quantized measurements (7) as

$$\begin{cases} \dot{x}(t) = -\breve{D}(t)x(t) + \breve{A}(t) f(x(t)) \\ \qquad +u(x) + F(u), \quad t \geq 0, \quad t \neq t_k, \\ x(t_k^+) = J_k(x^-) + G(J_k), \quad t = t_k, \end{cases} \tag{9}$$

where $F(u) = F(u(x)) = q_\mu(u(x)) - u(x), G(J_k) = G(J_k(x)) = q_\upsilon(J_k(x)) - J_k(x)$.

Remark 1. In previous results, some papers considered the stability and synchronization of SDNNs under impulses (Duan *et al.*, 2017; Yang *et al.*, 2016; Zhu *et al.*, 2018), but they all assumed that the transmission and exhibit processing of data have zero transmission delay and infinite precision. Because of the bandwidth constraint in SDNNs, the observations of the system state can only be transmitted to the controller through the finite rate network after quantization and coding, so that the influence of quantization and coding on the performance of the control system cannot be ignored. To the best of the authors' knowledge, the stability of SDNNs under quantization impulses has not been taken into account in previous papers. Therefore, the quantization technique is used in hybrid systems to quantify the external disturbances and impulses, which makes the results more practical.

Definition 2. (*Xu and Long, 2012*) For compact sets $\varphi, \phi \in \mathbb{R}^n$, $\{\varphi, \phi\}$ is called a quasi-invariant pair of (9) if, for any initial value $x_0 \in \varphi$, the solution $x(t, t_0, x_0)$ of (9) belongs to the set ϕ for $t \geq t_0$. In particular, the set φ is called an invariant set of (9), when $\varphi \equiv \phi$.

Definition 3. (*Xu and Long, 2012*) For $S, D \subset \mathbb{R}^n$, the set S is called an attracting set of the system (9) and D is called an attracting basin of S if, for any initial value $x_0 \in D$, the solution $x(t, t_0, x_0)$ converges to S as $t \to +\infty$, that is,

$$\operatorname{dist}(x(t, t_0, x_0), S) \to 0 \quad \text{as } t \to +\infty,$$

where

$$\operatorname{dist}(x, S) = \inf_{y \in S} \operatorname{dist}(x, y),$$
$$\operatorname{dist}(x, y) = \sup_{s \in (-\infty, 0]} |x(s) - y(s)|.$$

Definition 4. The ISDNN is called input-to-state stable (ISS) with the form of (9) if there exist functions $\xi \in \mathscr{K}\mathscr{L}$ and $\gamma \in \mathscr{K}_\infty$ such that, for every initial condition and every input (u, J_k), the solution corresponding to (9) exists globally and satisfies

$$|x(t)| \le \xi\big(|x_0|, t - t_0\big) + \gamma\big(\, \|(u, J_k)\|_{[t_0,t)}\,\big), \quad t \ge t_0,$$

for any forward complete solution $x(t) = x(t, t_0, x_0, u, J_k)$ of (9), where $\| \ (u, J_k)\ \|_{[t_0,t)}$ denotes the norm on the interval $[t_0, t)$ defined by

$$\|(u, J_k)\|_{[t_0,t)} := \max\Big\{\text{ess.} \sup_{s \in [t_0,t)} |u(s)|, \sup_{t_k \in [t_0,t)} |J_k(t_k)|\Big\}.$$

Lemma 1. (Huang *et al.*, 2018a) *The ISDNN (9) is ISS with respect to the disturbance $(F(u), G(J_k))$ if there exists a candidate exponential ISS-Lyapunov function $V : \mathbb{R}^n \to \mathbb{R}$ that satisfies*

$$\alpha(\|x\|) \le V(x) \le \beta(\|x\|), \tag{10}$$

$$\nabla V(x)\dot{x}(t) \le -cV(x) + \vartheta(\|F(u)\|), \tag{11}$$

$$V(x_k^+) \le e^{-d} V(x) + \vartheta(\|G(J_k)\|), \tag{12}$$

$$-dN(t, s) - (c - \lambda)(t - s) \le r, \quad t \ge s \ge 0, \tag{13}$$

where $x \in \mathbb{R}^n, \alpha, \beta, \vartheta \in \mathscr{K}_\infty, F(u) \in \mathscr{U}, G(J_k) \in \mathscr{V}, N(t, s)$ denotes the number of impulse instants t_k in $(s, t], c, d, r, \lambda$ are constants and $r, \lambda > 0$.

Remark 2. ISS is a stability problem based on perturbed systems, which was first proposed by Sontag (2002). Because some unexpected abrupt or instantaneous disturbances often occur in SDNNs (Wang *et al.*, 2016), it is interesting to consider the external disturbances and impulses and quantify them in this paper. However, few results have been reported in the literature.

Lemma 2. (Yu *et al.*, 2014) *Let P be an $n \times n$ symmetrical and positive definite matrix; then, for any $x(t)$ in $\mathbb{R}^n$, the following inequality holds:*

$$\lambda_{\min}(P) x^T(t) x(t) \le x^T(t) P x(t) \le \lambda_{\max}(P) x^T(t) x(t).$$

Lemma 3. (Yang *et al.*, 2011) *For any vectors $x(t), y(t) \in \mathbb{R}^n$, the inequality*

$$2x^T(t) y(t) \le x^T(t) Q x(t) + y^T(t) Q^{-1} y(t)$$

holds, in which Q is an $n \times n$ matrix or a constant with $Q > 0$.

Assumption 1. For $j = 1, 2, \dots, n$, the activation functions $f_j(u)$ are bounded and there exist nonnegative scalars σ_j such that, for $\forall x_1, x_2 \in \mathbb{R}, x_1 \ne x_2$,

$$0 \le \frac{f_j(x_1) - f_j(x_2)}{x_1 - x_2} \le \sigma_j.$$

Assumption 2. For $j = 1, 2, \dots, n$, the jump operators $J_{ik}(x)$ satisfy the Lipschitz condition with $J_{ik}(0) = 0$, i.e., there exist nonnegative scalars ρ_{ik} such that, for $\forall x_1, x_2 \in \mathbb{R}, x_1 \ne x_2$,

$$|J_{ik}(x_1) - J_{ik}(x_2)| \le \rho_{ik} |x_1 - x_2|.$$

Assumption 3. For $i = 1, 2, \dots, n$, the disturbance input $u_i(x_i(t))$ satisfies the Lipschitz condition with $u_i(0) = 0$, i.e., there exist nonnegative scalars L_i such that, for $\forall x_1, x_2 \in \mathbb{R}, x_1 \ne x_2$,

$$|u_i(x_1) - u_i(x_2)| \le L_i |x_1 - x_2|.$$

3. Main results

In this section, we analyze some properties like the quasi-invariant pair and attracting set for ISDNNs with quantized measurement under Assumptions 1–3, ensuring that the impulsive control systems (9) are globally asymptotically stable.

Theorem 1. *Under Assumptions 1–3, the ISDNN with quantized input (9) is ISS if there exist positive constants r, λ , positive definite matrix C, symmetric and positive definite matrix P such that*

$$(t - s)(\varrho^{-1} \ln w_k + \lambda - c) \le r, \quad t \ge s \ge 0, \tag{14}$$

where

$$c = -\lambda_{\max}\Big\{P\tilde{A}C^{-1}\tilde{A}^T + PC^{-1} + I - 2\underline{D} + \alpha^{-1}(\Lambda C \Lambda + \Xi C \Xi)\Big\}$$

with

$$\alpha = \lambda_{\min}(P), \quad \Lambda = \operatorname{diag}\{L_1, L_2, \cdots, L_n\},$$

$$\Xi = \operatorname{diag}\{\sigma_1, \sigma_2, \ldots, \sigma_n\},$$
$$w_k = \frac{\lambda_{\max}(\rho_k P \rho_k)}{\alpha}$$

with

$$\rho_k = \operatorname{diag}\{\rho_{1k}, \rho_{2k}, \ldots, \rho_{nk}\},$$
$$\varrho = \begin{cases} \rho, & 0 < w_k < 1, \\ \theta, & w_k \geq 1. \end{cases}$$

Proof. Consider the following candidate exponential ISS-Lyapunov functional:

$$V(x) = V(x(t)) = x^T(t)Px(t).$$

According to Lemma 2, we get the estimate

$$\alpha \, \|x(t)\| \leq V(x) \leq \beta \, \|x(t)\|,$$

where $\alpha = \lambda_{\min}(P), \beta = \lambda_{\max}(P)$, i.e., (10) holds.

Differentiating the functional $V(t)$ along the solution of the ISDNNs with quantized input (9), we have

$$\begin{aligned} \dot{V}(x) = &-2x^T(t)\breve{D}(t)Px(t) + 2f^T(x(t))\breve{A}^T(t)Px(t) \\ &+ 2u^T(x(t))Px(t) + 2F^T(u)Px(t). \end{aligned} \tag{15}$$

From Lemma 3 and Assumptions 1–3 it follows that

$$\begin{aligned} 2f^T(x(t))\breve{A}^T(t)Px(t) &\leq x^T(t)P\tilde{A}(t)C^{-1}\tilde{A}^T Px(t) \\ &\quad + f^T(x(t))Cf(x(t)) \\ &\leq x^T(t)P\tilde{A}(t)C^{-1}\tilde{A}^T Px(t) \\ &\quad + x^T(t)\Xi C\Xi x(t), \\ 2u^T(x(t))Px(t) &\leq u^T(x(t))Cu(x(t)) \\ &\quad + x^T(t)PC^{-1}Px(t) \\ &\leq x^T(t)\Lambda C\Lambda x(t) \\ &\quad + x^T(t)PC^{-1}Px(t), \\ 2F^T(u)Px(t) &\leq F^T(u)PF(u) + x^T(t)Px(t). \end{aligned}$$

From the above inequalities and (15), we obtain

$$\begin{aligned} \dot{V}(x) &= -2x^T(t)\breve{D}Px(t) + x^T(t)P\breve{A}C^{-1}\breve{A}^T Px(t) \\ &\quad + x^T(t)\Xi C\Xi x(t) + x^T(t)\Lambda C\Lambda x(t) + x^T(t)P \\ &\quad \times C^{-1}Px(t) + x^T(t)Px(t) + F^T(u)PF(u) \\ &\leq x^T(t)\Big(P\tilde{A}C^{-1}\tilde{A}^T P + \frac{\Xi C\Xi P}{\alpha} + \frac{\Lambda C\Lambda P}{\alpha} \\ &\quad + PC^{-1}P + P - 2\underline{D}P\Big)x(t) + F^T(u)PF(u) \\ &= \lambda_{\max}\Big\{P\tilde{A}C^{-1}\tilde{A}^T + PC^{-1} + I - 2\underline{D} \\ &\quad + \alpha^{-1}(\Lambda C\Lambda + \Xi C\Xi)\Big\}x^T(t)Px(t) \\ &\quad + \beta\|F(u)\|^2 \\ &= -cV(x) + \beta\|F(u)\|^2. \end{aligned} \tag{16}$$

When $t = t_k$,

$$\begin{aligned} V(x(t_k^+)) &= J_k^T(x(t_k^-))PJ_k^T(x(t_k^-)) \\ &\quad + G^T(J_k(x(t_k^-)))PG(J_k(x(t_k^-))) \\ &\leq x^T(t_k^-)\rho_k P\rho_k x(t_k^-) \\ &\quad + G^T(J_k(x(t_k^-)))PG(J_k(x(t_k^-))) \\ &\leq \frac{\lambda_{\max}(\rho_k P\rho_k)}{\alpha}x^T(t_k^-)Px(t_k^-) \\ &\quad + G^T(J_k(x(t_k^-)))PG(J_k(x(t_k^-))) \\ &= w_k V(x(t_k^-)) + \beta\|G(J_k)\|^2, \end{aligned} \tag{17}$$

where

$$w_k = \frac{\lambda_{\max}(\rho_k P\rho_k)}{\alpha},$$

which implies that (11) and (12) hold.

We know that

$$\begin{aligned} (\ln w_k)N(t,s) &- (c-\lambda)(t-s) \\ &\leq (t-s)(\varrho^{-1}\ln w_k + \lambda - c), \end{aligned}$$

where $N(t,s)$ is the number of impulse instants t_k in $(s,t]$. Combined with (14), we can obtain that (13) is valid.

In summary, all the conditions of Lemma 1 are satisfied, which gives us the desired result. ■

Proposition 1. *If all the conditions in Theorem 1 are valid and there exist positive constants Δ, M satisfying $a < b$, where*

$$\begin{aligned} a &= \beta\Delta^2\Big(\mu^2\lambda^{-1} + \upsilon^2(1-e^{-\lambda\theta})^{-1}\Big), \\ b &= \alpha e^{-r}\min\Big\{\bar{u}^{-1}(\mu M), \bar{J_k}^{-1}(vM)\Big\}, \end{aligned} \tag{18}$$

then $\{\Re(z^*), \Re(e^r z^*)\}$ is a quasi-invariant set pair of (9) for any $z^* \in (a,b]$.

Proof. For any initial condition $x(t_0) = x_0$, the solution $x(t) = x(t, t_0, x_0)$ exists at least locally for the ISDNN (9). Let $\Omega = [t_0, t_m)$ be the maximum existing interval, where $t_0 < t_m \leq \infty$. Take $V(t) = V(x(t)) = x^T(t)Px(t)$, and from (16) and (17), we have

$$\begin{cases} \dot{V}(x) \leq -cV(x) + \beta\,\|F(u)\|^2, & t \neq t_k, \\ V(x_k^+) \leq w_k V(x) + \beta\,\|G(J_k)\|^2, & t = t_k. \end{cases}$$

By using an impulsive-type comparison result

(Lakshmikantham *et al.*, 1989), we have

$$V(x(t)) \le \Phi(t,t_0)V(x(t_0)) + \beta\int_{t_0}^{t}\Phi(t,s)\,\|F(u(s))\|^2\ \mathrm{d}s + \beta\sum_{t_0<t_k\le t}\Phi(t,t_k)\,\|G(J_k(t_k^-))\|^2,$$

where $\Phi(t,s)$ is the Cauchy matrix of

$$\begin{cases}\dot\Phi(t) = -c\Phi(t), & t \ne t_k\\ \Phi(t_k^+) = e^{\ln w_k}\Phi(t), & t = t_k.\end{cases}$$

From the condition (14), it follows that

$$\begin{aligned}\Phi(t,s) &= e^{(\ln w_k)N(t,s)-c(t-s)}\\ &\le e^{(t-s)(\varrho^{-1}\ln w_k - c)}\\ &\le e^r e^{-\lambda(t-s)},\end{aligned}$$

for $t_0 \le s \le t$. Hence, we get

$$\begin{aligned}&V(x(t))\\ &\le e^r e^{-\lambda(t-t_0)}V(x(t_0))\\ &\quad + \beta e^r\int_{t_0}^{t} e^{-\lambda(t-s)}\,\|F(u(s))\|^2\ \mathrm{d}s\\ &\quad + \beta e^r\sum_{t_0<t_k\le t} e^{-\lambda(t-t_k)}\,\|G(J_k(t_k^-))\|^2 .\end{aligned}\tag{19}$$

From $a < b$, there exists a sufficiently small constant $\varepsilon > 0$ such that $z := z^* - \varepsilon < b$, which satisfies

$$z - a > 0.\tag{20}$$

We claim that if $V(t_0) \le z$, then

$$V(t) \le e^r z, \quad t \ge t_0.\tag{21}$$

Otherwise, since $V(t)$ is a piecewise continuous function, there must be a $t^* > t_0$ such that

$$V(t^*) \ge e^r z, \quad V(t) \le e^r z, \quad t_0 \le t < t^*.\tag{22}$$

From (8), (10) and (22) it follows for $t_0 \le t < t^*$, that

$$\begin{aligned}\|u(x(t))\| &\le \bar u(\,\|x(t)\|\,)\\ &\le \frac{\bar u}{\alpha}V(x) \le \frac{\bar u}{\alpha}(e^r b) \le \mu M,\\ \|J_k(x(t))\| &\le \bar J_k(\,\|x(t)\|\,)\\ &\le \frac{\bar J_k}{\alpha}V(x) \le \frac{\bar J_k}{\alpha}(e^r b) \le \upsilon M.\end{aligned}$$

From the definition of quantizers, we can easily obtain that

$$\begin{aligned}\|q_\mu(u(x(t))) - u(x(t))\| &\le \mu\Delta,\\ \|q_\upsilon(J_k(x(t))) - J_k(x(t))\| &\le \upsilon\Delta\end{aligned}\tag{23}$$

for $t_0 \le t < t^*$. From the inequality (2) and $\lambda > 0$,

$$\begin{aligned}&\sum_{t_0<t_k\le t} e^{-\lambda(t-t_k)}\\ &\le \sum_{t_0<t_k\le t} e^{-\lambda(t-t_k)}\\ &\quad\times(1-e^{-\lambda(t_k-t_{k-1})})(1-e^{-\lambda\theta})^{-1}\\ &\le (1-e^{-\lambda(t-t_0)})(1-e^{-\lambda\theta})^{-1}.\end{aligned}\tag{24}$$

Combining (19), (20), (23) and (24), we get

$$\begin{aligned}V(t^*) &\le e^r e^{-\lambda(t^*-t_0)}V(t_0)\\ &\quad + \beta e^r\int_{t_0}^{t^*} e^{-\lambda(t^*-s)}\,\|F(u(s))\|^2\ \mathrm{d}s\\ &\quad + \beta e^r\sum_{t_0<t_k\le t} e^{-\lambda(t^*-t_k)}\,\|G(J_k)\|^2\\ &\le e^r e^{-\lambda(t^*-t_0)}V(t_0)\\ &\quad + \beta e^r(\mu\Delta)^2\int_{t_0}^{t^*} e^{-\lambda(t^*-s)}\,\mathrm{d}s\\ &\quad + \beta e^r(\upsilon\Delta)^2(1-e^{-\lambda(t^*-t_0)})(1-e^{-\lambda\theta})^{-1}\\ &\le e^r e^{-\lambda(t^*-t_0)}\Big[z - \frac{\beta(\mu\Delta)^2}{\lambda}\\ &\quad - \beta(\upsilon\Delta)^2(1-e^{-\lambda\theta})^{-1}\Big]\\ &\quad + e^r\Big(\frac{\beta(\mu\Delta)^2}{\lambda} + \beta(\upsilon\Delta)^2(1-e^{-\lambda\theta})^{-1}\Big)\\ &\le e^r z,\end{aligned}$$

which contradicts (22). This implies that the assertion of (21) holds. Assuming $\varepsilon \to 0$, from the estimate (21), we arrive at the solution $x(t) \in \mathfrak{R}(e^r z)$. The proof is completed. ∎

Proposition 2. *If all the conditions in Theorem 1 are valid, then $\mathfrak{R}(b)$ is the attracting basin and $\mathfrak{R}(e^r a)$ is the attracting set of (9). Furthermore, for any $\varepsilon > 0$, all trajectories of (9) that start in $\mathfrak{R}(b)$ enter $\mathfrak{R}((a+\varepsilon)e^r)$ in a finite time, which is less than or equal to*

$$T := \max\left\{0, \frac{\ln(b-a) - \ln\varepsilon}{\lambda}\right\}.\tag{25}$$

Proof. For any given $x_0 \in \mathfrak{R}(b)$, according to Theorem 1, we obtain $V(t) \le e^r b, t \ge t_0$. Then $\| u(x(t)) \| \le \mu M$,

$\|J_k(x(t))\| \leq vM$. Namely, (23) holds for $t \geq t_0$. Thus,

$$\begin{aligned} V(t) &\leq e^r e^{-\lambda(t-t_0)} V(x(t_0)) \\ &\quad + \beta e^r \int_{t_0}^{t} e^{-\lambda(t-s)} \|F(u(s))\|^2 \, \mathrm{d}s \\ &\quad + \beta e^r \sum_{t_0 < t_k \leq t} e^{-\lambda(t-t_k)} \|G(J_k(t_k^-))\|^2 \\ &\leq b e^r e^{-\lambda(t-t_0)} + \beta e^r (\mu\Delta)^2 \int_{t_0}^{t^*} e^{-\lambda(t^*-s)} \, \mathrm{d}s \\ &\quad + \beta e^r (v\Delta)^2 (1-e^{-\lambda(t^*-t_0)})(1-e^{-\lambda\theta})^{-1} \\ &\leq e^r(b-a)e^{-\lambda(t-t_0)} + e^r a \\ &\leq e^r(b-a)e^{-\lambda T} + e^r a \\ &\leq (a+\varepsilon)e^r, \quad t \geq t_0. \end{aligned}$$

Let $\varepsilon \to 0$. We get $x(t) \in \mathfrak{R}(e^r a)$, which implies the conclusion. ■

Remark 3. These two properties tell us that there are two nested invariant regions in SDNNs, allowing all tracks of the quantization system (9) starting from the larger region to enter the smaller region in a finite time. However, it is not sure that further convergence can be achieved. Then we will adopt an adjustable quantizer to discuss further whether the system can achieve global asymptotic stability.

Theorem 2. *If all the conditions in Theorem 1 hold, the quantized range M and quantized error Δ satisfy*

$$\beta(\mu\Delta)^2[\lambda^{-1} + (1-e^{-\lambda\theta})^{-1}] < e^{-2r}\alpha(\eta(\mu M)), \quad (26)$$

where $\eta(s) := \min\left\{\bar{u}^{-1}(s), \bar{J}_k^{-1}(s)\right\} \in \mathscr{K}_\infty$. Then there exists a hybrid quantized input feedback control scheme guaranteeing the system (9) is globally asymptotically stable.

Proof. For simplicity, we set $\mu(t) = v(t)$. Note that μ is a piecewise constant function and the parameters given in (18) are

$$\begin{aligned} a(\mu) &= \beta(\mu\Delta)^2\Big[\lambda^{-1} + (1-e^{-\lambda\theta})^{-1}\Big], \\ b(\mu) &= e^{-r}\alpha(\eta(\mu M)). \end{aligned} \quad (27)$$

From (26), we take

$$\varepsilon(\mu) = \varepsilon_0(e^{-r}b(\mu) - a(\mu)) > 0, \quad (28)$$

where $0 < \varepsilon_0 < 1$. Thus, T given in (25) satisfies

$$T(\mu) = \frac{\ln[b(\mu) - a(\mu)] - \ln\varepsilon(\mu)}{\lambda} > \frac{-\ln(\varepsilon_0)}{\lambda} > 0.$$

The control law is determined in the stages that follow.

Stage 1. (Zooming-out) In this stage, take $u(x) = J_k(x) = 0$ so that the system (9) becomes open-loop. Our aim is to make the system state $x(t, 0, x_0)$ enter the quantized control domain by increasing μ. Let τ be a positive number. Set $\mu(t) = 1$ for $[0, \tau)$. For $t \in [(k-1)\tau, k\tau), k \in \mathcal{N}$, we take

$$\mu(t) = \frac{1}{M}\eta^{-1}(\alpha^{-1}(e^r(\beta \max_{\|z\|, s \leq k\tau} \|\varsigma(s, 0, z)\|))), \quad (29)$$

where $\varsigma(t, 0, z)$ is the solution of impulsive system (9) with $u(x) = J_k(x) = 0$. For any given x_0, there must be $t_0 \in [(k-1)\tau, k\tau)$ such that

$$\|x(t_0, 0, x_0)\| \leq \max_{\|z\|, s \leq k\tau} \| \varsigma(s, 0, z) \| .$$

From (10), (27) and (29), we have

$$\begin{aligned} V(x(t_0, x, x_0)) &\leq \beta \, \|x(t_0, 0, x_0)\| \\ &\leq e^{-r}\alpha(\eta(M\mu(t_0))) \\ &= b(\mu(t_0)). \end{aligned} \quad (30)$$

Then, $x(t_0, 0, x_0) \in \mathfrak{R}(b(\mu(t_0)))$, and so the system state enters the quantized control domain. Hence a controller can effectively control the state.

Stage 2. (Zooming-in) Let u and J_k satisfy the conditions in (8). Then (9) becomes closed-loop.

For $t \in [t_0, t_0 + T_0)$, let $\mu(t) = \mu_0 = \mu(t_0)$, where $T_0 = T(\mu_0)$ is given by (25). We know that $x(t_0) \in \mathfrak{R}(b(\mu_0))$, according to Propositions 1 and 2 with $a = a(\mu_0), b = b(\mu_0), \varepsilon = \varepsilon(\mu_0)$ in (27) and (28); we have $x(t) \in \mathfrak{R}(e^r(b(\mu_0)))$ and $x(t_0 + T_0) \in \mathfrak{R}(e^r(a(\mu_0) + \varepsilon(\mu_0))))$.

For $t \in [t_0 + T_0, t_0 + T_0 + T_1)$, let

$$\mu(t) = \mu_1 = \Pi(\mu_0), \quad (31)$$

where $T_1 = T(\mu_1)$ and

$$\Pi(\mu_0) = \frac{1}{M}\eta^{-1}(\alpha^{-1}(e^{2r}(a(\mu_0) + \varepsilon(\mu_0)))).$$

In view of (27), we obtain

$$\Pi(\mu_0) = b^{-1}(e^r(a(\mu_0) + \varepsilon(\mu_0))). \quad (32)$$

From $0 < \varepsilon_0 < 1$ in (28), we arrive at

$$\begin{aligned} &b^{-1}\Big(e^r\Big(a(\mu_0) + \varepsilon(\mu_0)\Big)\Big) \\ &\quad < b^{-1}\Big[e^r\Big(a(\mu_0) + \frac{\varepsilon(\mu_0)}{\varepsilon_0}\Big)\Big] = u_0, \end{aligned}$$

which implies $\Pi(\mu_0) < \mu_0$. By (31) and (32), we get

$$\mu(t) = \mu_1 < \mu_0, \quad t \in [t_0 + T_0, t_0 + T_0 + T_1).$$

Hence, we have $\mu(t_0 + T) < \mu(t_0)$. Combining this with (31) and (32), we have $\Re(b(\mu_1)) = \Re(b(\Pi(\mu_0))) = \Re(e^r(a(\mu_0)+\varepsilon(\mu_0)))$, which means that we can continue the analysis for $t \geq t_0 + T$ as before. Then $x(t_0 + T_0) \in \Re(b(\mu_1))$, using Propositions 1 and 2, we obtain that $x(t) \in \Re(e^r(b(\mu_1)))$ and $x(t_0+T_0+T_1) \in \Re(e^r(a(\mu_1)+\varepsilon(\mu_1)))$. For $t \in [t_0 + T_0 + \cdots + T_{k-1}, t_0 + T_0 + \cdots + T_k), k \in \mathcal{N}$, let $\mu(t) = \mu_k = \Pi(\mu_{k-1})$, where $T_k = T(\mu_k)$ and

$$\begin{aligned}\Pi(s) &= \frac{1}{M}\eta^{-1}(\alpha^{-1}(e^{2r}(a(s) + \varepsilon(s)))) \\ &= b^{-1}(e^r(a(s) + \varepsilon(s))), \quad s \geq 0.\end{aligned}$$

Repeating the process, we obtain that $\Re(b(\mu_k)) = \Re((a(\mu_{k-1}) + \varepsilon(\mu_{k-1}))e^r)$ and hence $x(t_0 + T_0 + \cdots + T_{k-1}) \in \Re(b(\mu_k))$. Applying Propositions 1 and 2, we obtain that $x(t) \in \Re(e^r b(\mu_k))$ and $x(t_0+T_0+\cdots+T_k) \in \Re((a(\mu_k)+\varepsilon(\mu_k))e^r)$. From $0 < \varepsilon_0 < 1$ in (28), we have

$$\begin{aligned}\Pi(s) &= b^{-1}(e^r(a(s) + \varepsilon(s))) \\ &\leq b^{-1}(e^r(a(s) + \frac{\varepsilon(s)}{\varepsilon_0})) = s,\end{aligned}$$

which implies $0 < \Pi(s) < s$ for $s \geq 0$. Then $\mu_k \to 0$ as $k \to \infty$, which implies $x(t) \to 0$ as $t \to \infty$ from $T_k > (-\ln \varepsilon_0)/\lambda > 0$. Therefore, the conclusion is easily obtained. ■

Remark 4. In fact, the scaling of μ is performed at $t = t_0 + T_0, t_0 + T_0 + T_1, \ldots$, which is not the only set of time series. We can replace it with any set of time series $t_1, t_2, \ldots$ satisfying $t_{i+1} - t_i \geq T_i, \quad i \geq 0$. We can also obtain $\mu(t) \to 0$ as $t \to 0$.

Remark 5. Recently, Wang *et al.* (2016) studied the stability criteria for impulsive MNNs by constructing a Lyapunov–Krasovskii-type functional. In order to improve the practicability of the obtained results, we consider the bandwidth constraint problem caused by too many nodes and too much data transmission. We add quantization control to external inputs and impulses. By discretely adjusting the quantization controller, we obtain that ISDNNs are ISS and globally asymptotically stable.

4. Numerical example

Consider the nonlinear ISDNN with nonlinear disturbance inputs

$$\begin{cases} \dot{x}_1(t) = -d_1(x_1(t))x_1(t) + a_{11}(x_1(t))f_1(x_1(t)) \\ \qquad +a_{12}(x_1(t))f_2(x_2(t)) + u_1(x_1(t)), \quad t \neq t_k, \\ \dot{x}_2(t) = -d_2(x_2(t))x_2(t) + a_{21}(x_2(t))f_1(x_1(t)) \\ \qquad +a_{22}(x_2(t))f_2(x_2(t)) + u_2(x_2(t)), \quad t \neq t_k, \\ x_i(t_k^+) = J_{ik}(x_i(t_k^-)), \quad k \in \mathcal{N}^+, \quad t = t_k, \end{cases} \tag{33}$$

where $i = 1, 2$, $f_i(x) = \tanh(x)$, and

$$d_1(x_1) = \begin{cases} 1.5, & |x_1| \leq 1, \\ 1.1, & |x_1| > 1, \end{cases}$$

$$d_2(x_2) = \begin{cases} 1.1, & |x_2| \leq 1, \\ 1.5, & |x_2| > 1, \end{cases}$$

$$a_{11}(x_1) = \begin{cases} 0.2, & |x_1| \leq 1, \\ -0.2, & |x_1| > 1, \end{cases}$$

$$a_{12}(x_1) = \begin{cases} 0.25, & |x_1| \leq 1, \\ -0.25, & |x_1| > 1, \end{cases}$$

$$a_{21}(x_2) = \begin{cases} \frac{1}{6}, & |x_2| \leq 1, \\ -\frac{1}{6}, & |x_2| > 1, \end{cases}$$

$$a_{22}(x_2) = \begin{cases} \frac{1}{8}, & |x_2| \leq 1, \\ -\frac{1}{8}, & |x_2| > 1, \end{cases}$$

We can obtain

$$\underline{D} = \begin{pmatrix} 1.1 & 0 \\ 0 & 1.1 \end{pmatrix}$$

and

$$\tilde{A} = \begin{pmatrix} \frac{1}{5} & \frac{1}{4} \\ \frac{1}{6} & \frac{1}{8} \end{pmatrix}.$$

Let $u_i(x_i(t)) \equiv 0, J_{ik}(x_i(t_k)) \equiv 0$. Under these conditions, the system (33) without disturbance input and impulse effects is stable, which can be seen in Figs. 1–3.

When we consider nonlinear disturbance inputs and impulsive effects, let $u_i(x_i(t)) = \sin(x_i(t)), J_{ik}(x_i(t_k)) = \operatorname{atan}(x_i(t_k))$. We choose $t_k - t_{k-1} = 0.1$, so that $\theta = \rho = \varrho = 0.1$. We can see that the system (33) is unstable when we add impulses and disturbance inputs in Figs. 4–6.

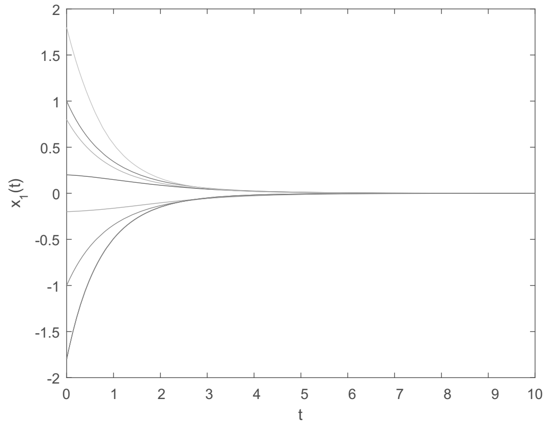

Fig. 1. State $x_1(t)$ of the ISDNN (33) without external disturbances and impulses.

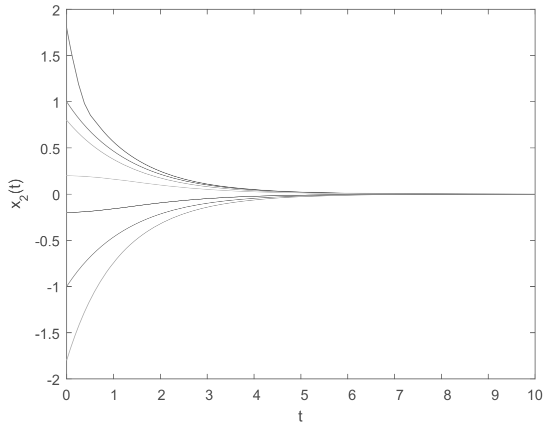

Fig. 2. State $x_2(t)$ of the ISDNN (33) without external disturbances and impulses.

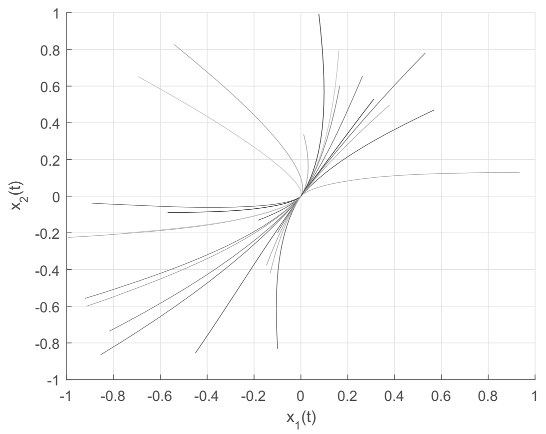

Fig. 3. Phase curves of the SDNN (33) without external disturbances and impulses at different initial values.

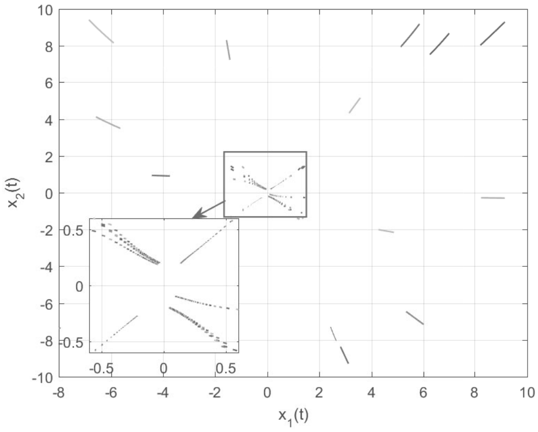

Fig. 4. Phase curves of the ISDNN (33) with nonlinear external disturbances and generated impulsive switching at $t = 0.1k, k \in \mathcal{N}$, under different initial values.

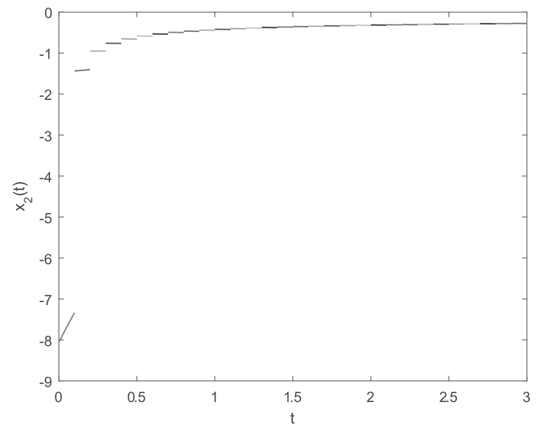

Fig. 5. State $x_1(t)$ of the ISDNN (33) generates impulsive switching at $t = 0.1k, k \in N$.

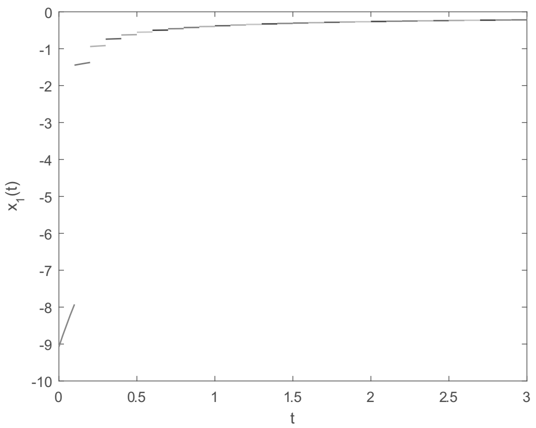

Fig. 6. State $x_2(t)$ of the ISDNN (33) generates impulsive switching at $t = 0.1k, k \in N$.

Now we consider the quantized hybrid input feedback $q_\mu(u_i(x_i(t)))$ and $q_\nu(J_{ik}(x_i(t)))$. Choose the quantizer $q_\mu(z)$

$$q_\mu(z) = \begin{cases} \mu M\Delta, & \frac{z}{\mu} > (M+0.5)\Delta, \\ -\mu M\Delta, & \frac{z}{\mu} < -(M+0.5)\Delta, \\ \mu\Delta[\frac{z}{\mu\Delta}], & -(M+0.5)\Delta \le \frac{z}{\mu} \le (M+0.5)\Delta, \end{cases}$$

where $M, \Delta > 0$, $[\cdot]$ is the rounding operation. It can be seen that M is the quantization range and Δ is te quantization error of the quantizer according to definition in Section 2.

Set $P = C = I, \alpha = \beta = 1, \vartheta(s) = s^2, \rho_{ik} = 0.9, L_i = \sigma_i = 1, \lambda = 0.1, r = 0, i = 1, 2, k \in N$. After calculation we get $\Lambda = \Xi = I, c = -1.944, w_k = 0.81$. We obtain

$$\begin{aligned} &(t-s)(\varrho^{-1}\ln w_k + \lambda - c) \\ &= 1 \times \Big(\frac{\ln(0.81)}{0.1} + 0.1 + 1.944\Big) = -0.0630 \le 0. \end{aligned}$$

We can verify that (14) holds in Theorem 1; then the ISDNN with quantized input is ISS. This shows that we can still prove that the system (9) with quantized measurements is ISS.

Taking $M = 2, \Delta = 0.1, \mu = \nu = 1$, we can calculate the parameters $a = 1.1050, b = 2$, which were used in Theorem 2. Then from Propositions 1 and 2, $\{\mathfrak{R}(z^*), \mathfrak{R}(z^*)\}$ is a quasi-invariant set pair of (33) for any $z^* \in (1.1051, 2]$ and all the trajectories starting in $\mathfrak{R}(1.1050)$ approach $\mathfrak{R}$ (2).

We take the hybrid quantized control policy by making discrete on-line adjustments of the quantizer parameters in Theorem 2. We set $\bar{u}_i(s) = s, \bar{J}_{ik}(s) = s$, which implies that (8) holds and $\eta(s) = s$. In the zooming-out stage, the system is open-loop and the zoom variable follows the formulation (29). When $t_0 = 0.2375$ and $\mu(t_0) = 1.5, x(t_0, 0, x_0) \in \mathfrak{R}$ (3) and the system state enters the quantized control domain. Then the first control stage is over at $t_0 = 0.2375$. In the zooming-in stage, the flexible quantizer $q_\mu(u_i(x))$ and $q_\nu(J_{ik}(x))$ with adjustable parameters gives

$$\begin{aligned} \mu(t) = \nu(t) = \mu_k &= \Pi(\mu_k) \\ &= \frac{1}{M}(a(\mu_{k-1}) + \varepsilon(\mu_{k-1})) \\ &= \frac{1}{M}(\varepsilon_0 b(\mu_{k-1}) + (1-\varepsilon_0)a(\mu_{k-1}) \\ &= 0.95\mu_{k-1} + 0.0276\mu_{k-1}^2, \\ &t \in [t_0 + T_0 + \cdots + T_{k-1}, t_0 + T_0 + \cdots + T_k], \end{aligned}$$

where $a(\mu_k) = 1.1050\mu_k^2, b(\mu_k) = 2\mu_k, \varepsilon(\mu_k) =$

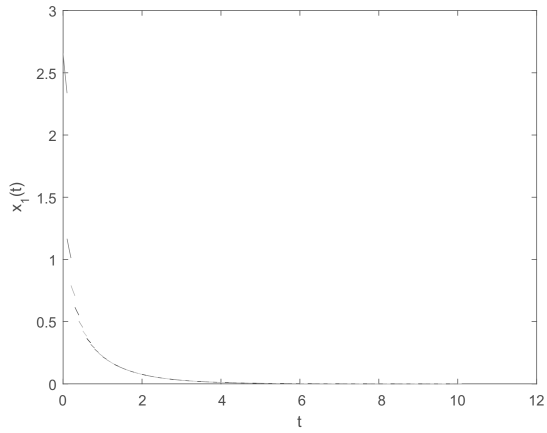

Fig. 7. State $x_1(t)$ of the system ISDNN (33) quantized by external disturbances and impulses.

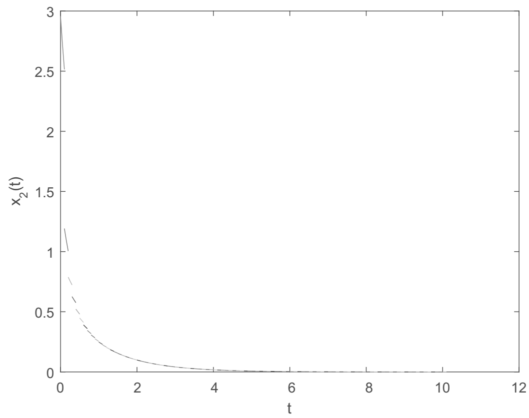

Fig. 8. State $x_2(t)$ of the system ISDNN (33) quantized by external disturbances and impulses.

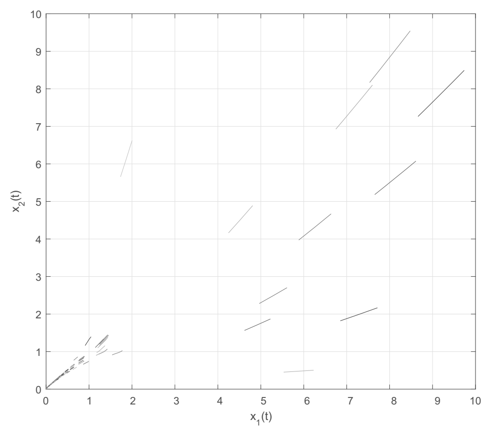

Fig. 9. Phase curves of the system ISDNN (33) quantized by external disturbances and impulses at different initial values.

$0.95\big(b(\mu_k)-a(\mu_k)\big)$ with $\varepsilon_0 = 0.95, \varepsilon_0 = 0.95, a(\mu_k) = 1.1078\mu_{k-1} + 0.00076\mu_{k-1}^2, b(\mu_k) = 1.9\mu_{k-1} + 0.5525\mu_{k-1}^2$, which implies $a(\mu_k) < e^{-r}b(\mu_k), T_k > -\ln(\varepsilon_0)/\lambda = 0.5129$, satisfying the condition (26). Figures 7–9 show the asymptotical stability behavior of quantized impulsive system.

5. Conclusion

By using the concepts of a quasi-invariant set and attraction, we considered the stability of ISDNNs in the case of quantization of nonlinear external inputs. Based on the ISS-Lyapunov function, differential inclusion theories and some inequality techniques, we obtain the ISS stability theorem and the global asymptotic stability theorem for ISDNNs with nonlinear impulsive controllers.

This paper does not discuss the impact of time delay on the ISDNNs. The time delay will make the system unstable. Therefore, we can consider the stability and synchronization of ISDNNs under the time-varying delay using quantized feedback.

Acknowledgment

This work was supported by the National Natural Science Foundation of China under the grant 61573005 and the Natural Science Foundation of the Fujian Province under the grants 2018J01417 and 2019J01330.

References

Aubin, J.P. and Cellina, A. (1984). *Differential Inclusions: Set-Valued Maps and Viability Theory*, Springer-Verlag, Berlin.

Chang, X., Zhang, X. and Yan, L. (2013). Synchronization of complex dynamical networks via sampled-data quantized feedback, *32nd Chinese Control Conference, Xi'an, China*, pp. 6531–6536.

Chua, L.C. (1971). Memristor—The missing circuit element, *IEEE Transactions on Circuit Theory* **18**(5): 507–519.

Duan, S., Wang, H., Wang, L., Huang, T. and Li, C. (2017). Impulsive effects and stability analysis on memristive neural networks with variable delays, *IEEE Transactions on Neural Networks and Learning Systems* **28**(2): 476–481.

Fan, Y., Wang, W. and Jiang, X. (2016). Synchronization for chaotic Lur'e systems based on adaptive quantized state measurements control design, *35th Chinese Control Conference, Chengdu, China*, pp. 913–915.

Guo, Z., Liu, L. and Wang, J. (2018). Multistability for recurrent neural networks with piecewise-linear radial basis functions and state-dependent switching parameters, *IEEE Transactions on Systems, Man and Cybernetics: Systems* **30**(7): 2052–2066.

Hao, Z., Yan, H., Yang, F. and Chen, Q. (2011). Quantized control design for impulsive fuzzy networked systems, *IEEE Transactions on Fuzzy Systems* **19**(6): 1153–1162.

Hong, Y., Bin, H. and Huang, Z. (2019). Synchronization of state-switching Hopfield-type neural networks: A quantized level set approach, *Chaos, Solitons and Fractals* **129**: 16–24.

Huang, C., Su, R., Cao, J. and Xiao, S. (2019). Asymptotically stable high-order neutral cellular neural networks with proportional delays and d operators, *Mathematics and Computers in Simulation* **5**(155): 41–56.

Huang, Z., Bin, H., Cao, J. and Wang, B. (2018a). Synchronization networks with proportional delays based on a class of q-type allowable time scales, *IEEE Transactions on Neural Networks and Learning Systems* **29**(8): 3418–3428.

Huang, Z., Cao, J. and Raffoul, Y.N. (2018b). Hilger-type impulsive differential inequality and its application to impulsive synchronization of delayed complex networks on time scales, *Science China (Information Sciences)* **61**(7): 244–246.

Lakshmikantham, V., Bainov, D.D. and Simeonov, P.S. (1989). *Theory of Impulsive Differential Equations*, World Scientific, Singapore.

Qian, Y., Cui, B. and Lou, X. (2012). Quantized feedback control for hybrid impulsive systems, *12th International Conference on Control, Automation and Systems, JeJu Island, South Korea*, pp. 1–6.

Sontag, E.D. (2002). Smooth stabilization implies coprime factorization, *IEEE Transactions on Automatic Control* **34**(4): 435–443.

Sun, H., Hou, L., Zong, G. and Yu, X. (2019). Fixed-time attitude tracking control for spacecraft with input quantization, *IEEE Transactions on Aerospace and Electronic Systems* **55**(1): 124–134.

Wan, Y., Cao, J. and Wen, G. (2017). Quantized synchronization of chaotic neural networks with scheduled output feedback control, *IEEE Transactions on Neural Networks and Learning Systems* **28**(11): 2638–2647.

Wang, H., Duan, S., Huang, T., Li, C. and Wang, L. (2016). Novel stability criteria for impulsive memristive neural networks with time-varying delays, *Circuits Systems and Signal Processing* **35**(11): 3935–3956.

Wu, Z.G., Yong, X., Pan, Y.J., Su, H. and Yang, T. (2018). Event-triggered control for consensus problem in multi-agent systems with quantized relative state measurements and external disturbance, *IEEE Transactions on Circuits and Systems I: Regular Papers* **65**(7): 2232–2242.

Xu, D. and Long, S. (2012). Attracting and quasi-invariant sets of non-autonomous neural networks with delays, *Neurocomputing* **77**(1): 222–228.

Yang, D., Qiu, G. and Li, C. (2016). Global exponential stability of memristive neural networks with impulse time window and time-varying delays, *Neurocomputing* **171**: 1021–1026.

Yang, X., Cao, J. and Lu, J. (2011). Synchronization of delayed complex dynamical networks with impulsive and stochastic effects, *Nonlinear Analysis Real World Applications* **12**(4): 2252–2266.

Yang, X., Qiang, S., Cao, J. and Lu, J. (2019). Synchronization of coupled Markovian reaction–diffusion neural networks with proportional delays via quantized control, *IEEE Transactions on Neural Networks and Learning Systems* **30**(3): 951–958.

Yu, J., Cheng, H., Jiang, H. and Teng, Z. (2014). Stabilization of nonlinear systems with time-varying delays via impulsive control, *Neurocomputing* **125**(3): 68–71.

Zhang, G. and Shen, Y. (2015). Novel conditions on exponential stability of a class of delayed neural networks with state-dependent switching, *Neural Networks* **71**: 55–61.

Zhang, W., Yang, S., Li, C. and Li, Z. (2019). Finite-time and fixed-time synchronization of complex networks with discontinuous nodes via quantized control, *Neural Processing Letters* **6**: 1–14.

Zhu, W., Wang, D., Liu, L. and Feng, G. (2018). Event-based impulsive control of continuous-time dynamic systems and its application to synchronization of memristive neural networks, *IEEE Transactions on Neural Networks and Learning Systems* **29**(8): 3599–3609.

Yaxian Hong received a BS degree in mathematics from Jimei University, Xiamen, China, in 2017. She is currently pursing an MS degree with the School of Science, Jimei University. Her present research interests include complex and state-dependent neural networks.

Honghua Bin received a BS degree in mathematics from Huazhong Normal University, Wuhan, China, in 1987, an MS degree in mathematics from Hunan University, Changsha, China, in 2002, and a PhD degree in mathematics from Hunan University in 2006. She is currently a professor and the dean of the School of Sciences, Jimei University Xiamen, China. She has authored or co-authored over 30 papers in refereed journals. Her present research interests include complex networks, switched systems, and nonlinear systems.

Zhenkun Huang received a BS degree in mathematics from Jimei University, Xiamen, China, in 2001, an MS degree in mathematics from Fuzhou University, China, in 2004, and a PhD degree in mathematics from Zhejiang University, Hangzhou, China, in 2007. He is currently a professor with the School of Science, Jimei University. He has published over 15 papers in refereed journals. His present research interests include nonlinear systems, neural networks, and stability analysis in dynamic systems including continuous, discrete, and impulsive systems.

Received: 6 June 2019
Revised: 28 September 2019
Re-revised: 8 November 2019
Accepted: 25 January 2020