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THE GEOMETRY OF DARLINGTON SYNTHESIS
(in memory of W. Cauer)

PATRICK DEWILDE*

We revisit the classical problem of 'Darlington synthesis', or Darlington embedding. Although traditionally it is solved using analytic means, a more natural way to approach it is to use the geometric properties of a well-chosen Hankel map. The method yields surprising results. In the first place, it allows us to formulate necessary and sufficient conditions for the existence of the embedding in terms of systems properties of the transfer operation to be embedded. In addition, the approach allows us to extend the solution to situations where no analytical transform is available. The paper has a high review content, as all the results presented have been obtained during the last twenty years and have been published. However, we make a systematic attempt at formulating them in a geometric way, independent of an accidental parametrization. The benefit is clarity and generality.

Keywords: Darlington synthesis, Hankel operator, time-varying systems, contractive operators, coprime factorization

1. Introduction

Cauer is best known for the characterization of rational lossless immitances. His interests went further: the properties of functions describing the physical behaviour of electrical circuits interested him in general and he wrote papers on many topics connected with their mathematical modelling and their ability to propagate and filter signals. For example, (Cauer, 1932) is of particular relevance to the subject I wish to considere here, known as 'Darlington synthesis'. Although this topic has strong connections with problems Cauer considered, I wish to focus on a relatively new aspect of it: the geometry of related state spaces. Geometrical considerations of the kind presented here were non-existent in Cauer's time, the treatment was purely of an analytic kind and related to properties of complex functions meromorphic in the complex plane. State space theory has changed that, and although circuit theorists such as Youla and Belevitch (1968) were keenly aware of its relation to

* DIMES, Faculty of Electrical Engineering, Delft University of Technology, Mekelweg 4, 2628 CD Delft, The Netherlands, e-mail: `dewilde@DIMES.tudelft.nl`

their theories, its full geometric power for circuit synthesis only became apparent in the mid-seventies, thanks to the pioneering efforts of Newcomb (1966), and Anderson and Vongpanitlerd (1973). In this presentation I shall give a survey of the impact geometric considerations on state spaces have for Darlington synthesis theory. At the end of the paper I shall make some connections with the classical Cauer problem. I follow a recently published paper (Dewilde, 1999).

Traditional Darlington synthesis (or Darlington embedding) is concerned with the realization of a rational and 'bounded real' transfer function $S(s)$ with bound one as a partial transfer operator of a 'lossless' transfer matrix $\Sigma(s)$ of the form

$$\Sigma(s) = \begin{bmatrix} S(s) & \Sigma_{12}(s) \\ \Sigma_{21}(s) & \Sigma_{22}(s) \end{bmatrix}$$

and in which the Smith-McMillan degree of Σ is equal to that of S (it is also equal to the dimension of the state-space of a minimal system-theoretical realisation for S). 'Losslessness' in this context means that $\Sigma(s)$ is analytic in the right half complex plane, bounded in norm by one uniformly over s in that region, and unitary on the imaginary axis. Also, the additional number of input and output signals introduced by the definition of $\Sigma(s)$ must be finite. Within these constraints, I want to consider the existential question: When does the Darlington synthesis exist? It is known classically that when $S(s)$ is rational, the Darlington embedding does indeed exist. But what if it is not rational? And what if non-stationary systems, which do not have a Laplace transform, are considered? The norm constraints make perfect sense 'in the time domain', but analytic properties are lacking. It turns out that Darlington synthesis is not always possible, and that its existence is strongly related to geometric properties of the state space of the system.

In the sequel we shall not use the Laplace variable s; we shall rather work with the related bilinear transform $z \triangleq (1-s)/(1+s)$. The advantage of this is putting time-continuous and time-discrete theory on an equal footing without essential changes. The right half plane gets mapped into the unit disc, and the more comfortable Hardy space theory on the unit disc can be used. All the properties we shall consider are unaffected by such a change in the stationary case.

2. The Non-Rational Case

Suppose that an $m \times m$ causal and contractive transfer function $S(z)$ is given (we may assume it to be square but non-rational). Let $T_*(z)$ indicate for any transfer operator $T(z)$ the para-hermitian conjugate, i.e. $[T_*(z)]_{i,j} = \overline{[T(1/\bar{z})]}_{j,i}$. It is the analytical continuation of the Hermitian conjugate of $T(e^{i\theta})$ to the outside of the unit disc. We solve the Darlington problem in two steps: first we compute Σ_{12} such that $\Sigma_{21}\Sigma_{12*}(z) = I - S(z)S_*(z)$ is a spectral factorization, which makes $[S(z)\ \Sigma_{12}(z)]$ isometric, and next, we compute a unitary extension if it exists. The first step is of course necessary; necessary and sufficient conditions for its existence have been derived first for the scalar case by Szegö (1920), and for the matrix-function case by Wiener and Masani (1957)—an insightful treatment is to be found in (Helson, 1964). The necessary and sufficient condition is given in terms of the 'spectral density' $W(e^{i\theta}) = I - S(e^{i\theta})S(e^{i\theta})^*$ as a property of the entropy or Szegö integral

$$\int_{-\pi}^{\pi} \log\det W(e^{i\theta})\frac{\mathrm{d}\theta}{2\pi} > -\infty.$$

Clearly not all contractive S have a Darlington synthesis; for example, an ideal low-pass filter (magnitude one in the passband, epsilon in the stopband) will not have one, since the Szegö condition for the spectral factor will not be satisfied. But what if the Szegö condition is satisfied? It turns out that even then the embedding does not necessarily exist. We explore the question in the next section.

3. State Space Geometry

The central operator determining state space geometry is the Hankel operator. In the discrete time setting a signal u can be decomposed into a part u_p belonging to the strict past and a part u_f belonging to present and future ($u = u_p + u_f$). In the time-continuous case a similar decomposition can be made, but then it is done with respect to countable bases for the input and the output space, which respect the causality. In our context, a signal space is endowed with a quadratic norm (it is a Hilbert space), u_f can be viewed as a projection ($\mathbf{P}$) of u 'on the future' $\mathcal{U}_f$, and u_p can be viewed as a projection 'on the strict past' $\mathcal{U}_p$. If S is a system map, then its Hankel map is defined as

$$H_S\colon\ u_p \in \mathcal{U}_p \mapsto u_f = u_p H_S \in \mathcal{U}_f.$$

The geometry of the Hankel map is closely related to the geometry of the system. Its range is the minimal observability space, while its co-range is the minimal reachability space. The orthogonal complements of these spaces are the kernels of the map and the co-map. They have important invariance properties. Let $\mathbf{k}$ be the kernel of H_S. Its Fourier transform $\mathbf{k}^\sim$ is a z^{-1}-shift invariant space. By the Beurling-Lax theorem of Hardy space theory, there will exist an isometric causal transfer function U_ℓ such that $\mathbf{k}^\sim = \mathcal{U}_p^\sim U_\ell^*$. The dimensions of U_ℓ will be $k \times m$ where m is the (local) dimension of the input space while k may range from 0 to m. It is a 'wide' isometric matrix. If $m = k$, then it is said that U_ℓ has 'full range'. It is then causal unitary. We reserve the term 'inner' for that case, following (Helson, 1964). We can now state the main results for this section (Dewilde, 1971; 1976):

Theorem 1. *Let $S(z)$ be a matrix-valued function of dimension $m \times m$ which is analytic in the open unit disc and contractive there. Then $S(z)$ will possess a Darlington embedding iff:*

1. *the Szegö integral $\int_{-\pi}^{\pi} \log\det[I - S(e^{i\theta} S(e^{i\theta})^*]\frac{\mathrm{d}\theta}{2\pi}$ converges;*
2. *$\mathbf{k}^\sim$ has full dimension m a.e. on the unit circle or, equivalently, U_ℓ is (causal) unitary.*

The two conditions mentioned in the theorem are independent; there exist cases which satisfy either one but not the other, none, or both; see the last section for examples. Further remarks are as follows:

- Condition (2) is equivalent to the existence of 'external' (or coprime) factorizations for S, i.e. there exist causal and unitary (in mathematical terms: inner) transfer functions U_ℓ and U_r and causal transfer functions Δ_ℓ and Δ_r such that

  $$S = U_\ell \Delta_\ell^* = \Delta_r^* U_r.$$

  U_ℓ and U_r characterize the kernels of the Hankel operator. It turns out that the Darlington synthesis simply amounts to a right external factorization on $[S \;\; \Sigma_{12}]$.

- Condition (2) is equivalent to the existence of a pseudo-meromorphic continuation for S, see in this context (Arov, 1971; Fuhrmann, 1981). This last condition is an analytic characterization, which is useful in checking specific cases. A more intrinsic characterization is given in the next section and will allow the treatment of a more general case, namely, time-varying systems, for which no analytic transform theory exists.

4. The Time-Varying Case

In the time-varying case, S is simply a contractive operator acting in the time domain between square summable series (note that for time-varying systems the discrete and continuous time cases cannot easily be put on equal footing, here we consider only the discrete time case). As before, a Hankel operator can be defined which characterizes the dynamic behavior of the system. It does not act between simple spaces of input and output sequences, but between their extended versions large enough to characterize the state space behaviour at each time point i. Since the transfer map changes from time point to time point, we take as input space an infinite stack of input sequences, one per time point, and similarly, we take as output space an infinite stack of resulting outputs, one per time point. On these extended input and output spaces we keep working with an overall quadratic norm, called a 'Hilbert Schmidt norm'. Signals belonging to the strict past form strictly lower triangular (extended) matrices $U_p \in \mathcal{U}_p$, while signals belonging to the future form upper triangular schemes $U_f \in \mathcal{U}_f$. Again, $\mathbf{P}$ denotes projection on causal operators, here represented by uppers, and the Hankel operator becomes

$$Y_f = U_p H_S = \mathbf{P}(U_p S).$$

Viewing it as an operator on inputs and outputs split according to their past-future dichotomy, we find a block decomposition of the operator S as

$$[U_p \; U_f] \begin{bmatrix} K_S & H_S \\ 0 & T_S \end{bmatrix} = [Y_p \; Y_f], \tag{1}$$

where H_S is seen as the Hankel operator, K_S may be called the 'past-to-past' operator, and T_S is a half-finite Toeplitz-like operator derived from S (it is not Toeplitz because of time-variance). We shall encounter K_S later in the development. Darlington embedding can now follow a similar geometric development as in the previous section. It will consist of two parts:

1. the spectral factorization
$$\Sigma_{12}\Sigma_{12}^* = I - SS^*,$$
2. the embedding of the isometric system $\Sigma \triangleq [S \;\; \Sigma_{12}]$.

In this more general case, the two problems will also be unrelated. For the first problem, to our knowledge, there is no general solution known. However, when S has a finite dimensional system representation, Part 1 can be solved explicitly as is shown in (Dewilde and van den Veen, 1998, p. 345); we reproduce the result. Suppose indeed that S can be written as

$$S = D + BZ(I - AZ)^{-1}C, \tag{2}$$

where $\{A, B, C, D\}$ are diagonal operators of appropriate dimensions and Z is a 'shift operator', i.e. its action on a sequence $u = [u_i]$ is given by $uZ = [u_{i-1}]$.

The property of having a finite dimensional realization reduces to the property that the entries of S can be characterized as $S_{i,i} = D_i$ for the diagonal entries and $S_{i,j} = B_i A_{i+1} \cdots A_{j-1} C_j$ for $i < j$. Let $A = \mathrm{diag}[A_i]$, and suppose that A_i has dimensions $\delta_i \times \delta_{i+1}$. Then δ_i is known as the 'minimal state dimension of S at point i'. The theory to construct the operators $\{A, B, C, D\}$ from the entries of S is known as realization theory. (Recently it has also been discovered how approximate realizations in the strong Hankel norm can be found for operators which do not have a realization in terms of a finite state dimension (Dewilde and van den Veen, 1998)). Σ_{12} can be written as

$$\Sigma_{12} = D_{12} + BZ(I - AZ)^{-1}C_2 \tag{3}$$

with the same A, B as for S, and C_2, $D_{1,2}$ expressed in terms of sub-operators of S as

$$\begin{aligned} D_{1,2} &= ([I - DD^* - BMB^*]^\dagger)^{1/2}, \\ C_2 &= (CD^* - AMB^*)(D_{1,2}^*)^\dagger, \end{aligned} \tag{4}$$

where M is given by (the dagger indicates the Moore-Penrose pseudoinverse)

$$M = \mathcal{O}(I - K_S^* K_S)^\dagger \mathcal{O}^*. \tag{5}$$

In this latter equation, $\mathcal{O}$ is the observability operator for S, $\mathcal{O} = (I - AZ)^{-1}C$ and K_S is the 'past-to-past operator': $K_S = \mathbf{P}'(\cdot S)|_{U_p}$, which can be expressed as a matrix in terms of the entries of S, see (Dewilde and van den Veen, 1998, p. 342) for details. The conclusion of this part of the embedding procedure is that a locally finite and contractive operator S always has a spectral cofactor. It is not too hard to see that the pseudo-inverses involved always exist. As for Part 2, the Darlington embedding, the geometry of H_Σ comes into play again and, remarkably, new phenomena occur. Let $\mathcal{H}_o(\Sigma)$ be the observability space of the isometric Σ. Then it is not hard to show that the causal output space $\mathcal{Y}_f$ of Σ decomposes into three components:

$$\mathcal{Y}_f = \mathcal{H}_o(\Sigma) \oplus \mathcal{U}_f \Sigma \oplus \ker(\cdot\Sigma^*)|_{\mathcal{Y}_f}.$$

This is a direct generalization of the time-invariant case. The kernel in the formula plays an important role in the Darlington theory. It is a part of a larger kernel of the operator Σ^* *acting over the entire output space* $\mathcal{Y} = \mathcal{Y}_p \oplus \mathcal{Y}_f$, and so are all its backward shifts, $Z^{-k}\ker(\cdot\Sigma^*)|_{\mathcal{Y}_f}$, as well as their union, which we call $\mathcal{K}_o'$ (*warning: the restriction in the formulas is absolutely essential!).* However, there is possibly an additional subspace in the global kernel, which we call the defect space $\mathcal{K}_o^{"} \triangleq \ker(\cdot\Sigma^*) \ominus \mathcal{K}_o'$. Even (isometric) systems with finite dimensional state spaces may have a non-trivial defect space, in contrast to the time-invariant case where this phenomenon cannot occur. The 'geometric' Darlington theorem then takes the following form:

Theorem 2. (Generalized Darlington) *Let Σ be a causal isometric transfer operator, $\mathcal{H}_o(\Sigma)$ its observability space and $\mathcal{K}_o^{"}$ its defect space. Then there exists an isometric operator Σ_2 with the same observability space $\mathcal{H}_o$ such that*

$$\ker(\cdot S^*)|_{\mathcal{Y}_f} = \mathcal{U}_f \Sigma_2.$$

The operator

$$\Sigma_t = \begin{bmatrix} \Sigma \\ \Sigma_2 \end{bmatrix}$$

will be isometric as well and such that $\ker(\cdot\Sigma_t^*)|_{\mathcal{Y}_f} = \{0\}$. Σ_t will be unitary (a Darlington embedding) iff the defect space $\mathcal{K}_o^{"}$ is zero.

Hence we are again able to characterize the possibility of Darlington synthesis entirely in geometric terms. As before, the Darlington embedding is computable when a state space representation for S is known, see (Dewilde, 1999) for details.

5. Concluding Remarks

Before formulating our final conclusions, we still have to provide some examples which show that the spectral factorization and the embedding problem are independent. An ideal causal low pass filter $S(z)$ such that $|S(e^{i\theta})| = 1$ in an interval $-\theta_0 \leq \theta \leq \theta_0$ (the 'passband') while zero elsewhere on the unit circle (the 'stopband') does not satisfy Szegö's condition. There does not even exist an analytic and contractive function in the unit circle meeting that specification, because of the properties of H_∞ functions. It we relax somewhat and ask that it equals epsilon in the stopband, then by outer continuation one finds indeed an analytic and contractive $S(z)$, but its spectral cofactor will not exist since the latter does not meet Szegö's condition.

However, a slight change in the amplitude specification will produce a function such that it satisfies the Szegö condition as well as its cofactor. Let again $0 < \epsilon \ll 1$ be a small positive quantity, and let $S(z)$ be such that $|S(e^{i\theta}| = 1 - \epsilon$ for $-\theta_0 \le \theta \le \theta_0$ while $|S(e^{i\theta})| = \epsilon$ for θ outside the interval $[-\theta_0,\, \theta]$; then $S(z)$ will be contractive and it will have a genuine, analytic and contractive spectral cofactor.

It is also fairly easy to produce contractive causal transfer functions that meet the Szegö condition yet cannot be further embedded. A standard example is $S(z) = \sqrt{1/(z+2)}$, which is clearly analytic and contractive in the unit disc. However, it does not have a pseudo-meromorphic continuation, as required for an embeddable function, since the square root shows an essential branching point at $z = -2$ (there is no meromorphic function outside the unit disc that can produce the branching). The state space of this transfer function fills the complete input space and it 'remembers' everything from the past inputs. However, we also know that it can be approximated as closely as one wishes by a rational transfer function, which does forget almost everything from the past. In other words: such functions are approximately embeddable but not exactly. Time-varying examples are much easier to construct. It is indeed possible to cook up time-varying examples with finite state spaces and for which the Darlington condition is violated—something that cannot occur with time-invariant systems.

The Darlington synthesis, when it exists, succeeds in representing a lossy operator as part of a lossless one. The latter can be realized by any known synthesis method, in particular by the Cauer synthesis. Although the cascade form is the most common one, there are other attractive filter or system structures that have found important applications; we mention in particular the Jauman synthesis. This insight has led to important characterizations of 'stable' functions and efficient algorithms to compute orthonormal polynomials and system realizations (Delsarte and Genin, 1986). Lastly, remark I would like to state that the famed PR- and BR-lemmas can be constructed as a fairly direct consequence of the geometric Darlington theory (Dewilde and van den Veen, 1998).

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