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DISCRETE–TIME OUTPUT OBSERVERS FOR BOUNDARY CONTROL SYSTEMS

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The paper studies the output observer design problem for a linear infinite-dimensional control plant modelled as an abstract boundary input/output control system. It is known that such models lead to an equivalent state space description with unbounded control (input) and observation (output) operators. For this class of infinite-dimensional systems we use the Cayley transform to approximate the sophisticated infinite-dimensional continuous-time model by a discrete-time infinite-dimensional one with all involved operators bounded. This significantly simplifies mathematical aspects of the observer design procedure. As is well known, the essential feature of the Cayley transform is that it preserves various system theoretic properties of the control system model, which may be useful in analysis. As an illustration, we consider an example of designing an output observer for the one-dimensional heat equation with measured controls (inputs) in the Neumann boundary conditions, measured outputs in the Dirichlet boundary conditions and an unmeasured output at a fixed point within the domain. Numerical simulations of this example show that the interpolated continuous-time signal, obtained from the discrete-time observer, can be successfully used for tracking the continuous-time plant output.

Keywords: boundary control systems, output observers, infinite-dimensional discrete-time systems.

1. Introduction

The aim of this paper is to develop an efficient method of designing output observers for linear infinite-dimensional control systems described by an abstract boundary input/output model. Here by an output observer we mean an observer which makes use of available (measured) plant inputs and outputs to track asymptotically another plant output which is unavailable for measurement. Such observers are also called functional observers (Trinh and Fernando, 2012).

The output or state observer for a linear, infinite-dimensional control system is usually an infinite-dimensional system and its design is mathematically very intricate. However, a practical implementation of this type of observer almost always relies on finite-dimensional approximations. The are two main ways to approach the problem. One is to develop an observer as an infinite-dimensional system, e.g., in the state space or boundary input/output form, and then apply finite-dimensional approximations, both in spatial and temporal variables. As a starting point, this general approach requires a sophisticated theory on how to design an infinite-dimensional observer and one can use various methods described in the literature. In the case of boundary control systems or, more precisely, systems described by partial differential equations with boundary control and observation, the popular backstepping method seems to be most successful (e.g., Smyshlayev and Krstic, 2005; 2008; Hasana *et al.*, 2016). More direct methods, exploring the relation between the boundary control models and the more familiar state space models or simply dealing with the latter can be found in the works of Demetriou and Rosen (2005),Vries *et al.* (2010), Demetriou (2013), Ferrante *et al.* (2020), Emirsajłow (2012; 2020; 2021) and the references cited therein. For a short survey on the observer design methods for infinite-dimensional control systems, see the work of Hidayat *et al.* (2011).

Another way to approach the observer design problem is to start with an approximation of the infinite-dimensional plant model, both in the spatial and temporal variables (see, e.g., Bartecki, 2020; Oprzędkiewicz and Mitkowski, 2018) and then design a finite-dimensional observer using well-developed finite-dimensional techniques (see, e.g., the work of Trinh and Fernando (2012) and the references cited therein).

In this paper we propose in some sense a third way by starting with a partial approximation of an infinite-dimensional boundary control system by means of the Cayley transform, which leads to an infinite-dimensional discrete-time model, and then solve the general output observation problem for this simplified, but still infinite-dimensional, plant. As a justification of this approach we claim that in the digital era it seems inevitable that all practical implementations will eventually require algorithms that are discrete in time.

This way of simplifying an infinite-dimensional control model is already known in the literature but interest of researchers has been rather focused on system theoretic properties preserved under the Cayley transformation and not on using such models in developing control and observation algorithms (e.g., Ober and Montgomery-Smith, 1990; Curtain and Oostveen, 1997; Guo and Zwart, 2006; Havu and Malinen, 2007). However, recently this approach has been also successfully used by Dubljevic and Humaloja (2020) to handle model predictive control of infinite-dimensional systems and by Xie *et al.* (2021) to handle output regulation of fluid flow systems. The idea was also involved in the discrete-time stabilization problem of a heat equation by Mitkowski *et al.* (2017). In the present paper we show that the approach based on the Cayley transform as the initial step leads to a successful design procedure of output observers for infinite-dimensional control systems described by boundary input/output models.

The paper is organized into five sections. Section 1 is an introduction to the subject. In Section 2 the boundary input/output model used for the infinite-dimensional plant is characterized together with its relation to the familiar state space model with unbounded control and observation operators. In Section 3 we introduce the Cayley transform and explain what approximation scheme is behind it and then apply this approximation to the time-continuous infinite-dimensional plant which results in a simplified discrete-time infinite-dimensional plant model with all operators in the model being bounded. Section 4 is devoted to the discrete-time infinite-dimensional output observer design where the so-called observer equation is derived. The obtained general results are numerically tested in Section 5 on an example of designing an output observer for the plant described by a one-dimensional heat equation with boundary inputs and outputs. The paper is completed with Section 6 containing some conclusions.

2. Plant model

A large class of practically important infinite-dimensional systems is described by partial differential equations with control and observation in the boundary conditions. For those readers who want to become familiar with the sophisticated control theory of such systems, we recommend the recent textbook by Curtain and Zwart (2020) and the more specialized monograph by Tucsnak and Weiss (2009). Since we want the above class to be covered by our continuous-time control plant model we choose the abstract boundary input/output system description. Our approach follows Emirsajłow and Townley (2000) as well as Tucsnak and Weiss (2009).

2.1. Boundary input/output model. In order to describe the plant mathematical model precisely, we need to introduce the following spaces (they all are Hilbert spaces with appropriate scalar products $\langle\cdot,\cdot\rangle$ and induced norms $\|\cdot\| := \langle\cdot,\cdot\rangle^{1/2}$; identified with their duals): X is the *state space of the plant*, U is the *space of the control input* , Y is the *space of the measured output*, Y_z is the *space of the umeasured output*. The spaces U, Y and Y_z can be infinite- or finite-dimensional, but the state space X is always assumed to be infinite-dimensional.

The *plant* is described by the following abstract boundary input/output system

$$\Sigma_P: \begin{cases} \dot{x}(t) = Mx(t), \quad x(0) = x_0, \\ Kx(t) = u(t), \\ z(t) = C_1x(t), \\ y(t) = C_2x(t). \end{cases} \tag{1}$$

where $(x(t))_{t\geq 0} \subset X$ is the *state*, $(u(t))_{t\geq 0} \subset U$ is the *control input* (available to the observer), $(z(t))_{t\geq 0} \subset Y_z$ is the *unmeasured output* (unavailable to the observer) and $(y(t))_{t\geq 0} \subset Y$ is the *measured output* (available to the observer). The control input signal $u(t)$ enters the system at the boundary and the outputs $y(t)$ and $z(t)$ leave the system at the boundary.

Next, we impose several assumptions on the operators involved in the model (1). These assumptions will guarantee the existence of a strong solution to the differential equation with boundary conditions and this will be obtained by transforming the boundary input/output system description to the more familiar abstract state space model (Emirsajłow and Townley, 2000).

In the following two sections we apply the Cayley transform to obtain a discrete-time infinite-dimensional approximation of the obtained continuous-time infinite-dimensional state space model (Havu and Malinen, 2007). Then, for the discrete-time state space model we consider the general output observation problem which is also called the functional observation problem (Trinh and Fernando, 2012).

First we start with imposing the following assumptions:

1. $(M, \mathscr{D}(M))$ is a linear, unbounded and closed operator

on X, $Z = \mathscr{D}(M)$ is a Banach space equipped with the graph norm, so $M \in \mathscr{L}(Z,X)$ and is called the *plant maximal system operator*, where $\mathscr{L}(Z,X)$ denotes the Banach space of linear, bounded operators from Z to X.

2. $K \in \mathscr{L}(Z,U)$ is surjective and called the *plant input boundary operator*.

3. $C_1 \in \mathscr{L}(Z,Y_z)$ is called the *unmeasured output boundary operator*.

4. $C_2 \in \mathscr{L}(Z,Y)$ is called the *measured output boundary operator*.

5. A is a linear, unbounded operator on X, called the *plant system operator*, defined as follows: $\mathscr{D}(A) := \ker K$, $Ah := Mh$, $h \in \mathscr{D}(A)$, A generates a strongly continuous semigroup $(T(t))_{t\geq 0} \subset \mathscr{L}(X)$, $\rho(A) \subset \mathbb{C}$ denotes its resolvent set.

6. $X_1 := \mathscr{D}(A)$ is a Hilbert space, equipped with the scalar product $\langle \cdot,\cdot\rangle_{X_1} := \langle (\mu I - A)\,\cdot, (\mu I - A)\,\cdot\rangle_X$, where $\mu \in \rho(A)$. Part of A in X_1, denoted by A_1, generates a semigroup $(T_1(t))_{t\geq 0} \subset \mathscr{L}(X_1)$, a restriction of $(T(t))_{t\geq 0}$ to X_1.

7. X_{-1} is a Hilbert space, defined here as the completion of X with respect to the scalar product $\langle \cdot,\cdot\rangle_{X_{-1}} := \langle (\mu I - A)^{-1}\,\cdot, (\mu I - A)^{-1}\,\cdot\rangle_X$. Then $(A_{-1}, \mathscr{D}(A_{-1}) = X)$ and $(T_{-1}(t))_{t\geq 0} \subset \mathscr{L}(X_{-1})$ are extensions of A and $(T(t))_{t\geq 0}$ to X_{-1}.

It follows that $X_1 \subset Z$ and it is a closed subspace of Z. Moreover, both embeddings $Z \subset X \subset X_{-1}$ are dense and continuous.

With the operators (M,K) of the plant model (1) we associate a time-invariant abstract *boundary value problem*. In order to state it, we assume that $\mu \in \rho(A)$, $f \in X$, $u \in U$ and ask for a solution $x \in Z$ of the following system of equations:

$$\left\{ \begin{aligned} (\mu I - M)x &= f, \\ Kx &= u. \end{aligned} \right. \tag{2}$$

One can prove (e.g., Emirsajłow and Townley, 2000; Tucsnak and Weiss, 2009) that for every $u \in U$ and $f \in X$ there exists a unique solution $x \in Z$ which can be expressed in the form

$$x = R(\mu)f + G(\mu)u, \tag{3}$$

where $R(\mu) := (\mu I - A)^{-1} \in \mathscr{L}(X, X_1)$ is just a *resolvent* of A and $G(\mu) \in \mathscr{L}(U,Z)$ is called an *abstract Green map*. Both these operators play an important role in our considerations and in practice $R(\mu)$ can be effectively computed by solving the problem (2) with $u = 0$ and $G(\mu)$ can be computed by solving (2) with $f = 0$.

For our purposes, it is convenient to consider a strong solution to (1).

Definition 1. Let $x_0 \in X$ and $u(\cdot) \in C([0,\infty);U)$. A function $(x(t))_{t\geq 0} \subset X$ is said to be a *strong solution* of the initial-boundary value problem

$$\left\{ \begin{aligned} \dot{x}(t) &= Mx(t), \quad x(0) = x_0, \\ Kx(t) &= u(t), \end{aligned} \right. \tag{4}$$

if $x(\cdot) \in C^1([0,\infty);X)$, $(x(t))_{t\geq 0} \subset Z$ and (4) holds for every $t \geq 0$.

It is known (e.g., Emirsajłow and Townley, 2000; Tucsnak and Weiss, 2009) that an efficient way of characterizing the strong solution is in terms of the associated semigroup $(T(t))_{t\geq 0} \subset \mathscr{L}(X)$ and its generator $(A, \mathscr{D}(A))$. This leads to the following fundamental result concerning the state $(x(t))_{t\geq 0}$ and the outputs $(z(t))_{t\geq 0}$, $(y(t))_{t\geq 0}$ of the plant Σ_P.

Theorem 1. (Emirsajłow and Townley, 2000) *If $u(\cdot) \in C^2([0,\infty);U)$, $u(0) = 0$ and $x_0 \in X_1$, then there exists a unique function*

$$x(\cdot) \in C([0,\infty);Z) \cap C^1([0,\infty);X), \tag{5}$$

satisfying (4) *for every $t \in [0,\infty)$. This function is explicitly given in the form*

$$\begin{aligned} x(t) &= G(\mu)u(t) + T(t)x_0 \\ &\quad + \int_0^t T(t-r)G(\mu)(\mu u(r) - \dot{u}(r))\,\mathrm{d}r, \end{aligned} \tag{6}$$

which can be simplified to the form

$$x(t) = T(t)x_0 + (\mu I - A)\int_0^t T(t-r)G(\mu)u(r)\,\mathrm{d}r. \tag{7}$$

Consequently, the plant outputs satisfy

$$z(\cdot) \in C([0,\infty);Y_z), \quad y(\cdot) \in C([0,\infty);Y). \tag{8}$$

If the assumptions of Theorem 1 hold, then the boundary input/output system Σ_P is said to be *internally well posed*. This basically means that we are working with strong solutions of differential equations and continuous outputs. It is also known (e.g., Cheng and Morris, 2003) that by taking the Laplace transform of (1) and then using the results for the abstract boundary value problem (2), we easily obtain that for $\mu \in \rho(A)$ the system Σ_P has a well-defined transfer function, given by

$$\begin{bmatrix} H_1(\mu) \\ H_2(\mu) \end{bmatrix} := \begin{bmatrix} C_1G(\mu) \\ C_2G(\mu) \end{bmatrix} \in \mathscr{L}(U, \begin{bmatrix} Y_z \\ Y \end{bmatrix}), \tag{9}$$

where

$$\begin{bmatrix} Y_z \\ Y \end{bmatrix} := Y_z \times Y.$$

2.2. State space model. It is easy to see that (7) can be written as

$$\begin{aligned} x(t) &= T(t)x_0 \\ &\quad + \int_0^t T_{-1}(t-r)(\mu I - A_{-1})G(\mu)u(r)\,\mathrm{d}r. \end{aligned} \tag{10}$$

If we now define the operator

$$B := (\mu I - A_{-1})G(\mu) \in \mathscr{L}(U, X_{-1}), \tag{11}$$

then it is clear that the plant state trajectory $(x(t))_{t\geq 0} \subset Z \subset X$ can be also interpreted as a *mild solution* of the differential equation

$$\dot{x}(t) = A_{-1}x(t) + Bu(t), \quad x(0) = x_0, \tag{12}$$

which is understood in the larger space X_{-1} (see Emirsajłow and Townley, 2000) with the system operator $(A_{-1}, \mathscr{D}(A_{-1}) = X)$ and outputs

$$\begin{aligned} z(t) &= C_1x(t), \\ y(t) &= C_2x(t). \end{aligned} \tag{13}$$

The last three equations (i.e. (12) and (13)) form the infinite-dimensional *state space model* with an unbounded control operator B and unbounded observation operators C_1 and C_2. Since it has been derived from the boundary input/output model (1), we can easily extend the notion of the well-posedness of the boundary input-output system to allow for $x_0 \in X$, $u(\cdot) \in L^2_{\mathrm{loc}}([0,\infty);U)$, $z(\cdot) \in L^2_{\mathrm{loc}}([0,\infty);Y_z)$ and $y(\cdot) \in L^2_{\mathrm{loc}}([0,\infty);Y)$. It can be done by adopting appropriate admissibility notions for B, C_1, C_2 and the input-output maps $u(\cdot) \mapsto z(\cdot)$, $u(\cdot) \mapsto y(\cdot)$ (for details, see the works of Tucsnak and Weiss (2009), Cheng and Morris (2003), or Grabowski (2021). One can easily see that in terms of the operators of the state space model the transfer function (9) of Σ_P is expressed in the very familiar form of

$$\begin{aligned} \begin{bmatrix} H_1(\mu) \\ H_2(\mu) \end{bmatrix} &= \begin{bmatrix} C_1(\mu I - A_{-1})^{-1}B \\ C_2(\mu I - A_{-1})^{-1}B \end{bmatrix} \\ &\in \mathscr{L}(U, \begin{bmatrix} Y_z \\ Y \end{bmatrix}). \end{aligned} \tag{14}$$

3. Cayley transform and time discretization of the plant model

The Cayley transform, which is sometimes called the Cayley–Tustin transform, is a mapping of functions $H(s)$ from the complex s-domain (like continuous-time transfer functions) to functions $H^d(z)$ in the complex z-domain (like discrete-time transfer functions) by simple substitution

$$s = \frac{z-1}{z+1}\mu,$$

where $\mu > 0$ is a parameter. This mapping is also reversible by back substitution

$$z = \frac{\mu + s}{\mu - s}.$$

Note that for a particular value of $\mu = 1$ it is also called the bilinear transformation (e.g., Curtain and Oostveen, 1997; Ober and Montgomery-Smith, 1990). It follows that the Cayley transform is a tool which allows us to convert continuous-time linear time-invariant state-space models to discrete-time ones and, what is important for us, it also applies to infinite-dimensional systems (e.g., Havu and Malinen, 2007; Guo and Zwart, 2006). It turns out that if we convert a continuous-time state space model with unbounded control and observation operators then the obtained discrete-time state space model remains infinite-dimensional. However, all the operators in the model become bounded and hence the model is mathematically easier to handle. The essential feature of the Cayley transform is that it preserves various system theoretic properties of the control system. For details see the works of Ober and Montgomery-Smith (1990), Curtain and Oostveen (1997), Guo and Zwart (2006), and the references cited therein.

It is well known (e.g., Havu and Malinen, 2007) that there is a simple time discretization scheme which, when applied to the continuous-time infinite-dimensional state space model, leads to the same discrete-time infinite-dimensional state space model as formally obtained by the Cayley transformation. In this paper we apply this time discretization scheme to a continuous-time infinite-dimensional plant modelled as an abstract boundary input/output system and derive its infinite-dimensional discrete-time approximation in the form of a state-space model with all operators bounded. It turns out that the formulae for operators in the discrete-time model depend on the resolvent $R(\mu)$ and the Green map $G(\mu)$ and for this reason they can be easily computed by solving the boundary value problem (2). For this reason our approach is well tailored for plants described by partial differential equations with controls and observations in the boundary conditions.

In order to start, we assume that $x_0 \in X_1$, $u(\cdot) \in C^2([0,\infty);U)$ with $u(0) = 0$, and then we have $x(\cdot) \in C([0,\infty);Z) \cap C^1([0,\infty);X)$ (see Theorem 1). In order to discretize time, we introduce a time step $h > 0$ and integrate equations

$$\Sigma_P : \begin{cases} \dot{x}(t) = Mx(t), \quad x(0) = x_0, \\ Kx(t) = u(t), \\ z(t) = C_1x(t), \\ y(t) = C_2x(t). \end{cases} \tag{15}$$

on time intervals $[(k-1)h, kh]$, where $k = 1, 2, 3, \ldots$.

Thus we obtain

$$\begin{aligned}\int_{(k-1)h}^{kh} \dot{x}(t)\ \mathrm{d}t &= M\int_{(k-1)h}^{kh} x(t)\,\mathrm{d}t, \quad x(0)=x_0,\\ K\int_{(k-1)h}^{kh} x(t)\ \mathrm{d}t &= \int_{(k-1)h}^{kh} u(t)\,\mathrm{d}t,\end{aligned}$$

$$\begin{aligned}\int_{(k-1)h}^{kh} z(t)\ \mathrm{d}t &= C_1\int_{(k-1)h}^{kh} x(t)\,\mathrm{d}t,\\ \int_{(k-1)h}^{kh} y(t)\ \mathrm{d}t &= C_2\int_{(k-1)h}^{kh} x(t)\,\mathrm{d}t,\end{aligned} \tag{16}$$

where $k = 1, 2, 3, \ldots$.

Introducing

$$\begin{aligned} x_k &:= x(kh) = x(t)\big|_{t=kh} \in Z \subset X,\\ \int_{(k-1)h}^{kh} x(t)\,\mathrm{d}t &\approx \frac{h}{2}(x_{k-1}+x_k) \in Z,\\ u_k &:= \frac{1}{\sqrt{h}}\int_{(k-1)h}^{kh} u(t)\,\mathrm{d}t \in U,\\ z_k &:= \frac{1}{\sqrt{h}}\int_{(k-1)h}^{kh} z(t)\,\mathrm{d}t \in Y_z,\\ y_k &:= \frac{1}{\sqrt{h}}\int_{(k-1)h}^{kh} y(t)\,\mathrm{d}t \in Y,\end{aligned} \tag{17}$$

where for the integral of $x(t)$ we used the trapezoidal rule, we can rewrite (16) in the form

$$\begin{aligned} x_k - x_{k-1} &\approx \frac{h}{2}M(x_{k-1}+x_k), \quad x(0)=x_0,\\ \frac{h}{2}K(x_{k-1}+x_k) &\approx \sqrt{h}u_k,\\ \sqrt{h}z_k &\approx \frac{h}{2}C_1(x_{k-1}+x_k),\\ \sqrt{h}y_k &\approx \frac{h}{2}C_2(x_{k-1}+x_k).\end{aligned}$$

Replacing them by equalities and introducing a real parameter $\mu := 2/h > 0$, with the meaning of a double sampling frequency, we obtain

$$\begin{aligned}(\mu I - M)(x_{k-1}+x_k) &= 2\mu x_{k-1},\\ K(x_{k-1}+x_k) &= \sqrt{2\mu}u_k,\\ z_k &= \frac{1}{\sqrt{2\mu}}C_1(x_{k-1}+x_k),\\ y_k &= \frac{1}{\sqrt{2\mu}}C_2(x_{k-1}+x_k).\end{aligned} \tag{18}$$

If the time step $h > 0$ is small enough, then $\mu \in \rho(A)$ and we can use formula (3) to express the sum $(x_{k-1} + x_k) \in Z$ as a solution of the first two equations of (18), i.e.,

$$x_{k-1} + x_k = 2\mu R(\mu)x_{k-1} + \sqrt{2\mu}G(\mu)u_k. \tag{19}$$

From this equation we compute x_k and also substitute (19) into the last two equations of (18). Finally, we get

$$\begin{aligned} x_k &= (-I + 2\mu R(\mu))x_{k-1} + \sqrt{2\mu}G(\mu)u_k,\\ z_k &= \sqrt{2\mu}C_1R(\mu)x_{k-1} + C_1G(\mu)u_k,\\ y_k &= \sqrt{2\mu}C_2R(\mu)x_{k-1} + C_2G(\mu)u_k.\end{aligned} \tag{20}$$

Defining the operators

$$\left[\begin{array}{c|c} A^d & B^d \\ \hline C_1^d & D_1^d \\ C_2^d & D_2^d \end{array}\right] := \left[\begin{array}{c|c} -I + 2\mu R(\mu) & \sqrt{2\mu}G(\mu) \\ \hline \sqrt{2\mu}C_1R(\mu) & C_1G(\mu) \\ \sqrt{2\mu}C_2R(\mu) & C_2G(\mu) \end{array}\right], \tag{21}$$

we arrive at the following infinite-dimensional *discrete-time state space model* Σ_P^d:

$$\Sigma_P^d: \quad \left[\begin{array}{c} x_k \\ \hline z_k \\ y_k \end{array}\right] = \left[\begin{array}{c|c} A^d & B^d \\ \hline C_1^d & D_1^d \\ C_2^d & D_2^d \end{array}\right] \left[\begin{array}{c} x_{k-1} \\ \hline u_k \end{array}\right], \tag{22}$$

where $k = 1, 2, 3, \ldots$, $(x_k)_{k\in\mathbb{N}} \subset Z \subset X$ is the state, $(u_k)_{k\in\mathbb{N}} \subset U$ is the measured input, $(y_k)_{k\in\mathbb{N}} \subset Y$ is the measured output and $(z_k)_{k\in N} \subset Y_z$ is the *unmeasured* output. The model (22) can be written shortly as $\Sigma_P^d(A^d, B^d, C_1^d, C_2^d, D_1^d, D_2^d)$. Since we have $A^d \in \mathscr{L}(X)$, $B^d \in \mathscr{L}(U, X)$, $C_1^d \in \mathscr{L}(X, Y_z)$, $C_2^d \in \mathscr{L}(X, Y)$, $D_1^d \in \mathscr{L}(U, Y_z)$, $D_2^d \in \mathscr{L}(U, Y)$, all the operators of Σ_P^d are bounded and it is a well-defined discrete-time state space model on X allowing states $(x_k)_{k\in\mathbb{N}} \subset X$. The model (22) is a *discrete-time approximation of the infinite-dimensional continuous-time model* (15) and, equivalently, (12) and (13). One can also see that the direct feedthrough terms $D_1^d = C_1G(\mu) \in \mathscr{L}(U, Y_z)$ and $D_2^d = C_2G(\mu) \in \mathscr{L}(U, Y)$ coincide with the continuous-time transfer functions $H_1(\mu)$ and $H_2(\mu)$, respectively (see (9)). These terms have appeared in the discrete-time model although there are no feedthrough terms in the continuous-time models.

It is clear that the state $(x_k)_{k\in\mathbb{N}}$ and the outputs $(z_k)_{k\in\mathbb{N}}$ and $(y_k)_{k\in\mathbb{N}}$ can be written explicitly as

$$\begin{aligned} x_k &= (A^d)^k x_0 + \sum_{j=0}^{k}(A^d)^{k-j}B^d u_j,\\ &\quad k = 0, 1, 2, 3, \ldots, \quad u_0 = 0,\\ z_k &= C_1^d(A^d)^{k-1}x_0 + \sum_{j=1}^{k-1} C_1^d(A^d)^{k-1-j}B^d u_j\\ &\quad + D_1^d u_k, \quad k = 1, 2, 3, \ldots,\\ y_k &= C_2^d(A^d)^{k-1}x_0 + \sum_{j=1}^{k-1} C_2^d(A^d)^{k-1-j}B^d u_j\\ &\quad + D_2^d u_k, \quad k = 1, 2, 3, \ldots.\end{aligned} \tag{23}$$

It is also known (Curtain and Oostveen, 1997) that if A is an unbounded generator of an exponentially stable strongly continuous semigroup then its discrete-time counterpart A^d cannot be power stable but if A generates a strongly stable semigroup then A^d is also strongly stable. For this reason we restrict ourselves to the concept of strong stability of discrete-time systems. Just for the sake of completeness we recall the definition: we say that A^d is *strongly stable* if for every $x \in X$ we have

$$\lim_{k\to\infty} \|(A^d)^k x\|_X = 0.$$

The operators in the model (22) involve the resolvent $R(\mu) \in \mathscr{L}(X, X_1)$ and the abstract Green map $G(\mu) \in \mathscr{L}(U, Z)$, defined for the boundary value problem (2). As has been already mentioned, they can be effectively computed by solving (2), which will be explored in the example considered in Section 5.

4. Discrete-time observer design

We can now use the discrete-time infinite-dimensional state space model (22) as an approximation of our continuous-time boundary input/output model of the plant and design a discrete-time infinite-dimensional output observer. One can go further and use such a discrete-time observer for the continuous-time plant (15). A similar idea has been recently used by Dubljevic and Humaloja (2020) as well as Xie *et al.* (2021), where the discrete-time infinite-dimensional models, obtained by the Cayley transformation, have been used to develop control algorithms for continuous-time infinite-dimensional state space plant models. We emphasize that in our paper we start with a continuous-time plant model in the form of a boundary input-output system in contrast to the state space description investigated by Dubljevic and Humaloja (2020) or Xie *et al.* (2021). Our choice of the plant model shows the advantage of using the resolvent $R(\mu)$ and the Green map $G(\mu)$ in the discrete-time model.

4.1. Observer model. In order to describe an infinite-dimensional discrete-time output observer, we introduce a Hilbert space V as the *observer, state space*, and define six linear bounded operators:

$$\begin{aligned} &A_O \in \mathscr{L}(V), && B_O \in \mathscr{L}(Y, V), \\ &E_O \in \mathscr{L}(U, V), && C_O \in \mathscr{L}(V, Y_z), \\ &D_O \in \mathscr{L}(Y, Y_z), && F_O \in \mathscr{L}(U, Y_z). \end{aligned}$$

The observer is described by the following discrete-time state space model:

$$\Sigma_O: \quad \left[\begin{array}{c} v_k \\ \hline z_{Ok} \end{array}\right] = \left[\begin{array}{c|cc} A_O & B_O & E_O \\ \hline C_O & D_O & F_O \end{array}\right] \left[\begin{array}{c} v_{k-1} \\ \hline y_k \\ u_k \end{array}\right], \tag{24}$$

where $k = 1, 2, 3, \ldots$, $(v_k)_{k\in\mathbb{N}} \subset V$ is the observer state, $(u_k)_{k\in\mathbb{N}} \subset U$ and $(y_k)_{k\in\mathbb{N}} \subset Y$ are the measured input and output of the plant (22), respectively, and $(z_{Ok})_{k\in\mathbb{N}} \subset Y_z$ is the observer output. The model (24) can be written shortly as $\Sigma_O(A_O, B_O, C_O, D_O, E_O, F_O)$.

Our *objective* is to track asymptotically the plant unmeasured output $(z_k)_{k\in\mathbb{N}}$ by the observer output $(z_{Ok})_{k\in\mathbb{N}}$ in the sense that the observer error $(e_k)_{k\in\mathbb{N}} \subset Y_z$ satisfies the condition

$$\lim_{k\to\infty} \|e_k\|_{Y_z} = \lim_{k\to\infty} \|z_{Ok} - z_k\|_{Y_z} = 0, \tag{25}$$

which we call the *output observation condition*.

In order to characterize the required observer, we derive the so-called *observer equation*. A similar idea, but in the continuous-time context, has been used by Emirsajłow (2021) who described the plant and the observer by infinite-dimensional state space models with bounded control and observation operators.

4.2. Observer equation. Collect all formulae (22), (24) and (25):

$$\begin{aligned} x_k &= A^d x_{k-1} + B^d u_k, \\ z_k &= C_1^d x_{k-1} + D_1^d u_k, \\ y_k &= C_2^d x_{k-1} + D_2^d u_k, \\ v_k &= A_O v_{k-1} + B_O y_k + E_O u_k, \\ z_{Ok} &= C_O v_{k-1} + D_O y_k + F_O u_k, \\ e_k &= z_{Ok} - z_k. \end{aligned} \tag{26}$$

After some manipulations, the interconnection of the plant Σ_P^d and the observer Σ_O can be described as

$$\begin{aligned} v_k &= A_O v_{k-1} + B_O C_2^d x_{k-1} + (B_O D_2^d + E_O) u_k, \\ x_k &= A^d x_{k-1} + B^d u_k, \\ e_k &= C_O v_{k-1} + (D_O C_2^d - C_1^d) x_{k-1} \\ &\quad + (D_O D_2^d + F_O - D_1^d) u_k, \end{aligned} \tag{27}$$

where $k = 1, 2, 3, \ldots$ and the *output observation condition* will say that for all initial states $v_0 \in V$, $x_0 \in X$ and all inputs $(u_k)_{k\in\mathbb{N}} \subset U$

$$\begin{aligned} \lim_{k\to\infty} e_k = \lim_{k\to\infty} \big[&C_O v_{k-1} + (D_O C_2^d - C_1^d) x_{k-1} \\ &+ (D_O D_2^d + F_O - D_1^d) u_k \big] = 0. \end{aligned} \tag{28}$$

In order to solve the problem under consideration, we introduce new state variables for the interconnection (27),

$$\begin{bmatrix} p_k \\ x_k \end{bmatrix} = \begin{bmatrix} I & -\Pi \\ 0 & I \end{bmatrix} \begin{bmatrix} v_k \\ x_k \end{bmatrix}, \tag{29}$$

where $\Pi \in \mathscr{L}(X, V)$ is to be specified. It is easy to verify that the transformation (29) is always boundedly invertible. Then the interconnection with new state variables takes the form

$$\begin{aligned} p_k &= A_O p_{k-1} + (A_O\Pi - \Pi A^d + B_O C_2^d)x_{k-1} \\ &\quad + (B_O D_2^d - \Pi B^d + E_O)u_k, \quad p_0 = v_0 - \Pi x_0, \\ x_k &= A^d x_{k-1} + B^d u_k, \\ e_k &= C_O p_{k-1} + (C_O\Pi + D_O C_2^d - C_1^d)x_{k-1} \\ &\quad + (D_O D_2^d - D_1^d + F_O)u_k, \end{aligned} \tag{30}$$

and the output observation condition says that for all $v_0 \in V$, $x_0 \in X$ ($p_0 = (v_0 - \Pi x_0) \in V$) and all inputs $(u_k)_{k\in\mathbb{N}} \subset U$ we have

$$\begin{aligned} &\lim_{k\to\infty} e_k \\ &\quad = \lim_{k\to\infty} \big[C_O p_{k-1} + (C_O\Pi + D_O C_2^d - C_1^d)x_{k-1} \\ &\qquad + (D_O D_2^d - D_1^d + F_O)u_k\big] = 0. \end{aligned} \tag{31}$$

Without the loss of generality we can assume that E_O and F_O satisfy the equalities

$$\begin{aligned} E_O &= \Pi B^d - B_O D_2^d, \\ F_O &= D_1^d - D_O D_2^d. \end{aligned} \tag{32}$$

This assumption is allowed since the choice of A_O, B_O, C_O and D_O is independent of E_O and F_O. Then the interconnection of Σ_P^d and Σ_O simplifies as follows:

$$\begin{aligned} p_k &= A_O p_{k-1} + (A_O\Pi - \Pi A^d + B_O C_2^d)x_{k-1}, \\ x_k &= A^d x_{k-1} + B^d u_k, \\ e_k &= C_O p_{k-1} + (C_O\Pi + D_O C_2^d - C_1^d)x_{k-1}, \end{aligned} \tag{33}$$

and the output observation condition says that for all $v_0 \in V$, $x_0 \in X$ ($p_0 = (v_0 - \Pi x_0) \in V$) and all inputs $(u_k)_{k\in\mathbb{N}} \subset U$ we have

$$\begin{aligned} &\lim_{k\to\infty} \|e_k\|_{Y_z} \\ &\quad = \lim_{k\to\infty} \|C_O p_{k-1} + (C_O\Pi + D_O C_2^d - C_1^d)x_{k-1}\|_{Y_z} \\ &\quad = 0. \end{aligned} \tag{34}$$

We are ready to complete the above considerations with the following important result.

Theorem 2. *Let the interconnection of the plant Σ_P^d and the observer Σ_O be described by Eqn.* (30)*. If the parameters $(A_O, B_O, C_O, D_O, E_O, F_O)$ of the observer are such that the following equation, called the* observer equation*,*

$$\begin{cases} A_O\Pi - \Pi A^d + B_O C_2^d = 0, \\ C_O\Pi + D_O C_2^d - C_1^d = 0, \end{cases} \tag{35}$$

has a solution $\Pi \in \mathscr{L}(X, V)$, and

$$\begin{aligned} E_O &= \Pi B^d - B_O D_2^d, \\ F_O &= D_1^d - D_O D_2^d, \end{aligned} \tag{36}$$

then for all $v_0 \in V$, $x_0 \in X$ and all inputs $(u_k)_{k\in\mathbb{N}} \subset U$ the observer error $(e_k)_{k\in\mathbb{N}} \subset Y_z$ is described as follows

$$\begin{aligned} p_k &= A_O p_{k-1}, \quad p_0 = (v_0 - \Pi x_0) \in V, \\ e_k &= C_O p_{k-1}. \end{aligned} \tag{37}$$

If additionally, the observer system operator A_O is strongly stable, then

$$\lim_{k\to\infty} \|e_k\|_{Y_z} = 0, \tag{38}$$

i.e., the output observation condition holds.

Proof. If the relations (35) and (36) hold, then Eqns. (33) simplifies to the form (37). If the observer is strongly stable, then for every $p_0 \in V$ we have

$$\lim_{k\to\infty} \|p_k\|_V = \lim_{k\to\infty} \|(A_O)^k p_0\|_V = 0,$$

and since $C_O \in \mathscr{L}(V, Y_z)$ this implies (38). ■

Theorem 2 provides very general conditions characterizing the output observer Σ_O. In the infinite-dimensional setup the task of choosing both the observer state space V and the parameters $(A_O, B_O, C_O, D_O, E_O, F_O)$ to fulfill these conditions seems rather difficult. Nevertheless, under the assumption that the pair (C_2^d, A^d) of the discrete-time plant $\Sigma_P^d(A^d, B^d, C_1^d, C_2^d, D_1^d, D_2^d)$ is strongly detectable, where the pair (C_2^d, A^d) is said to be *strongly detectable* if there exists $L \in \mathcal{L}(Y, X)$ such that for every $x \in X$ we have

$$\lim_{k\to\infty} \|((A^d - LC_2^d)^k x\|_X = 0,$$

Theorem 2 provides the following rather simple procedure of constructing a stable output observer $\Sigma_O(A_O, B_O, C_O, D_O, E_O, F_O)$:

1. Choose $L \in \mathscr{L}(Y, X)$ such that the operator $A^d - LC_2^d$ is strongly stable.
2. Choose V and an arbitrary operator $\Pi \in \mathscr{L}(X, V)$ which is boundedly invertible, i.e., $\Pi^{-1} \in \mathscr{L}(V, X)$.
3. Choose $B_O \in \mathscr{L}(U, V)$ such that
$$B_O := \Pi L.$$
4. Choose $A_O \in \mathscr{L}(X)$ such that
$$A_O := \Pi(A^d - LC_2^d)\Pi^{-1},$$
which implies that A_O is strongly stable iff $A^d - LC_2^d$ is strongly stable.
5. Choose an arbitrary $D_O \in \mathscr{L}(U, Y_z)$.
6. Choose $C_O \in \mathscr{L}(X, Y_z)$ such that
$$C_O := (-D_O C_2^d + C_1^d)\Pi^{-1}.$$

7. Choose $E_O \in \mathscr{L}(U, V)$ and $F_O \in \mathscr{L}(Y, Y_z)$ such that

$$E_O := \Pi B^d - B_O D_2^d,$$
$$F_O := D_1^d - D_O D_2^d.$$

One can easily check that the above operators $(A_O, B_O, C_O, D_O, E_O, F_O)$ satisfy all the conditions of Theorem 2. The simplest choice is to take $V := X$, $\Pi := I$ (identity) and $D_O = 0$. In this case we get a Luenberger-like output observer in the form

$$\begin{aligned} v_k &= (A^d - LC_2^d)v_{k-1} + Ly_k + (B^d - LD_2^d)u_k, \\ z_{Ok} &= C_1^d v_{k-1} + D_1^d u_k, \end{aligned} \tag{39}$$

where

$$\begin{aligned} A^d - LC_2^d &= -I + 2\mu R(\mu) - L\sqrt{2\mu} C_2 R(\mu), \\ B^d - LD_2^d &= \sqrt{2\mu} G(\mu) - LC_2 G(\mu), \\ C_1^d &= \sqrt{2\mu} C_1 R(\mu), \\ D_1^d &= C_1 G(\mu). \end{aligned} \tag{40}$$

4.3. Special case. In our discrete-time observation problem we have assumed that we know the input $(u_k)_{k\in\mathbb{N}}$ and the output $(y_k)_{k\in\mathbb{N}}$ and we do not know the initial state x_0. The simplest situation is when the plant input $(u_k)_{k\in\mathbb{N}}$ is zero, i.e., $u_k = 0$ for $k \in \mathbb{N}$. In this case the plant assumes the form

$$\begin{aligned} x_k &= A^d x_{k-1}, \quad x_0 \neq 0, \\ z_k &= C_1^d x_{k-1}, \\ y_k &= C_2^d x_{k-1}, \end{aligned} \tag{41}$$

and the observer takes the form

$$\begin{aligned} v_k &= A_O v_{k-1} + B_O y_k, \quad v_0 = 0, \\ z_{Ok} &= C_O v_{k-1} + D_O y_k, \end{aligned} \tag{42}$$

where $v_0 = 0$ is assumed for simplicity. Moreover, in this case the Luenberger-like output observer (39) simplifies as follows:

$$\begin{aligned} v_k &= (A^d - LC_2^d)v_{k-1} + Ly_k, \quad v_0 = 0, \\ z_{Ok} &= C_1^d v_{k-1}, \end{aligned} \tag{43}$$

where $k = 1, 2, 3, \ldots$.

5. Example

As an illustration, we consider a numerical example of an output observation problem for the spatially one-dimensional heat equation. We assume $\xi \in [0, \pi]$, $t \geq 0$ and the plant model is described as follows:

$$\Sigma_P : \begin{cases} \dfrac{\partial x(\xi,t)}{\partial t} = \dfrac{\partial^2 x(\xi,t)}{\partial \xi^2}, \quad x(\xi,0) = x_0(\xi), \\ \dfrac{\partial x}{\partial \xi}(0,t) = u^0(t), \quad \dfrac{\partial x}{\partial \xi}(\pi,t) = u^\pi(t), \\ z(t) = x(a,t), \\ y^0(t) = x(0,t), \quad y^\pi(t) = x(\pi,t), \end{cases} \tag{44}$$

where $(x(\cdot,t))_{t\geq 0} \subset L_2(0,\pi)$ is a state, $x_0(\xi)$ is an unknown initial state,

$$\left(\begin{bmatrix} u^0(t) \\ u^\pi(t) \end{bmatrix} \right)_{t\geq 0} \subset \mathbb{R}^2$$

is a measured input, $a \in (0, \pi)$ is a fixed point and $(z(t))_{t\geq 0} \subset \mathbb{R}$ is an unmeasured output,

$$\left(\begin{bmatrix} y^0(t) \\ y^\pi(t) \end{bmatrix} \right)_{t\geq 0} \subset \mathbb{R}^2$$

is a measured output. This model may describe a temperature distribution $x(\xi, t)$ in a finite rod with an unknown initial distribution $x_0(\xi)$, known heat fluxes

$$\left(\begin{bmatrix} u^0(t) \\ u^\pi(t) \end{bmatrix} \right)_{t\geq 0} \subset \mathbb{R}^2$$

at both ends and with temperatures

$$\left(\begin{bmatrix} y^0(t) \\ y^\pi(t) \end{bmatrix} \right)_{t\geq 0} \subset \mathbb{R}^2$$

measured at these ends. We are interested in evaluation of a temperature $z(t) = x(a, t)$ unavailable for measurement.

For this plant we choose a small sampling period $h > 0$ such that $\mu = 2/h \in \rho(A)$ and then construct an approximate infinite-dimensional discrete-time plant model in the form (22). For this model we design a strongly stable infinite-dimensional discrete-time observer which uses samples $(y_k^0)_{k\in\mathbb{N}} \subset \mathbb{R}$ and $(y_k^\pi)_{k\in\mathbb{N}} \subset \mathbb{R}$ of measured temperatures and samples $(u_k^0)_{k\in\mathbb{N}} \subset \mathbb{R}$ and $(u_k^\pi)_{k\in\mathbb{N}} \subset \mathbb{R}$ of measured fluxes, and then produces a discrete-time signal $(z_{Ok})_{k\in\mathbb{N}} \subset \mathbb{R}$. The signal $(z_{Ok})_{k\in\mathbb{N}} \subset \mathbb{R}$ will track asymptotically the unmeasured output $(z_k)_{k\in\mathbb{N}} \subset \mathbb{R}$ of the discrete-time model. The rescaled samples $(z_{Ok}/\sqrt{h})_{k\in\mathbb{N}}$ can be used to obtain an interpolated continuous-time signal $(z_O(t))_{t\geq 0}$ estimating the temperature $(z(t))_{t\geq 0} \subset \mathbb{R}$ at $\xi = a$.

For the system (44) we introduce the spaces

$$X = L_2(0,\pi), \quad U = Y = \mathbb{R}^2, \quad Y_z = \mathbb{R},$$

and the operators

$$M = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}, \quad \mathscr{D}(M) = Z = H_2^2(0,\pi),$$

$$K = \left[\begin{array}{cc} \frac{\mathrm{d}}{\mathrm{d}\xi}\Big|_{\xi=0} & \frac{\mathrm{d}}{\mathrm{d}\xi}\Big|_{\xi=\pi} \end{array} \right]^T$$

(Neumann boundary operator),

$$C_1 = (\ )_{\xi=a} \quad \text{(pointwise evaluation operator)},$$

$$C_2 = \left[\begin{array}{cc} (\ )_{\xi=0} & (\ )_{\xi=\pi} \end{array} \right]^T$$

(Dirichlet boundary operator),

$$A = \frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\Big|_{\mathscr{D}(A)} \text{ with } \mathscr{D}(A) \text{ given by}$$
$$\mathscr{D}(A) = \{x(\xi) : x \in H_2^2(0,\pi), x'_\xi(0) = 0, \quad x'_\xi(\pi) = 0\},$$

where $H_2^2(0,\pi) = \{f(\cdot) \in L_2(0,\pi) : f'_\xi(\cdot), f''_\xi(\cdot) \in L_2(0,\pi)\}$. Under the above assumptions we obtain that $(A, \mathscr{D}(A))$ generates a strongly continuous semigroup on X with $\omega_0(T) = 0$, so $\mathbb{R}_+ \subset \rho(A)$ (e.g., Curtain and Zwart, 2020) and the system (44) fits into our framework. For the continuous-time plant (44) we now develop the discrete-time approximation (22), where an arbitrary sampling period $h > 0$ is allowed.

It is known from the literature (e.g., Emirsajłow and Townley, 2000) that for the above continuous-time problem the abstract Green map $G(\mu) \in \mathscr{L}(\mathbb{R}^2, H_2^2(0,\pi))$ is explicitly given by the formula

$$\begin{aligned} G(\mu)\left[\begin{array}{c} u^0 \\ u^\pi \end{array}\right] &= \left[\begin{array}{cc} g^0(\xi) & g^\pi(\xi) \end{array}\right]\left[\begin{array}{c} u^0 \\ u^\pi \end{array}\right] \\ &= \left[\begin{array}{cc} -\frac{\cosh\sqrt{\mu}(\pi-\xi)}{\sqrt{\mu}\sinh\sqrt{\mu}\pi} & \frac{\cosh\sqrt{\mu}\xi}{\sqrt{\mu}\sinh\sqrt{\mu}\pi} \end{array}\right]\left[\begin{array}{c} u^0 \\ u^\pi \end{array}\right], \end{aligned} \tag{45}$$

where $\left[\begin{array}{c} u^0 \\ u^\pi \end{array}\right] \in \mathbb{R}^2$, and from the work of Kythe (2011) it is known that the resolvent $R(\mu) \in \mathscr{L}(L_2(0,\pi))$ (in fact $R(\mu) \in \mathscr{L}(L_2(0,\pi), \mathscr{D}(A))$) can be also explicitly expressed in the form

$$(R(\mu)x)(\xi) := \int_0^\pi g(\xi,s)x(s)\,\mathrm{d}s, \quad \xi \in [0,\pi], \tag{46}$$

where $x(\cdot) \in L_2(0,\pi)$ and the Green function $g(\xi,s)$ has the form

$$g(\xi,s) = \begin{cases} \dfrac{\cosh(\sqrt{\mu}(\pi-\xi))\cosh(\sqrt{\mu}s)}{\sqrt{\mu}\sinh(\sqrt{\mu}\pi)} \\ 0 \le \xi < s, \\ \dfrac{\cosh(\sqrt{\mu}\xi)\cosh(\sqrt{\mu}(\pi-s))}{\sqrt{\mu}\sinh(\sqrt{\mu}\pi)} \\ s < \xi \le \pi. \end{cases} \tag{47}$$

Thus, using (21), we get explicit expressions for the operators $(A^d, B^d, C_1^d, C_2^d, D_1^d, D_2^d)$ in the infinite-dimensional discrete-time plant model (22)

$$A^d x = -x(\xi) + 2\mu\int_0^\pi g(\xi,s)x(s)\,\mathrm{d}s,$$

$$B^d u = \sqrt{2\mu}\left[\begin{array}{cc} g^0(\xi) & g^\pi(\xi) \end{array}\right]\left[\begin{array}{c} u^0 \\ u^\pi \end{array}\right],$$

$$C_1^d x = \sqrt{2\mu}\int_0^\pi g(a,s)x(s)\,\mathrm{d}s,$$

$$C_2^d x = \left[\begin{array}{c} \sqrt{2\mu}\int_0^\pi g(0,s)x(s)\,\mathrm{d}s \\ \sqrt{2\mu}\int_0^\pi g(\pi,s)x(s)\,\mathrm{d}s \end{array}\right], \tag{48}$$

$$D_1^d u = \left[\begin{array}{cc} g^0(a) & g^\pi(a) \end{array}\right]\left[\begin{array}{c} u^0 \\ u^\pi \end{array}\right],$$

$$D_2^d u = \left[\begin{array}{cc} g^0(0) & g^\pi(0) \\ g^0(\pi) & g^\pi(\pi) \end{array}\right]\left[\begin{array}{c} u^0 \\ u^\pi \end{array}\right],$$

and (22) can be explicitly written as

$$\begin{aligned} x_k(\xi) &= -x_{k-1}(\xi) + 2\mu\int_0^\pi g(\xi,s)x_{k-1}(s)\,\mathrm{d}s \\ &\quad + \sqrt{2\mu}(g^0(\xi)u_k^0 + g^\pi(\xi)u_k^\pi), \\ z_k &= \sqrt{2\mu}\int_0^\pi g(a,s)x_{k-1}(s)\,\mathrm{d}s \\ &\quad + g^0(a)u_k^0 + g^\pi(a)u_k^\pi, \\ y_k^0 &= \sqrt{2\mu}\int_0^\pi g(0,s)x_{k-1}(s)\,\mathrm{d}s \\ &\quad + g^0(0)u_k^0 + g^\pi(0)u_k^\pi, \\ y_k^\pi &= \sqrt{2\mu}\int_0^\pi g(\pi,s)x_{k-1}(s)\,\mathrm{d}s \\ &\quad + g^0(\pi)u_k^0 + g^\pi(\pi)u_k^\pi, \end{aligned} \tag{49}$$

where $k = 1,2,3,\dots$ and an initial condition $x_0(\xi)$ is unknown. For the plant (49) we can design a Luenberger-like output observer in the form (39), where

$$\begin{aligned} &(A^d - LC_2^d)v \\ &= -v(\xi) + 2\mu\int_0^\pi g(\xi,s)v(s)\,\mathrm{d}s \end{aligned}$$

$$- \begin{bmatrix} l^0(\xi) & l^\pi(\xi) \end{bmatrix} \begin{bmatrix} \sqrt{2\mu}\int_0^\pi g(0,s)v(s)\,\mathrm{d}s \\ \sqrt{2\mu}\int_0^\pi g(\pi,s)v(s)\,\mathrm{d}s \end{bmatrix},$$

$$Ly = \begin{bmatrix} l^0(\xi) & l^\pi(\xi) \end{bmatrix} \begin{bmatrix} y^0 \\ y^\pi \end{bmatrix},$$

$$\begin{aligned} &(B^d - LD_2^d)u \\ &= \Big(\sqrt{2\mu}\begin{bmatrix} g^0(\xi) & g^\pi(\xi) \end{bmatrix} \\ &- \begin{bmatrix} l^0(\xi) & l^\pi(\xi) \end{bmatrix} \begin{bmatrix} g^0(0) & g^\pi(0) \\ g^0(\pi) & g^\pi(\pi) \end{bmatrix}\Big) \begin{bmatrix} u^0 \\ u^\pi \end{bmatrix}, \end{aligned}$$

$$\begin{aligned} C_1^d v &= \sqrt{2\mu}\int_0^\pi g(a,s)v(s)\,\mathrm{d}s, \\ D_1^d u &= \begin{bmatrix} g^0(a) & g^\pi(a) \end{bmatrix} \begin{bmatrix} u^0 \\ u^\pi \end{bmatrix}, \end{aligned} \tag{50}$$

where $l^0(\cdot), l^\pi(\cdot) \in L_2(0,\pi)$. Hence, (39) can be explicitly written as

$$\begin{aligned} v_k(\xi) &= -v_{k-1}(\xi) + 2\mu\int_0^\pi g(\xi,s)v_{k-1}(s)\,\mathrm{d}s \\ &\quad - l^0(\xi)\sqrt{2\mu}\int_0^\pi g(0,s)v_{k-1}(s)\,\mathrm{d}s \\ &\quad - l^\pi(\xi)\sqrt{2\mu}\int_0^\pi g(\pi,s)v_{k-1}(s)\,\mathrm{d}s \\ &\quad + \sqrt{2\mu}g^0(\xi)u_k^0 + \sqrt{2\mu}g^\pi(\xi)u_k^\pi \\ &\quad + l^0(\xi)(y_k^0 - g^0(0)u_k^0 - g^\pi(0)u_k^\pi) \\ &\quad + l^\pi(\xi)(y_k^\pi - g^0(\pi)u_k^0 - g^\pi(\pi)u_k^\pi), \\ z_{Ok} &= \sqrt{2\mu}\int_0^\pi g(a,s)v_{k-1}(s)\,\mathrm{d}s \\ &\quad + g^0(a)u_k^0 + g^\pi(a)u_k^\pi, \end{aligned} \tag{51}$$

where $k = 1,2,3,\dots$ and the initial condition $v_0(\xi)$ can be arbitrary but known. In order to simplify formulae and related computations, we assume in the sequel that the inputs $(u_k^0)_{k\in\mathbb{N}}$ and $(u_k^\pi)_{k\in\mathbb{N}}$ are zero sequences. In this case the discrete-time plant (49) and the observer (51) are described by

$$\begin{aligned} x_k(\xi) &= -x_{k-1}(\xi) + 2\mu\int_0^\pi g(\xi,s)x_{k-1}(s)\,\mathrm{d}s \\ z_k &= \sqrt{2\mu}\int_0^\pi g(a,s)x_{k-1}(s)\,\mathrm{d}s \\ y_k^0 &= \sqrt{2\mu}\int_0^\pi g(0,s)x_{k-1}(s)\,\mathrm{d}s \\ y_k^\pi &= \sqrt{2\mu}\int_0^\pi g(\pi,s)x_{k-1}(s)\,\mathrm{d}s, \end{aligned} \tag{52}$$

and

$$\begin{aligned} v_k(\xi) &= -v_{k-1}(\xi) + 2\mu\int_0^\pi g(\xi,s)v_{k-1}(s)\,\mathrm{d}s \\ &\quad - l^0(\xi)\sqrt{2\mu}\int_0^\pi g(0,s)v_{k-1}(s)\,\mathrm{d}s \\ &\quad - l^\pi(\xi)\sqrt{2\mu}\int_0^\pi g(\pi,s)v_{k-1}(s)\,\mathrm{d}s \\ &\quad + l^0(\xi)y_k^0 + l^\pi(\xi)y_k^\pi \\ z_{Ok} &= \sqrt{2\mu}\int_0^\pi g(a,s)v_{k-1}(s)\,\mathrm{d}s, \end{aligned} \tag{53}$$

with $v_0(\xi) \equiv 0$ (for simplicity). In our stability analysis of (53) it will be covenient to use eigenvalues $(\lambda_n)_{n\in\mathbb{N}}$ and eigenfunctions $(\psi_n(\xi))_{n\in\mathbb{N}}$ of the operator A and for our problem the set $(\psi_n(\xi))_{n\in\mathbb{N}}$ forms an orthonormal basis in $X = L_2(0,\pi)$. One gets (e.g., Kythe, 2011)

$$\lambda_n = -n^2, \quad n = 0,1,2,3,\dots,$$

$$\psi_0(\xi) = \frac{1}{\sqrt{\pi}}, \quad \psi_n(\xi) = \sqrt{\frac{2}{\pi}}\cos(n\xi), \quad n \geq 1. \tag{54}$$

Consequently, the operator $A^d = (\mu I + A)(\mu I - A)^{-1}$ has eigenvalues $(\mu_n)_{n\in\mathbb{N}}$, which can be expressed in terms of μ and λ_n, and its eigenfunctions $(\psi_n(\xi))_{n\in\mathbb{N}}$ coincide with those of A, i.e., we have

$$\mu_n = \frac{\mu + \lambda_n}{\mu - \lambda} = \frac{\mu - n^2}{\mu + n^2}, \quad n = 0,1,2,3,\dots,$$

$$\psi_0(\xi) = \frac{1}{\sqrt{\pi}}, \quad \psi_n(\xi) = \sqrt{\frac{2}{\pi}}\cos(n\xi), \quad n \geq 1. \tag{55}$$

It is also convenient to use eigenfunction expansions for the Green function (see (47))

$$g(\xi,s) = \sum_{n\in\mathbb{N}} \frac{\psi_n(\xi)\psi_n(s)}{\mu - \lambda_n},$$

and also for

$$v(\xi) = \sum_{n\in\mathbb{N}} v_n\psi_n(\xi), \qquad v_n = \int_0^\pi v(\xi)\psi_n(\xi)\,\mathrm{d}\xi,$$
$$l^0(\xi) = \sum_{n\in\mathbb{N}} l_n^0\psi_n(\xi), \qquad l_n^0 = \int_0^\pi l^0(\xi)\psi_n(\xi)\,\mathrm{d}\xi,$$
$$l^\pi(\xi) = \sum_{n\in\mathbb{N}} l_n^\pi\psi_n(\xi), \qquad l_n^\pi = \int_0^\pi l^\pi(\xi)\psi_n(\xi)\,\mathrm{d}\xi.$$

In this case we obtain the expressions

$$A^d v = \begin{bmatrix} \psi_0(\xi) & \psi_1(\xi) & \cdots \end{bmatrix} \times \begin{bmatrix} \frac{\mu+\lambda_0}{\mu-\lambda_0} & 0 & \cdots \\ 0 & \frac{\mu+\lambda_1}{\mu-\lambda_1} & \cdots \\ & & \ddots \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \end{bmatrix} \tag{56}$$

and

$$LC_2^d v = \begin{bmatrix} \psi_0(\xi) & \psi_1(\xi) & \cdots \end{bmatrix} \begin{bmatrix} l_0^0 & l_0^\pi \\ l_1^0 & l_1^\pi \\ \vdots & \vdots \end{bmatrix} \times \begin{bmatrix} \frac{\sqrt{2\mu}\psi_0(0)}{\mu-\lambda_0} & \frac{\sqrt{2\mu}\psi_1(0)}{\mu-\lambda_1} & \cdots \\ \frac{\sqrt{2\mu}\psi_0(\pi)}{\mu-\lambda_0} & \frac{\sqrt{2\mu}\psi_1(\pi)}{\mu-\lambda_1} & \cdots \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \end{bmatrix}. \tag{57}$$

One can see that starting from $n = 1$ on all eigenvalues μ_n satisfy the condition $|\mu_n| < 1$ and for $n = 0$ the eigenvalue

$$\mu_0 = \frac{\mu+\lambda_0}{\mu-\lambda_0} = 1$$

implies that A^d is unstable, so we have to use an output injection operator L to make $A^d - LC_2^d$ strongly stable. The operator $A^d - LC_2^d$, with respect to the basis (55), allows the following matrix representation:

$$\begin{bmatrix} \mu_0 & 0 & \cdots & 0 & \cdots \\ 0 & \mu_1 & \cdots & 0 & \cdots \\ & & \ddots & & \\ 0 & 0 & \cdots & \mu_n & \cdots \\ & & & & \ddots \end{bmatrix} - \begin{bmatrix} l_0^0 & l_0^\pi \\ l_1^0 & l_1^\pi \\ \vdots & \vdots \\ l_n^0 & l_n^\pi \\ \vdots & \vdots \end{bmatrix} \begin{bmatrix} c_0^0 & c_1^0 & \cdots & c_n^0 & \cdots \\ c_0^\pi & c_1^\pi & \cdots & c_n^\pi & \cdots \end{bmatrix}, \tag{58}$$

which we shortly write as

$$\mathbb{M} - \mathbb{L}\mathbb{C},$$

where we used the notation

$$\mu_n = \frac{\mu+\lambda_n}{\mu-\lambda},$$
$$c_n^0 = \frac{\sqrt{2\mu}\psi_n(0)}{\mu-\lambda_n},$$
$$c_n^\pi = \frac{\sqrt{2\mu}\psi_n(\pi)}{\mu-\lambda_n}.$$

It is clear that we can express the state space $X = L_2(0,\pi)$ as a a direct sum $X = X_0 \oplus X_\infty$, where X_0 is a one-dimensional subspace spanned by $\psi_0(\xi)$ (basis in X_0) and X_∞ is an infinite-dimensional subspace spanned by $(\psi_n(\xi))_{n=1}^\infty$ (basis in X_∞).

According to this decomposition, we restrict ourselves to $\mathbb{L}$ with the first row containing non-zero entries and the remaining rows with zero entries, i.e.,

$$\mathbb{L} = \left[\begin{array}{c} \mathbb{L}_0 \\ \hline 0 \end{array}\right] = \left[\begin{array}{cc} l_0^0 & l_0^\pi \\ \hline 0 & 0 \\ \vdots & \vdots \end{array}\right]. \tag{59}$$

Consequently, using the partitions

$$\mathbb{M} = \left[\begin{array}{c|c} \mathbb{M}_0 & 0 \\ \hline 0 & \mathbb{M}_\infty \end{array}\right], \quad \mathbb{C} = \left[\begin{array}{c|c} \mathbb{C}_0 & \mathbb{C}_\infty \end{array}\right],$$

where, for simplicity, we use the notation

$$\mathbb{M}_0 = \begin{bmatrix} \mu_0 \end{bmatrix}, \quad \mathbb{M}_\infty = \begin{bmatrix} \mu_1 & 0 & \cdots & 0 \\ 0 & \mu_2 & \cdots & 0 \\ & & \ddots & \end{bmatrix}, \tag{60}$$

and

$$\mathbb{C}_0 = \begin{bmatrix} c_0^0 \\ c_0^\pi \end{bmatrix}, \quad \mathbb{C}_\infty = \begin{bmatrix} c_1^0 & c_2^0 & \cdots \\ c_1^\pi & c_2^\pi & \cdots \end{bmatrix}, \tag{61}$$

we can rewrite (58) as follows

$$\begin{aligned} \mathbb{M}_C &= \mathbb{M} - \mathbb{L}\mathbb{C} \\ &= \left[\begin{array}{c|c} \mathbb{M}_0 & 0 \\ \hline 0 & \mathbb{M}_\infty \end{array}\right] - \left[\begin{array}{c} \mathbb{L}_0 \\ \hline 0 \end{array}\right] \left[\begin{array}{c|c} \mathbb{C}_0 & \mathbb{C}_\infty \end{array}\right] \\ &= \left[\begin{array}{c|c} \mathbb{M}_0 - \mathbb{L}_0\mathbb{C}_0 & -\mathbb{L}_0\mathbb{C}_\infty \\ \hline 0 & \mathbb{M}_\infty \end{array}\right]. \end{aligned} \tag{62}$$

It is known (Huang *et al.*, 2016) that for the point spectrum σ_p of bounded operators we have

$$\sigma_p(\mathbb{M}_C) = \sigma_p(\mathbb{M}_0 - \mathbb{L}_0\mathbb{C}_0) \cup \sigma_p(\mathbb{M}_\infty),$$

and hence we obtain

$$\sigma_p(\mathbb{M}_C) = \{\mu_0 - l_0^0 c_0^0 - l_0^\pi c_0^\pi\} \cup (\mu_n)_{n=1}^\infty, \tag{63}$$

Since

$$\mu_0 - l_0^0 c_0^0 - l_0^\pi c_0^\pi = 1 - \sqrt{\frac{2}{\mu\pi}}(l_0^0 + l_0^\pi),$$

we can always choose $l_0^0 \in \mathbb{R}$ and $l_0^\pi \in \mathbb{R}$ such that

$$|1 - \sqrt{\frac{2}{\mu\pi}}(l_0^0 + l_0^\pi)| < 1. \tag{64}$$

For such l_0^0, l_0^π the point spectrum of $A^d - LC_2^d$, i.e.,

$$\begin{aligned}&\sigma_p(A^d - LC_2^d)\\ &= \sigma_p(\mathbb{M}_C) = \{1 - \sqrt{\frac{2}{\mu\pi}}(l_0^0 + l_0^\pi)\} \cup \Big(\frac{\mu - n^2}{\mu + n^2}\Big)_{n=1}^{\infty},\end{aligned} \tag{65}$$

guarantees the strong stability of $A^d - LC_2^d$. To this end in the discrete-time observer (53) we choose

$$l^0(\xi) = l_0^0 \psi_0(\xi) = \frac{l_0^0}{\sqrt{\pi}}, \quad l^\pi(\xi) = l_0^\pi \psi_0(\xi) = \frac{l_0^\pi}{\sqrt{\pi}}. \tag{66}$$

In order to verify the obtained results we carry out numerical computations for the following specific data:

1. The plant initial state $x_0(\xi) = \cos^2(\xi)$ and $a = \pi/2$.

2. The time step $h = 0.05$ and hence $\mu = 2/h = 40$.

3. The observer initial state $v_0(\xi) \equiv 0$ and the output injection

   $$l^0(\xi) = \frac{\sqrt{5}}{2}, \quad l^\pi(\xi) = \frac{\sqrt{5}}{2},$$

   so that

   $$\sigma_p(A^d - LC_2^d) = \{1/2\} \cup \Big(\frac{40 - n^2}{40 + n^2}\Big)_{n=1}^{\infty}.$$

Figure 1 shows the plant state trajectory $(x_k(\xi))_{k=0}^{30}$, computed from (52), and Fig. 2—the observer state trajectory $(v_k(\xi))_{k=0}^{30}$, computed from (53).

Figure 3 combines the plant output $(z_k)_{k=1}^{30}$, computed from (52), with the observer output $(z_{Ok})_{k=1}^{30}$, computed from (53).

Finally, Figure 4 compares the continuous-time plant unmeasured output $(z(t))_{t\in[0,1.5]}$ (see (44)) with the piecewise linear interpolated signal $(z_O(t))_{t\in[0,1.5]}$, defined as follows:

$$\begin{aligned}z_O(t) &= \frac{1}{\sqrt{h}}\big((z_{Ok} - z_{O(k-1)})(\frac{t}{h} - k) + z_{Ok}\big)\\ &= 2\sqrt{5}\big((z_{Ok} - z_{O(k-1)})(20t - k) + z_{Ok}\big)\end{aligned}$$

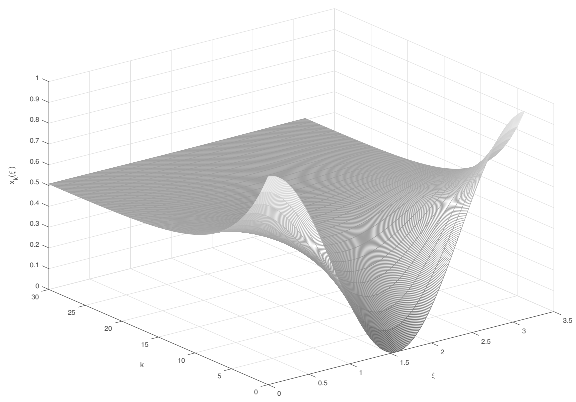

Fig. 1. Plant state trajectory $(x_k(\xi))_{k=0}^{30}$.

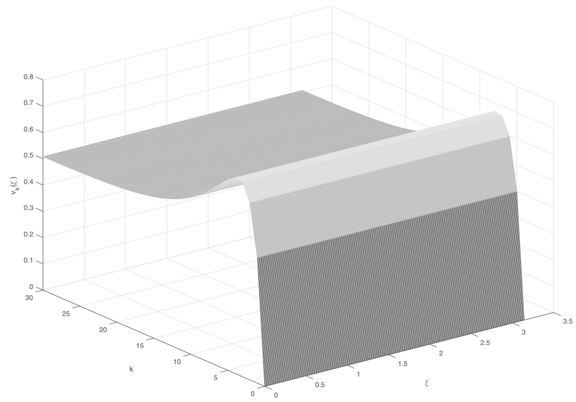

Fig. 2. Observer state trajectory $(v_k(\xi))_{k=0}^{30}$.

for $t \in [(k-1)0.05, k0.05]$, where $k = 1, 2, 3, \dots, 30$.

It follows from Fig. 3 that the observer output $(z_{Ok})_{k\in\mathbb{N}}$ tracks asymptotically the plant output $(z_k)_{k\in\mathbb{N}}$. In turn, Fig. 4 shows that the piecewise linear interpolated signal $(z_O(t))_{t\geq 0}$ can be used to estimate asymptotically the unmeasured output $(z(t))_{t\geq 0}$ of the continuous-time infinite-dimensional plant (44).

6. Conclusion

In the paper we developed a method of solving the general output observation problem for a linear infinite-dimensional system governed by boundary input/output model. By means of the Cayley transform the continuous-time infinite-dimensional system was approximated by a discrete-time infinite-dimensional system and then the procedure of designing a discrete-time output observer was developed. The general results were illustrated with an example of one-dimensional heat equation with boundary inputs and outputs and their numerical effectiveness was tested.

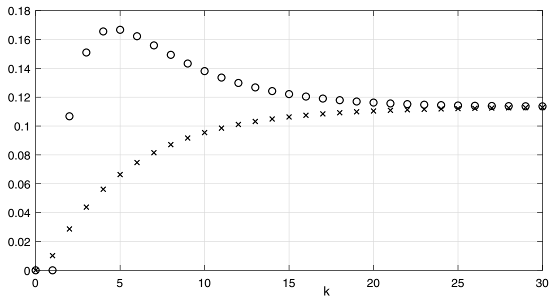

Fig. 3. Plant output $(z_k)_{k=0}^{30}$ (cross line) and observer output $(z_{Ok})_{k=0}^{30}$ (circle line).

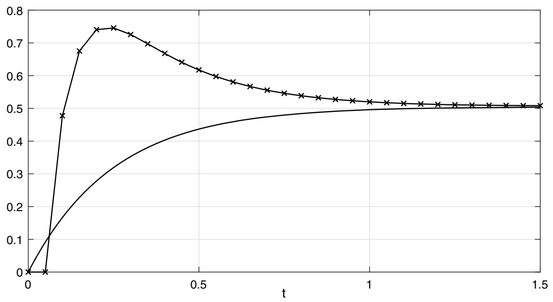

Fig. 4. Plant output $(z(t))_{t\in[0,1.5]}$ (solid line) and piecewise linear signal $(z_O(t))_{t\in[0,1.5]}$ (cross line).

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