Int. J. Appl. Math. and Comp. Sci., 2000, Vol.10, No.2, 345–356

SENSOR CHARACTERIZATION FOR REGIONAL BOUNDARY OBSERVABILITY

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The purpose of this paper is to report some original results on regional boundary observation and boundary strategic sensors. Characterization of such sensors is related to their spatial structure and location, and aims at achieving regional boundary observability for parabolic systems. An application to a two-dimensional diffusion process and various illustrative examples are demonstrated.

Keywords: parabolic systems, regional boundary observability, boundary strategic sensors

1. Introduction

In various distributed-parameter systems we are interested in the knowledge of system states (Dolecki, 1973; Rolewicz, 1972). This concept is closer to practical situations when we are only interested in the knowledge of the state in a given subregion of the system domain, or if the system is not observable on the whole domain. The notion of regional observability has been introduced only recently by El Jai and Zerrik and the results are finer than those regarding the usual observability (Amouroux *et al.*, 1993; El Jai *et al.*, 1994). A sensor which allows for a unique reconstruction of the state in a given domain (resp. in a given subregion of the domain) is said to be strategic, cf. (El Jai and Pritchard, 1988) (resp. regionally strategic, cf. Zerrik, 1993). The notion of regional observability was then extended to the case where the subregion is a part of the boundary of the system domain (Zerrik *et al.*, 1999).

In this paper, we present some results related to regional boundary strategic sensors and their characterization in connection with regional boundary observability. We then apply the results to a particular parabolic system.

The paper is organized as follows. Section 2 is devoted to the presentation of the system under consideration and preliminaries. We also give results related to regional boundary observability. In Section 3, we characterize this concept in terms of the sensor structure. In the last section, an application to a two-dimensional diffusion process is considered. Examples of various situations are also given and specific results are summarized in a tabular form.

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2. System Description and Preliminaries

Let Ω be an open regular subset of $\mathbb{R}^n$ with sufficiently regular boundary $\partial\Omega$ and Γ a non-empty simply-connected part of $\partial\Omega$. For a given time $T > 0$ let $Q = \Omega\times]0,T[$ and $\Sigma = \partial\Omega\times]0,T[$.

We consider the system described by the equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x,t) &= Ay(x,t) \quad \text{in } Q, \\ \dfrac{\partial y}{\partial \nu_A}(\xi,t) &= 0 \quad \text{on } \Sigma, \\ y(x,0) &= y_0(x) \quad \text{in } \bar{\Omega}, \end{cases} \tag{1}$$

where A is a linear differential operator, with compact resolvent, which generates a strongly continuous semigroup $(S(t))_{t\geq 0}$ on the Hilbert state space $H^1(\Omega)$. In the sequel, A^* stands for the adjoint operator of A. We assume here that $y_0 \in H^1(\bar{\Omega})$. Moreover, the measurements are given by

$$z(t) = Cy(x,t), \tag{2}$$

where $C : H^1(\bar{\Omega}) \longrightarrow \mathbb{R}^p$ is linear and depends on the structure of the sensors employed. The observation space is assumed to be $\mathcal{O} = L^2(0,T;\mathbb{R}^p)$.

System (1) is autonomous and note that (2) allows us to write

$$z(t) = CS(t)y_0(x).$$

Let us recall that a sensor is conventionally defined by a couple (D,f), where:

(i) D denotes a closed subset of $\bar{\Omega}$, which is the spatial support of the sensor,

(ii) f defines the spatial distribution of the sensing measurements on D.

According to the choice of the parameters D and f, we may have various types of sensors. A sensor may be pointwise when $D = \{b\}$ and $f = \delta(\cdot - b)$, where δ is the Dirac mass concentrated at b. The output function (2) can then be written in the form

$$z(t) = y(b,t). \tag{3}$$

In this case the operator C is unbounded and some precautions must be taken (El Jai and Pritchard, 1988).

A sensor may also be of zone type when $D \subset \bar{\Omega}$. The output function (2) can then be written in the form

$$z(t) = \int_D y(x,t)f(x)\,\mathrm{d}x. \tag{4}$$

In both the cases b and D may be internal in Ω or on the boundary $\partial\Omega$.

The concept of strategic sensors was introduced in (El Jai and Pritchard, 1988) and generalized to the regional case in (Zerrik, 1993). Let us recall the definitions:

1. A sensor (D, f) is said to be *strategic* if the observed system is approximately observable.
2. For an non-empty subregion ω which is internal to Ω, a sensor is said to be ω-*strategic* if the observed system is approximately ω-observable.

Consider

- the operator

$$K : \begin{array}{ccc} H^1(\Omega) & \longrightarrow & \mathcal{O} \\ h & \longrightarrow & CS(\cdot)h \end{array}$$

which is linear and bounded, with the adjoint operator defined by

$$K^* : \begin{array}{ccc} \mathcal{O} & \longrightarrow & H^1(\Omega) \\ z^* & \longrightarrow & \displaystyle\int_0^T S^*(\tau)C^*z^*(\tau)\,\mathrm{d}\tau, \end{array}$$

- the trace operator of order zero:

$$\gamma_0 : H^1(\Omega) \longrightarrow H^{\frac{1}{2}}(\partial\Omega)$$

which is linear, surjective and continuous, γ_0^* being its adjoint operator,

- the operator

$$\chi_\Gamma : \begin{array}{ccc} H^{\frac{1}{2}}(\partial\Omega) & \longrightarrow & H^{\frac{1}{2}}(\Gamma) \\ z & \longrightarrow & z_{|\Gamma} \end{array}$$

where $z_{|\Gamma}$ is the restriction of z to Γ (its adjoint is denoted by χ_Γ^*),

- the operator H from $\mathcal{O}$ to $H^{\frac{1}{2}}(\Gamma)$ defined by

$$H = \chi_\Gamma \gamma_0 K^*, \tag{5}$$

- a boundary initial state on Γ, $\tilde{y}_0 = \chi_\Gamma \gamma_0(y_0)$.

The problem of boundary observability is as follows: Given the autonomous system (1) and output (2), characterize the sensors which allow for a unique reconstruction of the state of system (1) in a subregion supposed to be located on the boundary of the system domain. Basic results were established by Zerrik *et al.* (1999). In this paper, we recall the most important of these results and show a relationship between regional boundary observability and boundary strategic sensors.

Regional boundary observability is a natural extension of the concept of regional observability. It concerns the state reconstruction in a given subregion which is a part of the boundary of the domain where the system is defined. For completness, we recall here some relevant results:

Definition 1. System (1) together with output (2) is exactly (resp. approximately) boundary regionally observable on Γ or exactly B-observable on Γ (resp. approximately B-observable on Γ) if $\operatorname{Im} H = H^{\frac{1}{2}}(\Gamma)$ (resp. $\overline{\operatorname{Im} H} = H^{\frac{1}{2}}(\Gamma)$).

The problem of reconstructing the initial state $\tilde{y}_0$ on Γ can be addressed using various approaches (Zerrik *et al.*, 1999). It is clear that

1. The approximate B-observability on Γ amounts to the condition $H^* z^* = 0 \Rightarrow z^* = 0$.
2. If a system is exactly B-observable on Γ, then it is approximately B-observable on Γ.
3. If a system is exactly (resp. approximately) B-observable on Γ, then it is exactly (resp. approximately) B-observable on every subset Γ_1 of Γ.
4. There exist states which are not observable on the whole domain Ω but B-observable on Γ. This is illustrated by the following example.

Example. Consider a two-dimensional system described by the diffusion equation

$$\begin{cases} \dfrac{\partial y}{\partial t}(x_1, x_2, t) = \dfrac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) & \text{in } Q, \\ y(x_1, x_2, 0) = y_0(x_1, x_2) & \text{in } \bar{\Omega}, \\ \dfrac{\partial y}{\partial n}(\xi, \eta, t) = 0 & \text{on } \Sigma, \end{cases} \tag{6}$$

where $\Omega =]0,1[\times]0,1[$, the time interval is $]0,T[$ and $\Gamma = [0,1] \times \{0\}$. The output function is given by

$$z(t) = \int_{\Gamma_0} y(\xi, \eta, t) f(\xi, \eta) \, d\xi \, d\eta, \tag{7}$$

where $f(\xi, \eta) = \cos \pi \eta$ and $\Gamma_0 = \{0\} \times [0,1]$. The operator

$$A = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

generates a semi-group $(S(t))_{t \geq 0}$ on $H^1(\Omega)$ given by

$$S(t)y = \sum_{i,j=0}^{\infty} e^{\lambda_{ij} t} \langle y, \varphi_{ij} \rangle_{H^1(\Omega)} \varphi_{ij},$$

where $\lambda_{ij} = -(i^2 + j^2)\pi^2$, $\varphi_{ij}(x_1, x_2) = 2a_{ij}\cos(i\pi x_1)\cos(j\pi x_2)$ and $a_{ij} = (1 - \lambda_{ij})^{-1/2}$. The state $y_0(x_1, x_2) = \cos(\pi x_1)\cos(2\pi x_2)$ is not weakly observable on Ω ($Ky_0 = 0$, cf. El Jai and Pritchard, 1988) but is approximately $\mathcal{B}$-observable on Γ ($H^* y_0 = 0$, cf. Zerrik *et al.*, 1999). ♦

3. Regional Boundary Strategic Sensors

In the following section, we introduce the notion of regional boundary strategic sensors and characterize same links between regional boundary observability and sensors' structure.

Definition 2. A sensor (D, f) is said to be Γ-strategic if the observed system is approximately B-observable on Γ.

Now, consider system (1) and assume that the measurements are taken by p sensors. The output function is then given by $z(t) = (z_1(t), \dots, z_p(t))$ with $z_i(t) = y(b_i, t)$, $b_i \in \Omega$ for $1 \le i \le p$ in the pointwise case and $z_i(t) = \int_{D_i} y(x,t) f_i(x)\,dx$, $D_i \subset \bar{\Omega}$ for $1 \le i \le p$ in the zone case.

Assume that there exists a complete set of eigenfunctions $(\varphi_i)_{i\in I}$ of A associated with the eigenvalues (λ_i) of multiplicities m_i and $m = \sup_{i\in I} m_i$ is finite. For $x = (x_1, \dots, x_n) \in \Omega$ and $i = (i_1, \dots, i_n) \in I = \mathbb{N}^n$, let $\bar{x} = (x_1, \dots, x_{n-1})$ and $\bar{i} = (i_1, \dots, i_{n-1})$. Suppose that the functions $(\psi_{\bar{i}})$ defined by $\psi_{\bar{i}}(\bar{x}) = \chi_\Gamma \gamma_0 \varphi_i(x)$, $i \in I$ form a complete set in $H^{\frac{1}{2}}(\Gamma)$. Then we have the following result:

Proposition 1. *System (1) together with output (6) is B-observable on Γ iff*

1. $p \ge m$, *and*
2. $\operatorname{rank} G_i = m_i$ *for* $i \in I$,
   where $G_i = (G_i)_{j,k}$, $1 \le j \le m_i$, $1 \le k \le p$,

$$(G_i)_{j,k} = \begin{cases} \varphi_{i_j}(b_k) & \text{in the pointwise case,} \\ \langle \varphi_{i_j}, f_k \rangle_{D_k} & \text{in the zone case.} \end{cases}$$

Proof. For brevity the proof is limited to the case of zone sensors. For $z^* \in H^{\frac{1}{2}}(\Gamma)$ we have

$$\begin{aligned} H^* z^* &= K \gamma_0^* \chi_\Gamma^* z^* \\ &= \Big(\sum_{i\in I} e^{\lambda_i t} \sum_{j=1}^{m_i} \langle \varphi_{i_j}, \gamma_0^* \chi_\Gamma^* z^* \rangle_\Omega \langle \varphi_{i_j}, f_1 \rangle_{D_1}, \\ &\qquad \dots, \sum_{i\in I} e^{\lambda_i t} \sum_{j=1}^{m_i} \langle \varphi_{i_j}, \gamma_0^* \chi_\Gamma^* z^* \rangle_\Omega \langle \varphi_{i_j}, f_p \rangle_{D_p} \Big) \end{aligned}$$

$$= \left(\sum_{i \in I} e^{\lambda_i t} \sum_{j=1}^{m_i} \langle \chi_\Gamma \gamma_0 \varphi_{i_j}, z^* \rangle_\Gamma \langle \varphi_{i_j}, f_1 \rangle_{D_1}, \right.$$

$$\left. \ldots, \sum_{i \in I} e^{\lambda_i t} \sum_{j=1}^{m_i} \langle \chi_\Gamma \gamma_0 \varphi_{i_j}, z^* \rangle_\Gamma \langle \varphi_{i_j}, f_p \rangle_{D_p} \right)$$

$$= \left(\sum_{i \in I} e^{\lambda_i t} \sum_{j=1}^{m_i} \langle \psi_{\bar{i}_j}, z^* \rangle_\Gamma \langle \varphi_{i_j}, f_1 \rangle_{D_1}, \ldots, \sum_{i \in I} e^{\lambda_i t} \sum_{j=1}^{m_i} \langle \psi_{\bar{i}_j}, z^* \rangle_\Gamma \langle \varphi_{i_j}, f_p \rangle_{D_p} \right)$$

If the system is not B-observable on Γ, there exists $z^* \neq 0$ such that $H^* z^* = 0$ which gives $\sum_{j=1}^{m_i} \langle \psi_{\bar{i}_j}, z^* \rangle_\Gamma \langle \varphi_{i_j}, f_k \rangle_{D_k} = 0$ for all $i \in I$, $1 \le k \le p$.

Consider $z_{i_j} = \langle \psi_{\bar{i}_j}, z^* \rangle_\Gamma$ and $z_i = (z_{i_1}, \ldots, z_{i_{m_i}})^t$. Then we obtain $G_i z_i = 0$ $\operatorname{rank} G_i \neq m_i$ for any i.

Conversely, if $\operatorname{rank} G_i \neq m_i$, there exists $i \in I$ such that $z_i = (z_{i_1}, \ldots, z_{i_{m_i}})^t \neq 0$ and $G_i z_i = 0$.

Let $z^* \in H^{\frac{1}{2}}(\Gamma)$ verify

$$\langle \psi_{\bar{l}_j}, z^* \rangle_\Gamma = 0 \qquad \text{for } l \neq i,$$

$$\langle \psi_{\bar{i}_j}, z^* \rangle_\Gamma = z_{i_j} \qquad \text{for } 1 \le j \le m_i.$$

Thus $z^* \neq 0$ and $H^* z^* = 0$, i.e. system (1) is not B-observable on Γ. ∎

Remark 1.

- Proposition 1 implies that the required number of sensors is greater than or equal to the largest multiplicity of the eigenvalues.
- By infinitesimally deforming the domain, the multiplicity of the eigenvalues can be reduced to one (El Jai and El Yacoubi, 1993). Consequently, the B-observability on Γ can be guaranteed by employing only one sensor.

4. Application

Consider the two-dimensional system defined on $\Omega =]0, b[\times]0, d[$ by

$$\begin{cases} \dfrac{\partial y}{\partial t}(x_1, x_2, t) = \dfrac{\partial^2 y}{\partial x_1^2}(x_1, x_2, t) + \dfrac{\partial^2 y}{\partial x_2^2}(x_1, x_2, t) & \text{in } Q, \\ \dfrac{\partial y}{\partial \nu}(\xi_1, \xi_2, t) = 0 & \text{on } \Sigma, \\ y(x_1, x_2, 0) = y_0(x_1, x_2) & \text{in } \bar{\Omega}. \end{cases} \tag{8}$$

Let $\Gamma =]0, b[\times\{0\}$ and assume that the output function is given by (4) or (5). The eigenfunctions associated with (8) are of the form

$$\varphi_{kj}(x_1, x_2) = \frac{2a_{kj}}{\sqrt{bd}} \cos\left(k\pi \frac{x_1}{b}\right) \cos\left(j\pi \frac{x_2}{d}\right)$$

with $a_{kj} = (1 + (k^2/b^2 + j^2/d^2)\pi^2)^{1/2}$.
They correspond to the eigenvalues

$$\lambda_{kj} = -\left(\frac{k^2}{b^2} + \frac{j^2}{d^2}\right)\pi^2$$

of multiplicity of one if $b^2/d^2 \notin \mathbb{Q}$ because if $b^2/d^2 = p/q$ for $p, q \in \mathbb{N}^*$, then the eigenvalue $\lambda_{p+1,q-1} (= \lambda_{p-1,q+1})$ is associated with two different eigenfunctions $\varphi_{p+1,q-1}$ and $\varphi_{p-1,q+1}$. In this case $I = \mathbb{N}^2$, $\overline{(k,j)} = k$ and $\overline{(x_1, x_2)} = x_1$.

The functions

$$\psi_k(x_1) = \frac{\sqrt{2}\, a_k}{\sqrt{b}} \cos\left(k\pi \frac{x_1}{b}\right), \quad k \in \mathbb{N}$$

form a complete set in $H^{\frac{1}{2}}(\Gamma)$.

For $0 < \alpha_1 < \alpha_2 < b$ and $0 < \beta_1 < \beta_2 < d$ we set

$$\eta_1 = \frac{\alpha_1 + \alpha_2}{2}, \quad \eta_2 = \frac{\beta_1 + \beta_2}{2}, \quad \mu_1 = \frac{\alpha_2 - \alpha_1}{2}, \quad \mu_2 = \frac{\beta_2 - \beta_1}{2}.$$

Now consider some examples.

4.1. Case of a Pointwise Sensor

Consider (8) with the output function $z(t) = y(\bar{b}, t)$, where $\bar{b}$ is the sensor location.

4.1.1. Boundary Pointwise Sensor

Suppose that $\bar{b} = (\alpha, 0)$ (cf. Fig. 1(a)) or $\bar{b} = (0, \beta)$ (cf. Fig. 1(b)) with $0 < \alpha < b$ and $0 < \beta < d$.

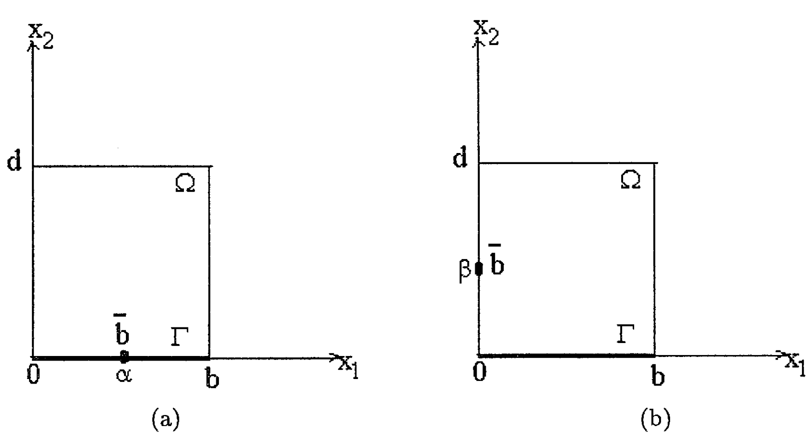

Fig. 1. Location of a boundary pointwise sensor.

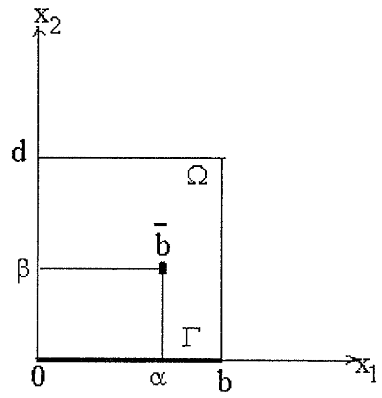

Fig. 2. Location of an internal pointwise sensor.

Corollary 1. *In the case of Fig. 1(a), the sensor is not Γ-strategic if there exists $k \in \mathbb{N}^*$ such that $2k\alpha/b$ is odd. In the case of Fig. 1(b) it is not Γ-strategic if there exists $l \in \mathbb{N}^*$ such that $2l\beta/d$ is odd.*

4.1.2. Internal Pointwise Sensor

Consider the sensor located inside the domain at a point $\bar{b} = (\alpha, \beta)$ (Fig. 2).

Corollary 2. *The sensor is not Γ-strategic if there exists $k, l \in \mathbb{N}^*$ such that $2k\alpha/b$ or $2l\beta/d$ is odd.*

4.2. Case of a Zone Sensor

Here we consider (8) with the output function $z(t) = \int_D y(x_1, x_2, t) f(x_1, x_2)\, dx_1\, dx_2$, where D is the sensor support.

4.2.1. Boundary Zone Sensor

The sensor is located on the boundary along $D =]\alpha_1, \alpha_2[\times \{d\}$ (cf. Fig. 3(a)) or $D = \{0\} \times]\beta_1, \beta_2[$ (Fig. 3(b)). In the case of Fig. 3(a) we have the following result:

Corollary 3.

(a) If f is uniformly distributed on $[\alpha_1, \alpha_2] \times \{0\}$ (or on $[\alpha_1, \alpha_2] \times \{d\}$), then the sensor is not Γ-strategic if $\mu_1/b \in \mathbb{Q}$ or there exists $k \in \mathbb{N}^*$ such that $2k\eta_1/b$ is odd.

(b) If f is symmetric with respect to the point $(\eta_1, 0)$ or with respect to the point (η_1, d), then the sensor is not Γ-strategic if $\eta_1/b \in \mathbb{Q}$.

(c) If f is symmetric with respect to the axis $x = \eta_1$, then the sensor is not Γ-strategic if there exists $k \in \mathbb{N}^*$ such that $2k\eta_1/b$ is odd.

In the case of Fig. 3(b) we get the following characterization.

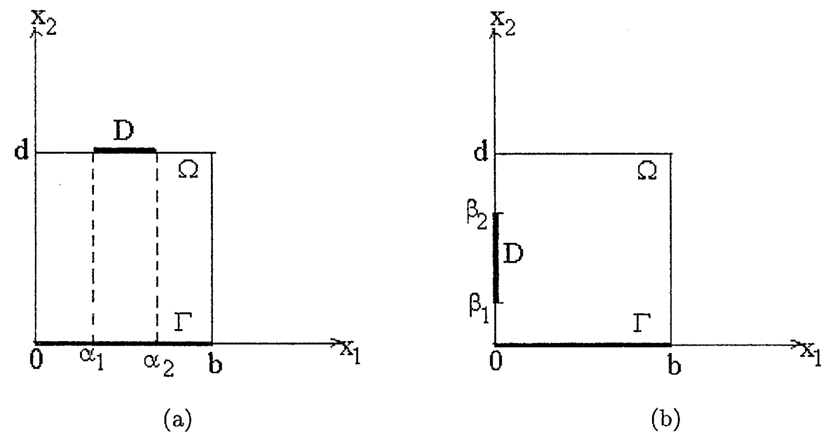

Fig. 3. Location of a boundary zone sensor.

Corollary 4.

(a) If f is uniformly distributed on $\{0\}\times]\beta_1,\beta_2[$ (or on $\{b\}\times]\beta_1,\beta_2[$), then the sensor is not Γ-strategic if $\mu_2/d \in \mathbb{Q}$ or there exists $l \in \mathbb{N}^*$ such that $2l\eta_2/d$ is odd.

(b) If f is symmetric with respect to the point $(0,\eta_2)$ or with respect to the point (b,η_2), then the sensor is not Γ-strategic if $\eta_2/d \in \mathbb{Q}$.

(c) If f is symmetric with respect to the axis $y = \eta_2$, then the sensor is not Γ-strategic if there exists $l \in \mathbb{N}^*$ such that $2l\eta_2/d$ is odd.

4.2.2. Internal Zone Sensor

Suppose that the sensor is located inside the domain over $D =]\alpha_1,\alpha_2[\times]\beta_1,\beta_2[$ (Fig. 4).

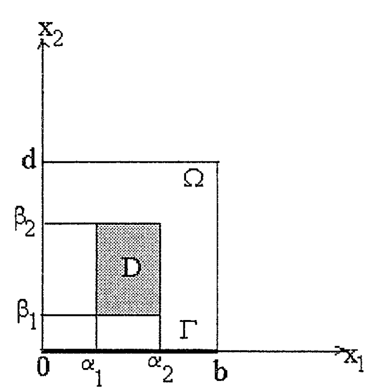

Fig. 4. Location of an internal zone sensor.

Corollary 5.

(a) *If f is uniformly distributed on $]\alpha_1, \alpha_2[\times]\beta_1, \beta_2[$, then the sensor is not Γ-strategic if one of the following properties is verified: $\mu_1/b \in \mathbb{Q}$ or $\mu_2/d \in \mathbb{Q}$ or there exists $k, l \in \mathbb{N}^*$ such that $2k\eta_1/b$ is odd or $2l\eta_2/d$ is odd.*

(b) *If f is symmetric with respect to the point (η_1, η_2), then the sensor is not Γ-strategic if $\eta_1/b \in \mathbb{Q}$ or $\eta_2/d \in \mathbb{Q}$.*

(c) *If f is symmetric with respect to the axis $x = \eta_1$ (or with respect to the axis $y = \eta_2$), then the sensor is not Γ-strategic if there exists $k \in \mathbb{N}^*$ such that $2k\eta_1/b$ is odd (resp. there exists $l \in \mathbb{N}^*$ such that $2l\eta_2/d$ is odd).*

Remark 2.

- The results obtained follow, in general, from symmetry considerations.
- If f is uniformly distributed, it also verifies the axial symmetry and the corollaries are compatible with this observation.

4.3. Recapitulating Table

Tables 1 and 2 summarize the results obtained in the paper.

Tab. 1. Pointwise sensor.

Sensor location	Non Γ-strategic cases
$\alpha \in]0, b[$ and $\beta = 0$ or $\beta = d$	$\exists k \in \mathbb{N}^* \mid 2k\alpha/b$ is odd
$\beta \in]0, d[$ and $\alpha = 0$ or $\alpha = b$	$\exists l \in \mathbb{N}^* \mid 2l\beta/d$ is odd
$\alpha \in]0, b[$ and $\beta \in]0, d[$	$\exists k \in \mathbb{N}^* \mid 2k\alpha/b$ is odd or $\exists l \in \mathbb{N}^* \mid 2l\beta/d$ is odd

Remark 3. From a practical point of view, the distributed system is most often approximated by a finite-dimensional system. Then the conditions of the B-observability on Γ can also be verified for the finite-dimensional system. For instance, in the pointwise sensor case (Fig. 1), if the system is approximated by a seven-dimensional system, then the condition of non B-observability on Γ is $\alpha/b \in I_7$ where $I_7 = \{1/6, 1/4, 1/2, 3/4, 5/6\}$.

5. Conclusion

The concept developed in this paper is related to the regional boundary state reconstruction in connection with the sensors structure (both the distribution and location of the support). It permits us to avoid some 'bad' sensor locations.

Various open questions are still under consideration. For example, the case of the optimal sensor location for the boundary observability problem.

Tab. 2. Zone sensor.

Sensor location	Non Γ-strategic cases
$D = [\alpha_1, \alpha_2] \times \{0\}$ or $D = [\alpha_1, \alpha_2] \times \{d\}$	• f uniformly distributed on D $\mu_1/b \in \mathbb{Q}$ or $\exists k \in \mathbb{N}^* \mid 2k\eta_1/b$ is odd • f symmetric with respect to $(\eta_1, 0)$ or (η_1, d) $\eta_1/b \in \mathbb{Q}$ • f symmetric with respect to the axis $x = \eta_1$ $\exists k \in \mathbb{N}^* \mid 2k\eta_1/b$ is odd
$D = \{0\} \times [\beta_1, \beta_2]$ or $D = \{b\} \times [\beta_1, \beta_2]$	• f uniformly distributed on D $\mu_2/d \in \mathbb{Q}$ or $\exists l \in \mathbb{N}^* \mid 2l\eta_2/d$ is odd • f symmetric with respect to $(0, \eta_2)$ or (b, η_2) $\eta_2/d \in \mathbb{Q}$ • f symmetric with respect to the axis $y = \eta_2$ $\exists l \in \mathbb{N}^* \mid 2l\eta_2/d$ is odd
$D =]\alpha_1, \alpha_2[\times]\beta_1, \beta_2[$	• f uniformly distributed on D $\mu_1/b \in \mathbb{Q}$ or $\mu_2/d \in \mathbb{Q}$ or $\exists k \in \mathbb{N}^* \mid 2k\eta_1/b$ is odd or $\exists l \in \mathbb{N}^* \mid 2l\eta_2/d$ is odd • f symmetric with respect to (η_1, η_2) $\eta_1/b \in \mathbb{Q}$ or $\eta_2/d \in \mathbb{Q}$ • f symmetric with respect to the axis $x = \eta_1$ or $y = \eta_2$ $\exists k \in \mathbb{N}^* \mid 2k\eta_1/b$ is odd or $\exists l \in \mathbb{N}^* \mid 2l\eta_2/d$ is odd

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Received: 15 May 1999
Revised: 8 January 2000