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OPTIMAL HARVESTING OF THE NONLINEAR POPULATION DYNAMICS

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This paper deals with an optimal harvesting problem for a nonlinear age-dependent population dynamics. The existence and uniqueness of a positive solution for the model considered is demonstrated. The existence of an optimal harvesting effort and the convergence of a certain fractional step scheme are investigated. Necessary optimality conditions for some approximating problems are established.

Keywords: population dynamics, Carathéodory solution, optimal harvesting, fractional step scheme, necessary optimality conditions

1. Introduction

For a single population species denote by $p(a,t)$ the density of individuals of age $a \in (0, a_\dagger)$, at the moment $t \in (0,T)$ (here $a_\dagger$, $T \in (0,+\infty)$; $a_\dagger$ is the maximal age for the considered population species). Consider the following model for the population dynamics:

$$\begin{cases} p_t + p_a + \mu(a,t)p + \Phi\big(t, P(t)\big)p = -u(t)p, & (a,t) \in Q, \\ p(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a,t)p(a,t)\,\mathrm{d}a, & t \in (0,T), \\ P(t) = \displaystyle\int_0^{a_\dagger} p(a,t)\,\mathrm{d}a, & t \in (0,T), \\ p(a,0) = p_0(a), & a \in (0,a_\dagger), \end{cases} \tag{1}$$

where $Q = (0,a_\dagger) \times (0,T)$. System (1) describes the evolution of an age-structured population subject to harvesting. Here $\beta(a,t)$ is the fertility rate, $\mu(a,t)$ is the mortality rate and $u(t)$ is the harvesting rate (effort).

Note that $P(t)$ is the total population so that in (1) the term $\Phi(t,P(t))$ represents an additional mortality rate (due to the crowding) which does not depend on the age (Gurtin and MacCamy, 1979). The harvesting effort acts as a mortality rate.

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We shall use the following hypotheses:

(H1) $\beta \in L^\infty(Q), \quad \beta(a,t) \geq 0, \quad \text{a.e. in } Q,$

(H2a) $\mu \in L^1_{\text{loc}}([0,a_\dagger) \times [0,T]), \quad \mu(a,t) \geq 0, \quad \text{a.e. in } Q,$

(H2b) $\displaystyle\int_0^{\min\{a_\dagger,t\}} \mu(a_\dagger - h, t-h)\,\mathrm{d}h = +\infty, \quad \text{a.e. } t \in (0,T),$

(H3) $\Phi\colon [0,T] \times [0,+\infty) \to [0,+\infty)$ is continuously differentiable,

and the initial density p_0 satisfies

(H4) $p_0 \in L^\infty(0,a_\dagger), \quad p_0(a) > 0, \quad \text{a.e. on } (0,a_\dagger).$

Hypotheses (H1), (H2a), (H3) and (H4) are all in accordance with practical observations of biological populations. We also refer to (Iannelli, 1995; Webb, 1985).

As regards hypothesis (H2b), let us observe that this is the necessary and sufficient condition for $a_\dagger$ to be the maximal age of the population species (i.e. $p(a_\dagger,t) = 0$ a.e. $t \in (0,T)$, where p is the solution to (1)). We shall sketch the proof for the case when $u := 0$ (otherwise we can put $\mu := \mu + u$). Indeed, if we denote by p the solution to (1) corresponding to $u := 0$, then we have

$$p(a_\dagger,t) = \exp\left\{-\int_0^{\min\{a_\dagger,t\}} \Big[\mu(a_\dagger - h, t-h) + \Phi\big(t-h, P(t-h)\big)\Big]\,\mathrm{d}h\right\}$$
$$\times\, p\big(a_\dagger - \min\{a_\dagger,t\}, t - \min\{a_\dagger,t\}\big)$$

for almost all $t \in (0,T)$ and, since the application $t \mapsto \Phi(t,P(t))$ is bounded and $p(a_\dagger - \min\{a_\dagger,t\}, t - \min\{a_\dagger,t\}) > 0$ for almost all $t \in (0,T)$, we conclude that

$$p(a_\dagger,t) = 0 \Leftrightarrow \exp\left\{-\int_0^{\min\{a_\dagger,t\}} \mu(a_\dagger - h, t-h)\,\mathrm{d}h\right\} = 0,$$

which is equivalent to (H2b).

Suppose that the harvesting effort (which is the control) u belongs to:

$$\mathcal{U} = \Big\{v \in L^\infty(0,T);\quad 0 \leq v(t) \leq L, \quad \text{a.e. on } (0,T)\Big\}$$

($L \in (0,+\infty)$). If we denote by p^u the solution to (1), we may formulate the optimal harvesting problem as:

(P_0) Maximize $\displaystyle\int_0^T \int_0^{a_\dagger} u(t)w(a)p^u(a,t)\,\mathrm{d}a\,\mathrm{d}t,$

subject to $u \in \mathcal{U}$. Here $w(a)$ is a certain weight (it is possible to consider it as the cost of an individual of age a) which satisfies

(H5) $w \in L^1(0,a_\dagger), \quad w(a) > 0, \quad \text{a.e. on } (0,a_\dagger).$

We deal here with a slightly more general problem than that in (Aniţa, 1998). Since the model (1) is separable (Aniţa, 1998) we can get a solution to (1) (in the sense precised in the above-mentioned paper) of the form

$$p(a,t) = y(t)\tilde{p}(a,t), \tag{2}$$

where $\tilde{p}$ is the solution to:

$$\begin{cases} \tilde{p}_t + \tilde{p}_a + \mu(a,t)\tilde{p} = 0, & (a,t) \in Q, \\ \tilde{p}(0,t) = \displaystyle\int_0^{a_\dagger} \beta(a,t)\tilde{p}(a,t)\,\mathrm{d}a, & t \in (0,T), \\ \tilde{p}(a,0) = p_0(a), & a \in (0,a_\dagger). \end{cases} \tag{3}$$

System (3) has a unique solution, i.e. $\tilde{p} \in L^\infty(Q)$ and

$$D\tilde{p}(a,t) = -\mu(a,t)\tilde{p}(a,t), \qquad \text{a.e. in } Q, \tag{4a}$$

$$\lim_{h\to 0^+} \tilde{p}(h,t+h) = \int_0^{a_\dagger} \beta(a,t)\tilde{p}(a,t)\,\mathrm{d}a, \qquad \text{a.e. } t \in (0,T), \tag{4b}$$

$$\lim_{h\to 0^+} \tilde{p}(a+h,h) = p_0(a), \qquad \text{a.e. on } (0,a_\dagger), \tag{4c}$$

which is strictly positive (Iannelli, 1995). Here $D\tilde{p}$ is the directional derivative

$$D\tilde{p}(a,t) = \lim_{h\to 0} \frac{1}{h}\Big[\tilde{p}(a+h,t+h) - \tilde{p}(a,t)\Big].$$

Note that by (4a) a solution $\tilde{p}$ to (4) must be an absolutely continuous function on almost every line of equation $a - t = k$, $(a,t) \in \bar{Q}$, $k \in \mathbb{R}$, so that (4b) and (4c) are meaningful.

Using now (2) and (1), we obtain

$$\begin{cases} \Big[y'(t) + \Phi\big(t, P_0(t)y(t)\big)y(t) + u(t)y(t)\Big]\tilde{p}(a,t) = 0, & \text{a.e. } (a,t) \in Q, \\ y(0) = 1, \end{cases}$$

and since $\tilde{p}(a,t) > 0$ a.e. in Q, we deduce that y is the solution to

$$\begin{cases} y'(t) + \Phi\big(t, P_0(t)y(t)\big)y(t) + u(t)y(t) = 0, & \text{a.e. } t \in (0,T), \\ y(0) = 1, \end{cases} \tag{5}$$

where $P_0(t) = \int_0^{a_\dagger} \tilde{p}(a,t)\,\mathrm{d}a$, $t \in [0,T]$.

It was proved in (Aniţa, 1998) that problem (1) has a unique solution which is strictly positive almost everywhere in Q. It was shown that this solution p^u satisfies (2) a.e., where y is the unique Carathéodory solution to (5).

If we denote by y^u the Carathéodory solution to (5), then Problem $(\mathrm{P_0})$ is equivalent to the following one:

$$\text{(P)} \qquad \text{Maximize} \int_0^T m(t)u(t)y^u(t)\,\mathrm{d}a\,\mathrm{d}t,$$

subject to $u \in \mathcal{U}$, where $m(t) = \int_0^{a_\dagger} w(a)\tilde{p}(a,t)\,\mathrm{d}a$, $t \in [0,T]$.

In conclusion, $(\mathrm{P_0})$ is equivalent to (P), because

$$\int_0^T \int_0^{a_\dagger} u(t)w(a)p^u(a,t)\,\mathrm{d}a\,\mathrm{d}t = \int_0^T m(t)u(t)y^u(t)\,\mathrm{d}t,$$

thus any result in this paper can be easily translated into a result for the original problem. We notice that (P) depends on the initial datum $p_0(a)$ via the term $P_0(t)$.

We mention that the optimal harvesting problem for a linear age-structured population with some assumptions on the structure of the problem was previously studied in (Aniţa, 1998; Brokate, 1985; Gurtin and Murphy, 1981; Murphy and Smith, 1990). The optimal harvesting effort for periodic linear age-dependent population dynamics was studied in (Aniţa *et al.*, 1998).

The paper is organized as follows. In Section 2, we prove the existence of an optimal control for (P). Section 3 concerns a fractional step scheme for Problem (P) and in Section 4 we obtain necessary optimality conditions for the approximating problems.

2. Existence of an Optimal Control for (P)

Consider the following optimal harvesting problem:

$$\text{(P)} \qquad \text{Maximize} \int_0^T m(t)u(t)y^u(t)\,\mathrm{d}t,$$

subject to $u \in \mathcal{U}$, y^u being the Carathéodory solution of

$$\begin{cases} y'(t) + \Phi\big(t, P_0(t)y(t)\big)y(t) = -u(t)y(t), & t \in (0,T), \\ y(0) = y_0 \in (0,+\infty). \end{cases} \tag{6}$$

This is a slightly more general problem than (P) in the previous section.

Theorem 1. *There exists at least one optimal control for (P).*

The proof is analogous to that of Theorem 3.1 in (Aniţa, 1998). First of all, we can prove the following result.

Lemma 1. *If* $\{u_n\} \subset \mathcal{U}$ *satisfies* $u_n \to u$ *weakly in* $L^2(0,T)$, *then*

$$y^{u_n} \to y^u \quad \text{in } L^2(0,T).$$

Proof. The Carathéodory solution to (6) corresponding to $u := u_n$ satisfies

$$y^{u_n}(t) = \exp\left[-\int_0^t \Big(u_n(s) + \Phi\big(s, P_0(s)y^{u_n}(s)\big)\Big)\,\mathrm{d}s\right] y_0, \tag{7}$$

for any $t \in [0,T]$ and this implies

$$0 \le y^{u_n}(t) \le y_0, \quad \text{for any } t \in [0,T].$$

If we denote by

$$v_n(t) = \Phi\big(t, P_0(t)y^{u_n}(t)\big), \quad \text{a.e. } t \in (0,T),$$

then we observe that

$$0 \le v_n(t) \le M, \quad \text{a.e. } t \in (0,T),$$

where $M \in (0,+\infty)$ is a constant. For a subsequence (also denoted by $\{v_n\}$) we have

$$v_n \to v, \quad \text{weakly in } L^2(0,T).$$

The last convergence and (7) allow us to conclude that

$$y^{u_n} \to \tilde{y}, \quad \text{in } L^2(0,T),$$

where $\tilde{y}$ is the Carathéodory solution to

$$\begin{cases} y'(t) + v(t)y(t) = -u(t)y(t), & t \in (0,T), \\ y(0) = y_0. \end{cases}$$

The last two convergence results imply that $v(t) = \Phi(t, P_0(t)\tilde{y}(t))$ for almost all $t \in (0,T)$ and consequently $\tilde{y} = y^u$. ■

Proof of Theorem 1. Consider now

$$d = \sup_{u \in \mathcal{U}} \int_0^T m(t)u(t)y^u(t)\,\mathrm{d}t.$$

It is obvious that $d \in [0,+\infty)$ and that there exist $u_n \in \mathcal{U}$ such that

$$d - \frac{1}{n} \le \int_0^T m(t)u_n(t)y^{u_n}(t)\,\mathrm{d}t \le d, \quad \forall\, n \in \mathbb{N}^*.$$

There exists a subsequence (also denoted by $\{u_n\}$) such that

$$u_n \to u^* \quad \text{weakly in } L^2(0,T)$$

and by Lemma 1 we obtain

$$y^{u_n} \to y^{u^*} \quad \text{in } L^2(0,T).$$

The last two convergence results imply that

$$my^{u_n} \to my^{u^*} \quad \text{in } L^2(0,T)$$

(because $m \in L^\infty(0,T)$), and so

$$\int_0^T m(t)u_n(t)y^{u_n}(t)\,\mathrm{d}t \to \int_0^T m(t)u^*(t)y^{u^*}(t)\,\mathrm{d}t$$

together with

$$d = \int_0^T m(t)u^*(t)y^{u^*}(t)\,\mathrm{d}t.$$

We thus conclude that (u^*, y^{u^*}) is an optimal pair for problem (P). ∎

3. A Fractional Step Scheme

We shall prove that Problem (P) can be 'approximated' (for $\varepsilon \to 0^+$) by the following sequence of optimal control problems:

$$(\mathrm{P}_\varepsilon) \qquad \text{Maximize} \int_0^T m(t)u(t)y_\varepsilon^u(t)\,\mathrm{d}t,$$

subject to $u \in \mathcal{U}$, y_ε^u being the Carathéodory solution to

$$\begin{cases} y'(t) + \gamma(t)y(t) = -u(t)y(t), & t \in (i\varepsilon, (i+1)\varepsilon), \\ y(i\varepsilon+) = F\big((i+1)\varepsilon-; i\varepsilon, y(i\varepsilon-)\big), & i = 0, 1, \ldots, N-1, \quad \varepsilon = T/N, \\ y(0-) = y_0, \end{cases}$$

where $F(t; i\varepsilon, x)$ is the Carathéodory solution to

$$\begin{cases} F'(t) + \Phi\big(t, P_0(t)F(t)\big)F(t) = \gamma(t)F(t), & t \in \big(i\varepsilon, (i+1)\varepsilon\big), \\ F(i\varepsilon+) = x. \end{cases}$$

Here $\gamma \in C([0,T])$ is arbitrary. For other results concerning some fractional step schemes we refer to (Aniţa, 1988; Barbu, 1988; 1994; Barbu and Iannelli, 1993).

Using an analogous argument as in the previous section it is possible to prove that (P_ε) has at least one optimal pair. In the same manner as in (Aniţa, 1998) we can prove the following result.

Lemma 2. *If $u_\varepsilon \to u$ weakly in $L^2(0,T)$ for $\varepsilon \to 0^+$ $(u_\varepsilon \in \mathcal{U})$, then*

$$y_\varepsilon^{u_\varepsilon} \to y^u \quad \text{in } BV([0,T]),$$

for $\varepsilon \to 0^+$.

Consider $\phi,\ \phi_\varepsilon : \mathcal{U} \to [0, +\infty)$ defined by

$$\phi(u) = \int_0^T m(t)u(t)y^u(t)\,\mathrm{d}t$$

and

$$\phi_\varepsilon(u) = \int_0^T m(t)u(t)y_\varepsilon^u(t)\,\mathrm{d}t$$

respectively, and u_ε^* as an optimal control for (P_ε). We conclude this section with the main result.

Theorem 2. *If u^* is a weak limit point of $\{u_\varepsilon^*\}$ in $L^2(0,T)$, then u^* is an optimal control for (P) and, in addition,*

$$\lim_{\varepsilon\to 0+} \phi(u_\varepsilon^*) = \phi(u^*) \tag{8}$$

and

$$\lim_{\varepsilon\to 0+} \phi_\varepsilon(u_\varepsilon^*) = \phi(u^*). \tag{9}$$

Proof. Since

$$\phi_\varepsilon(u_\varepsilon^*) = \int_0^T m(t)u_\varepsilon^*(t)y_\varepsilon^{u_\varepsilon^*}(t)\,\mathrm{d}t \ge \int_0^T m(t)u(t)y_\varepsilon^u(t)\,\mathrm{d}t, \quad \text{for any } u \in \mathcal{U},$$

using Lemma 2, we conclude that

$$\int_0^T m(t)u^*(t)y^{u^*}(t)\,\mathrm{d}t \ge \int_0^T m(t)u(t)y^u(t)\,\mathrm{d}t, \quad \text{for any } u \in \mathcal{U}.$$

This means that u^* is an optimal control for (P).

Now, since

$$u_\varepsilon^* \to u^* \quad \text{weakly in } L^2(0,T)$$

and

$$y_\varepsilon^{u_\varepsilon^*} \to y^{u^*} \quad \text{in } L^2(0,T),$$

we infer that (9) holds.

Using now the convergence

$$y^{u_\varepsilon^*} \to y^{u^*} \quad \text{in } L^2(0,T)$$

(see Section 2) we obtain relation (8). ■

4. The Maximum Principle for (P_ε)

We shall establish here the maximum principle for Problem (P_ε). For that purpose, suppose

(H6a) $\quad m \in C^1([0,T])$,

(H6b) $\quad \gamma - \dfrac{m'}{m}$ is not constant on any subset of a positive measure,

which is fullfilled under certain additional assumptions on p_0 (see Aniţa, 1998) ($\gamma \in C([0,T])$ is chosen in order to satisfy (H6b)).

The main result of this section is as follows:

Theorem 3. *If* $(u_\varepsilon, y_\varepsilon)$ *is an optimal pair for* (P_ε) *and if* q *is the Carathéodory solution to*

$$q'(s) - \gamma(s)q(s) + \frac{m'(s)}{m(s)}q(s) = u_\varepsilon(s)\big(1+q(s)\big), \qquad s \in \big(i\varepsilon, (i+1)\varepsilon\big), \tag{10a}$$

$$q(i\varepsilon-) = \frac{\partial F}{\partial x}\big((i+1)\varepsilon-; i\varepsilon, y_\varepsilon(i\varepsilon-)\big)q(i\varepsilon+), \tag{10b}$$

$$q(T+) = 0, \tag{10c}$$

then

$$u_\varepsilon(s) = \begin{cases} 0 & \text{if } 1+q(s) < 0, \\ L & \text{if } 1+q(s) > 0. \end{cases} \tag{11}$$

Proof. Since $(u_\varepsilon, y_\varepsilon)$ is an optimal pair, we have

$$\int_0^T m(s)u_\varepsilon(s)y_\varepsilon(s)\,\mathrm{d}s \ge \int_0^T m(s)(u_\varepsilon + \eta v)(s)y_\varepsilon^{u_\varepsilon+\eta v}(s)\,\mathrm{d}s$$

for any $v \in L^\infty(Q)$ such that $u_\varepsilon + \eta v \in \mathcal{U}$ and $\eta > 0$ small enough. Consequently, we have

$$\int_0^T m(s)u_\varepsilon(s)\frac{y_\varepsilon^{u_\varepsilon+\eta v} - y_\varepsilon}{\eta}(s)\,\mathrm{d}s + \int_0^T m(s)v(s)y_\varepsilon^{u_\varepsilon+\eta v}(s)\,\mathrm{d}s \le 0.$$

Passing to the limit $(\eta \to 0^+)$, we get

$$\int_0^T m(s)(u_\varepsilon z + v y_\varepsilon)(s)\,\mathrm{d}s \le 0, \tag{12}$$

where z is the Carathéodory solution to

$$z'(s) + \gamma(s)z(s) = -u_\varepsilon(s)z(s) - v(s)y_\varepsilon(s), \qquad s \in \big(i\varepsilon, (i+1)\varepsilon\big], \tag{13a}$$

$$z(i\varepsilon+) = \frac{\partial F}{\partial x}\big((i+1)\varepsilon-; i\varepsilon, y_\varepsilon(i\varepsilon-)\big)z(i\varepsilon-), \quad i = 0, 1, \ldots, N-1, \tag{13b}$$

$$z(0-) = 0. \tag{13c}$$

Let q be the solution to (10). Multiplying (10a) by $m(s)z(s)$ and integrating the result over $[0,T]$, we obtain:

$$\int_0^T q'(s)m(s)z(s)\,\mathrm{d}s - \int_0^T \gamma(s)q(s)m(s)z(s)\,\mathrm{d}s$$
$$+\int_0^T m'(s)q(s)z(s)\,\mathrm{d}s = \int_0^T u_\varepsilon(s)(1+q(s))m(s)z(s)\,\mathrm{d}s.$$

After an easy calculation involving (13a), we obtain

$$\sum_{i=0}^{N-1}\Big[m((i+1)\varepsilon)z((i+1)\varepsilon-)q((i+1)\varepsilon-) - m(i\varepsilon)z(i\varepsilon+)q(i\varepsilon+)\Big]$$
$$+\int_0^T m(s)q(s)(u_\varepsilon z+vy_\varepsilon)(s)\,\mathrm{d}s = \int_0^T u_\varepsilon(s)\big(1+q(s)\big)m(s)z(s)\,\mathrm{d}s.$$

Using now (13b)–(13c), we deduce that

$$\int_0^T m(s)q(s)v(s)y_\varepsilon(s)\,\mathrm{d}s = \int_0^T m(s)u_\varepsilon(s)z(s)\,\mathrm{d}s$$

and, via (12), we obtain

$$\int_0^T m(s)v(s)\big(1+q(s)\big)y_\varepsilon(s)\,\mathrm{d}s \le 0,$$

(for any $v \in L^\infty(Q)$ such that $u_\varepsilon + \eta v \in \mathcal{U}$, for $\eta > 0$ small enough), which is equivalent to (11). ■

Remark. If we choose γ such that

$$\gamma(t) > \frac{m'(t)}{m(t)},$$

for any $t \in [0,T]$, then in any interval $(i\varepsilon,(i+1)\varepsilon)$ $(i \in \{0,\ 1,\dots,N-1\})$ the function q has at most one point where it takes the value -1. Indeed, for any $\tau \in (i\varepsilon,(i+1)\varepsilon)$ such that $q(\tau) = -1$, eqn. (10a) implies $q'(t) < 0$ for any t in a neighbourhood of τ. This implies that there is at most one point with this property in the interval $\tau \in (i\varepsilon,(i+1)\varepsilon)$.

Consequently, $1+q$ has at most one zero in every interval $(i\varepsilon,(i+1)\varepsilon)$ and therefore u_ε has the form

$$u_\varepsilon(t) = \begin{cases} L & \text{if } t \in [i\varepsilon,\tau],\\ 0 & \text{if } t \in [\tau,(i+1)\varepsilon], \end{cases} \tag{14}$$

where τ is a point in $[i\varepsilon,(i+1)\varepsilon]$ (u_ε has at most one switching point in $[i\varepsilon,(i+1)\varepsilon]$).

5. Conclusion

The fractional step scheme we have used allows us to conclude that there is a sequence of bang-bang controllers with the structure as in (14) such that

$$\lim_{\varepsilon \to 0^+} \phi(u_\varepsilon^*) = \phi(u^*)$$

(the optimal harvest is 'approximated' by the harvest corresponding to the effort u_ε).

Relation (14) allows us to obtain excellent numerical results for the approximation of the optimal harvest, $\phi(u^*)$.

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