A QUADRATIC OPTIMAL CONTROL PROBLEM FOR A CLASS OF LINEAR DISCRETE DISTRIBUTED SYSTEMS

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A linear quadratic optimal control problem for a class of discrete distributed systems is analyzed. To solve this problem, we introduce an adequate topology and establish that optimal control can be determined though an inversion of the appropriate isomorphism. An example and a numerical approach are given.

Keywords: discrete distributed system, Hilbert uniqueness method, linear system, optimal control

1. Introduction

Most of the systems encountered in practice are continuous in time (Athans and Falb, 1966; Curtain and Pritchard, 1978; Curtain and Zwart, 1995; Kalman, 1960; Lasiecka and Triggiani, 2000). However, the analysis and control of a continuous system with a computer requires sampling and thus a discretization of the system considered. The importance of discrete systems lies in the fact that they are present in a large number of fields, such as engineering, economics, biomathematics, etc. The recourse to discrete models is often preferred by engineers since, on the one hand, some mathematical complexities such as the choice of a function space and regularity of the solution are avoided and, on the other the hand, they are better adapted to computer processing.

Let us start with a continuous distributed system:

$$x(t) = S(t)x_0 + \int_0^t S(t-r)Bu(r)\,\mathrm{d}r, \quad t \in [0,T], \tag{1}$$

where $x_0, x(t) \in \mathcal{X}$, $(S(t))_{t\geq 0}$ is a strongly continuous semigroup on $\mathcal{X}$, $B \in \mathcal{L}(\mathcal{U},\mathcal{X})$, and $\mathcal{X}$, $\mathcal{U}$ are Hilbert spaces. ($\mathcal{X}$ and $\mathcal{U}$ can be of finite dimensions, and then the system is lumped.)

One of the discretization procedures which is most often used (Lee *et al.*, 1972; Ogata, 1995; Rabah and Malabre, 1999) consists in partitioning the time horizon time $[0,T]$ using the instants $t_0 = 0$, $t_1 = \delta$, $t_2 = 2\delta, \dots$, $t_n = n\delta$, where $\delta = T/N$ and $N \in \mathbb{N}^*$, δ being the sampling period. Then we assume that the control u is constant over each interval $[t_i, t_{i+1}[$, i.e.,

$$u(t) = u_i, \quad \forall t \in [t_i, t_{i+1}[. \tag{2}$$

Thus, setting $x(t_i) = x_i$, we get

$$\begin{aligned} x_{i+1} &= x(t_{i+1}) \\ &= S(t_{i+1})x_0 \\ &\quad + \int_0^{t_{i+1}} S(t_{i+1}-r)Bu(r)\,\mathrm{d}r \\ &= S(\delta)S(t_i)x_0 \\ &\quad + \int_0^{t_i} S(\delta)S(t_i-r)Bu(r)\,\mathrm{d}r \\ &\quad + \int_{t_i}^{t_{i+1}} S(t_{i+1}-r)Bu(r)\,\mathrm{d}r \\ &= S(\delta)\Big[S(t_i)x_0 + \int_0^{t_i} S(t_i-r)Bu(r)\,\mathrm{d}r\Big] \\ &\quad + \int_{t_i}^{t_{i+1}} S(t_{i+1}-r)Bu(r)\,\mathrm{d}r, \end{aligned}$$

and then

$$x_{i+1} = S(\delta)x_i + \int_{t_i}^{t_{i+1}} S(t_{i+1}-r)Bu(r)\,\mathrm{d}r. \tag{3}$$

Using the hypothesis (2), we deduce that

$$\begin{aligned} x_{i+1} &= S(\delta)x_i + \Big[\int_{t_i}^{t_{i+1}} S(t_{i+1}-r)B\,\mathrm{d}r\Big]u_i \\ &= S(\delta)x_i + \Big(\int_0^{\delta} S(\tau)B\,\mathrm{d}\tau\Big)u_i, \end{aligned}$$

where $\tau = t_{i+1} - r$. Then

$$x_{i+1} = \Phi x_i + \mathcal{B}u_i, \tag{4}$$

with $\Phi = S(\delta)$ and $\mathcal{B} = \int_0^{\delta} S(\tau)B\,\mathrm{d}\tau$.

The discrete version (4) has been the subject of numerous works (Chraibi *et al.*, 2000; Dorato, 1993; Faradzhev *et al.*, 1986; Halkin, 1964; Klamka, 1995; 2002; Lee *et al.*, 1972; Lun'kov, 1980; Weiss, 1972). Our contribution in this context consists in studying the quadratic control problem for a linear discrete system. It is true that we are not the first to have examined this problem. Lee *et al.* (1972) demonstrated that optimal control and optimal cost can be obtained using a discrete Riccati equation. Zabczyk (1974) proved that the optimum can be computed using Lagrange multipliers. In (Karrakchou and Rachik, 1995; Karrakchou *et al.*, 1998), the Hilbert uniqueness method (HUM), set forth by Lions (1988a; 1988b), was used to prove that the optimum comes from solving an algebraic linear equation.

The originality of our work consists in adopting the discretization scheme described previously, but in the absence of the hypothesis (2), i.e., we assume that the control $u(\cdot)$ is not necessarily constant in the time interval $[t_i, t_{i+1}[$.

In fact, when the difference between two consecutive sampling instants is quite important (for example, if the system (1) is a compartmental model (Daley and Gani, 2001; Jolivet, 1983) describing the evolution of a long illness or a chronic illness, it is natural that the difference between two measurements, $t_{i+1} - t_i$, can be as long as several months), it does not make sense to suppose that $u(t)$ is constant between the instants t_{i+1} and t_i.

In order to overcome this obstacle, we reconsider the discretization of the system (1), but without the hypothesis (2), which yields the difference equation

$$x_{i+1} = \Phi x_i + \int_{t_i}^{t_{i+1}} \mathcal{B}_i(\theta) u(\theta)\,\mathrm{d}\theta, \tag{5}$$

with $\Phi = S(\delta)$ and $\mathcal{B}_i(\theta) = S(t_{i+1} - \theta)B$. To be more precise and classify our problem in a more general mathematical framework, in this paper we consider the discrete-time system

$$\begin{cases} x_{i+1} = \Phi x_i + \displaystyle\int_{t_i}^{t_{i+1}} B_i(\theta) u(\theta)\,\mathrm{d}\theta, \quad i \geq 0, \\ x_0 \in \mathcal{X}, \end{cases} \tag{6}$$

where $x_i \in \mathcal{X}$ is the state variable and $u(\theta) \in \mathcal{U}$.

Using a technique which is similar to the HUM (Lions, 1988a; 1988b; El Jai and Bel Fekih, 1990; El Jai and Berrahmoune, 1991), we introduce a suitable topology to prove that optimal control and optimal cost stem from the inversion of a coercive isomorphism and thus from an algebraic equation easy to solve by classical numerical methods.

To motivate the problem discussed in this paper, consider temperature distribution in an industrial oven whose simplified mathematical model is

$$\frac{\partial T}{\partial t}(x,t) = \alpha \frac{\partial^2 T}{\partial^2 x}(x,t) + \sum_{i=1}^{p} g_i \mathcal{X}_{\omega_i} u_i(t), \quad \forall t \geq 0, \tag{7}$$

where $T(\cdot, t)$ is the temperature profile at the time t. We suppose that the system is controlled by a variable control $u(t) = (u_1(t), \dots, u_p(t))^T$, where $u_i(t)$ acts on the zone $\omega_i \subset]0,1[$ according to a spatial distribution $g_i \in L^2(\omega_i)$.

The associated initial condition is supposed to be homogeneous, i.e.,

$$T(x,0) = T_0(x), \quad \forall x \in [0,1],$$

and the boundary condition is also homogeneous, i.e.,

$$T(0,t) = T(1,t) = 0, \quad \forall t \geq 0.$$

Equation (7) can be written as

$$\frac{\partial T}{\partial t}(x,t) = AT(x,t) + Bu(t), \quad \forall t \geq 0, \tag{8}$$

where A is the operator $\partial^2/\partial x^2$ whose domain $\mathcal{D}(A)$ and spectrum $\sigma(A)$ are respectively given by

$$\mathcal{D}(A) = \Big\{ f \in L^2(0,1) \;/\; f'' \in L^2(0,1) \text{ and } f(0) = f(1) = 0 \Big\},$$

and

$$\sigma(A) = \big\{ \lambda_n = -n^2\pi^2 \;/\; n \in \mathbb{N}^* \big\},$$

while the associated eigenfunctions are

$$\varphi_n(x) = \sqrt{2}\sin(n\pi x), \quad n = 1, 2, \dots,$$

$(\varphi_n)_{n\geq 1}$ being an orthonormal basis of $L^2(0,1)$. The bounded operator B is such that

$$B : \begin{cases} \mathbb{R}^p & \longrightarrow & L^2(0,1), \\ \begin{pmatrix} u_1 \\ \vdots \\ u_p \end{pmatrix} & \longrightarrow & \displaystyle\sum_{i=1}^{p} g_i \mathcal{X}_{\omega_i} u_i. \end{cases}$$

It is known that the mild solution of Eqn. (8) is

$$x(t) = S(t)x_0 + \int_0^t S(t-r)Bu(r)\,\mathrm{d}r, \quad t \in [0,T],$$

where $x(t) \in \mathcal{X} = L^2(0,1)$, and $(S(t))_{t\geq 0}$ is the strongly continuous semigroup generated by the operator A. Then the discretization of our system without the hypothesis (2) leads to the difference equation

$$x_{i+1} = \Phi x_i + \int_{t_i}^{t_{i+1}} \mathcal{B}_i(\theta) u(\theta)\,\mathrm{d}\theta. \tag{9}$$

The corresponding output is supposed to be a sequence of measurements taken at the instants $t_0 = 0$, $t_1 = \delta, \dots, t_N = N\delta = T$, i.e.,

$$y(t_i) = T(\cdot, t_i).$$

The control strategy consists in determining minimum-norm control allowing us to minimize the differences $\|y(t_N) - y_d\|$ and $(\|y(t_i) - r_i\|)_{0\le i\le N-1}$, where y_d is a desired state and $(r_i)_{1\le i\le N-1}$ is a given desired trajectory. Mathematically, solving this problem amounts to the minimization of the quadratic criterion

$$\begin{aligned} J(u) &= \langle y_N - y_d, G(y_N - y_d)\rangle \\ &\quad + \sum_{i=1}^{N-1} \langle y_i - r_i, M(y_i - r_i)\rangle \\ &\quad + \int_0^T \langle u(\theta), Ru(\theta)\rangle \,\mathrm{d}\theta. \end{aligned}$$

M, R and G are selected to weigh the relative importance of the performance measures caused by the vectors $(y_i)_i$, the control variable u and the final output y_N, respectively.

2. Some Useful Properties

In this section, we shall develop an optimality system to characterize some optimal control u^*. For this purpose, observe that the state $(x_k)_{1\le k\le N}$ can be written as follows:

$$x_k = \Phi^k x_0 + \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Phi^{k-j} B_{j-1}(\theta)u(\theta)\,\mathrm{d}\theta, \qquad k = 1, \dots, N. \tag{10}$$

If we introduce the bounded operator $\mathcal{H}$ defined by

$$\mathcal{H} : \begin{cases} L^2(0, T; \mathcal{U}) & \longrightarrow \quad l^2(1, 2, \dots, N; \mathcal{X}), \\ \qquad u & \longrightarrow \quad \mathcal{H}u = \big((\mathcal{H}u)_i\big)_{1\le i\le N}, \end{cases}$$

where

$$(\mathcal{H}u)_k = \sum_{j=1}^{k} \int_{t_{j-1}}^{t_j} \Phi^{k-j} B_{j-1}(\theta)u(\theta)\,\mathrm{d}\theta, \qquad \forall k = 1, 2, \dots, N, \tag{11}$$

then from (10) we establish that

$$x_k = \Phi^k x_0 + (\mathcal{H}u)_k, \quad \forall k = 1, 2, \dots, N.$$

The adjoint operator $\mathcal{H}^*$ is such that

$$\begin{aligned} &\langle \mathcal{H}u, (x_1, \dots, x_N)\rangle_{l^2(1,\dots,N;\mathcal{X})} \\ &= \sum_{k=1}^{N} \langle (\mathcal{H}u)_k, x_k\rangle \\ &= \sum_{k=1}^{N} \Big\langle \sum_{i=1}^{k} \int_{t_{i-1}}^{t_i} \Phi^{k-i} B_{i-1}(\theta)u(\theta)\,\mathrm{d}\theta, x_k \Big\rangle \\ &= \sum_{i=1}^{N}\sum_{k=i}^{N} \int_{t_{i-1}}^{t_i} \langle u(\theta), B^*_{i-1}(\theta){\Phi^*}^{k-i} x_k\rangle \,\mathrm{d}\theta \\ &= \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle u(\theta), \sum_{k=i}^{N} B^*_{i-1}(\theta){\Phi^*}^{k-i} x_k\rangle \,\mathrm{d}\theta. \end{aligned}$$

Setting

$$f(\theta) = \sum_{k=i}^{N} B^*_{i-1}(\theta){\Phi^*}^{k-i} x_k, \quad \forall \theta \in [t_{i-1}, t_i[, \qquad i = 1, 2, \dots, N,$$

we have

$$\langle \mathcal{H}u, (x_1, \dots, x_N)\rangle_{l^2(1,\dots,N;\mathcal{X})} = \int_0^T \langle u(\theta), f(\theta)\rangle \,\mathrm{d}\theta.$$

Then $\mathcal{H}^*(x_1, x_2, \dots, x_N) \in L^2(0, T; \mathcal{U})$ is given by

$$\mathcal{H}^*(x_1, x_2, \dots, x_N)(\theta) = \sum_{k=i}^{N} B^*_{i-1}(\theta){\Phi^*}^{k-i} x_k, \qquad \theta \in [t_{i-1}, t_i[\ \text{ for } \ i = 1, 2, \dots, N. \tag{12}$$

Consider the operator L defined by

$$L : \begin{cases} L^2(0, T; \mathcal{U}) \longrightarrow \mathcal{X}, \\ \qquad u \longrightarrow (\mathcal{H}u)_N \\ \qquad\quad = \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \Phi^{N-k} B_{k-1}(\theta)u(\theta)\,\mathrm{d}\theta, \end{cases}$$

so that

$$x_N = \Phi^N x_0 + Lu.$$

The adjoint operator $L^* : \mathcal{X} \longrightarrow L^2(0, T; \mathcal{U})$ is such that

$$\begin{aligned} \langle Lu, x\rangle &= \Big\langle \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \Phi^{N-k} B_{k-1}(\theta)u(\theta)\,\mathrm{d}\theta, x \Big\rangle \\ &= \sum_{k=1}^{N} \int_{t_{k-1}}^{t_k} \langle u(\theta), B^*_{k-1}(\theta){\Phi^*}^{N-k} x\rangle \,\mathrm{d}\theta \\ &= \int_0^T \langle u(\theta), g(\theta)\rangle \,\mathrm{d}\theta, \end{aligned}$$

and therefore

$$(L^*x)(\theta) = B_{k-1}^*(\theta){\Phi^*}^{N-k}x, \quad \forall \theta \in [t_{k-1}, t_k[, \quad k = 1, \dots, N.$$

3. Optimality System

Knowing that the functional to minimize in $L^2(0,T;\mathcal{U})$ is

$$\begin{aligned} J(u) &= \langle x_N - x_d, G(x_N - x_d)\rangle \\ &\quad + \sum_{i=1}^{N-1} \langle x_i - r_i, M(x_i - r_i)\rangle \\ &\quad + \int_0^T \langle u(\theta), Ru(\theta)\rangle \,\mathrm{d}\theta, \end{aligned} \tag{13}$$

we use the technical results established in the previous section to deduce that

$$\begin{aligned} &\langle x_N - x_d, G(x_N - x_d)\rangle \\ &\quad = \langle \Phi^N x_0 - x_d + Lu, G(\Phi^N x_0 - x_d) + GLu\rangle \\ &\quad = \langle \Phi^N x_0 - x_d, G(\Phi^N x_0 - x_d)\rangle + \langle u, L^*GLu\rangle \\ &\qquad + 2\langle Lu, G(\Phi^N x_0 - x_d)\rangle, \end{aligned}$$

and, for $i = 1, \dots, N-1$, we have

$$\begin{aligned} &\langle x_i - r_i, M(x_i - r_i)\rangle \\ &\quad = \langle \Phi^i x_0 - r_i + (\mathcal{H}u)_i, M(\Phi^i x_0 - r_i) + M(\mathcal{H}u)_i\rangle \\ &\quad = \langle \Phi^i x_0 - r_i, M(\Phi^i x_0 - r_i)\rangle + \langle (\mathcal{H}u)_i, M(\mathcal{H}u)_i\rangle \\ &\qquad + 2\langle (\mathcal{H}u)_i, M(\Phi^i x_0 - r_i)\rangle. \end{aligned}$$

We easily deduce that the functional J can be written as

$$J(u) = \mathrm{const} + J^*(u),$$

where

$$\begin{aligned} \mathrm{const} &= \langle \Phi^N x_0 - x_d, G(\Phi^N x_0 - x_d)\rangle \\ &\quad + \sum_{i=1}^{N-1} \langle \Phi^i x_0 - r_i, M(\Phi^i x_0 - r_i)\rangle, \end{aligned}$$

and

$$\begin{aligned} J^*(u) &= 2\Big(\langle Lu, G(\Phi^N x_0 - x_d)\rangle \\ &\quad + \sum_{i=1}^{N-1} \langle (\mathcal{H}u)_i, M(\Phi^i x_0 - r_i)\rangle\Big) \\ &\quad + \langle u, L^*GLu\rangle + \sum_{i=1}^{N-1} \langle (\mathcal{H}u)_i, M(\mathcal{H}u)_i\rangle \\ &\quad + \langle u, Ru\rangle. \end{aligned}$$

Consider the sequence $(a_k)_{1\le k\le N}$ and the operators D and $\overline{R}$ described by

$$\begin{cases} a_k = M(\Phi^k x_0 - r_k), & k = 1, 2, \dots, N-1, \\ a_N = G(\Phi^N x_0 - x_d), \end{cases}$$

$$D : \begin{cases} \mathcal{F} = l^2(1,2,\dots,N,\mathcal{X}) \longrightarrow \mathcal{F} = l^2(1,2,\dots,N,\mathcal{X}), \\ (x_1, x_2, \dots, x_N) \longrightarrow (Mx_1, Mx_2, \dots, Mx_{N-1}, Gx_N) \end{cases}$$

and

$$\overline{R} : \begin{cases} L^2(0,T;\mathcal{U}) \longrightarrow L^2(0,T;\mathcal{U}), \\ u \longrightarrow \overline{R}u, \end{cases}$$

where $(\overline{R}u)(\theta) = Ru(\theta)$. It is easy to see that

$$\begin{cases} D^* = D \text{ and } D \ge 0, \\ (\overline{R})^* = \overline{R}, \quad (\overline{R})^{-1} = \overline{R^{-1}} \\ \qquad \text{and } \langle \overline{R}u, u\rangle \ge \alpha \|u\|^2_{L^2(0,T;\mathcal{U})}. \end{cases} \tag{14}$$

Moreover, since $Lu = (\mathcal{H}u)_N$, the cost functional J^* can be written as

$$\begin{aligned} J^*(u) &= 2\langle \mathcal{H}u, (a_1, a_2, \dots, a_N)\rangle \\ &\quad + \langle \mathcal{H}u, D\mathcal{H}u\rangle + \langle u, \overline{R}u\rangle \\ &= 2\langle u, \mathcal{H}^*(a_1, a_2, \dots, a_N)\rangle + \langle u, (\mathcal{H}^*D\mathcal{H} + \overline{R})u\rangle \\ &= 2l(u) + B(u,u), \end{aligned}$$

where l is the linear form

$$l : \begin{cases} L^2(0,T;\mathcal{U}) \longrightarrow \mathbb{R}, \\ u \longrightarrow \langle u, \mathcal{H}^*(a_1, \dots, a_N)\rangle \end{cases}$$

and $B(\cdot,\cdot)$ is the symmetric bilinear form

$$B(\cdot,\cdot) : \begin{cases} L^2(0,T;\mathcal{U}) \times L^2(0,T;\mathcal{U}) \longrightarrow \mathbb{R}, \\ (u,v) \longrightarrow B(u,v) = \langle u, (\mathcal{H}^*D\mathcal{H} + \overline{R})v\rangle. \end{cases}$$

We have $B(u,u) = \langle \mathcal{H}u, D\mathcal{H}u\rangle + \langle u, \overline{R}u\rangle$ so $B(u,u) \ge \langle u, \overline{R}u\langle$ because $D \ge 0$. From (14) we deduce that $B(u,u) \ge \alpha\|u\|^2$.

Thus J^* is the sum of a continuous linear form l and a bilinear, continuous, symmetric and coercive form $B(\cdot,\cdot)$.

From the Lax-Milgram theorem (Brezis, 1987; Ciarlet, 1988; Lions, 1968)), it follows that J^* has a unique solution u^* in $L^2(0,T,\mathcal{U})$. Furthermore, u^* is characterized by

$$B(u^*, v) = -l(v), \quad \forall v \in L^2(0,T;\mathcal{U}),$$

i.e.,

$$\langle(\mathcal{H}^* D\mathcal{H} + \overline{R})u^*, v\rangle = -\langle\mathcal{H}^*(a_1, \dots, a_N), v\rangle.$$

Thus

$$\mathcal{H}^* D\mathcal{H}u^* + \overline{R}u^* = -\mathcal{H}^*(a_1, \dots, a_N).$$

The optimal control u^* is characterized by

$$u^* = -(\overline{R})^{-1}\big[\mathcal{H}^* D\mathcal{H}u^* + \mathcal{H}^*(a_1, a_2, \dots, a_N)\big].$$

Accordingly,

$$\begin{aligned} u^* &= -(\overline{R})^{-1}\mathcal{H}^* \\ &\quad\times\Big[\big(M(\mathcal{H}u^*)_1, \dots, M(\mathcal{H}u^*)_{N-1}, G(\mathcal{H}u^*)_N\big) \\ &\quad + \big(M(\Phi x_0 - r_1), \dots, M(\Phi^{N-1}x_0 - r_{N-1}), \\ &\quad G(\Phi^N x_0 - x_d)\big)\Big], \end{aligned}$$

which gives

$$\begin{aligned} u^* &= -\overline{R^{-1}}\mathcal{H}^*\big[(Mx_1^{u^*}, Mx_2^{u^*}, \dots, Mx_{N-1}^{u^*}, Gx_N^{u^*}) \\ &\quad - (Mr_1, Mr_2, \dots, Mr_{N-1}, Gx_d)\big]. \end{aligned}$$

Hence for $\theta \in [t_{i-1}, t_i[$, $i = 1, 2, \dots, N-1$, we have

$$\begin{aligned} u^*(\theta) &= -R^{-1}B_{i-1}^*(\theta)\Big(\sum_{k=i}^{N-1}\Phi^{*k-i}M(x_k^{u^*} - r_k) \\ &\quad + \Phi^{*N-i}G(x_N^{u^*} - x_d)\Big) \end{aligned}$$

and

$$u^*(\theta) = -R^{-1}B_{N-1}^*(\theta)G(x_N - x_d) \quad \text{if } \theta \in [t_{N-1}, t_N[.$$

Consider the signal $(p_i)_{1\le i\le N}$ defined by

$$\begin{cases} p_i = \displaystyle\sum_{k=i}^{N-1}\Phi^{*k-i}M(x_k - r_k) + \Phi^{*N-i}G(x_N - x_d), \\ \qquad\qquad i = 1, 2, \dots, N-1 \\ p_N = G(x_N - x_d). \end{cases}$$

We have

$$\begin{aligned} \Phi^* p_{i+1} &= \sum_{k=i+1}^{N-1}\Phi^{*k-i}M(x_k - r_k) + \Phi^{*N-i}G(x_N - x_d) \\ &= p_i - M(x_i - r_i), \quad i = 1, 2, \dots, N-2, \end{aligned}$$

and thus the signal p_i satisfies the following difference equation:

$$\begin{cases} p_i = \Phi^* p_{i+1} + M(x_i - r_i), \quad i = 1, 2, \dots, N-1, \\ p_N = G(x_N - x_d). \end{cases}$$

Finally, we deduce the following optimality system:

$$\begin{cases} u^*(\theta) = -R^{-1}B_{i-1}^*(\theta)p_i, \quad \theta \in [t_{i-1}, t_i[, \\ \qquad\qquad i = 1, 2, \dots, N, \\ p_i = \Phi^* p_{i+1} + M(x_i^{u^*} - r_i), \\ \qquad\qquad i = 1, 2, \dots, N-1, \\ p_N = G(x_N^{u^*} - x_d), \\ x_{i+1}^{u^*} = \Phi x_i^{u^*} + \displaystyle\int_{t_i}^{t_{i+1}} B_i(\theta)u^*(\theta)\,\mathrm{d}\theta, \\ \qquad\qquad i = 0, 1, \dots, N-1. \end{cases} \tag{15}$$

4. Convenient Topology

In this section, we develop a technique similar to the HUM. Indeed, let $f = (x_1, x_2, \dots, x_{N-1}, x_N) \in \mathcal{F} = l^2(1, 2, \dots, N; \mathcal{X})$ and the signal $z^f = (z_1^f, \dots, z_N^f)$ be described by the difference equation

$$\begin{cases} z_i^f = \Phi^{*N-i}G^{\frac{1}{2}}x_N + \displaystyle\sum_{k=i}^{N-1}\Phi^{*k-i}M^{\frac{1}{2}}x_k, \\ \qquad\qquad i = 1, 2, \dots, N-1, \\ z_N^f = G^{\frac{1}{2}}x_N. \end{cases} \tag{16}$$

We define the following functional on $l^2(1, \dots, N; \mathcal{X})$:

$$|||f|||^2 = \|f\|^2 + \sum_{k=1}^{N}\int_{t_{j-1}}^{t_j}\langle B_{j-1}^*(\theta)z_j^f, R^{-1}B_{j-1}^*(\theta)z_j^f\rangle\,\mathrm{d}\theta. \tag{17}$$

Lemma 1. *$|||\cdot|||$ is a norm on $\mathcal{F}$ equivalent to the usual one.*

Proof. From the linearity of the map $f \to z_j^f$, it is easy to deduce that $|||\cdot|||$ is a norm on the space $\mathcal{F}$. The equivalence is then immediate. ■

For $f = (x_1, x_2, \dots, x_{N-1}, x_N) \in \mathcal{F}$, we define $\Psi^f = (\Psi_i^f)_{0\le i\le N}$ by

$$\begin{cases} \Psi_{i+1}^f = \Phi\Psi_i^f + \displaystyle\int_{t_i}^{t_{i+1}} B_i(\theta)u_f(\theta)\,\mathrm{d}\theta, \\ \qquad\qquad i = 0, 1, \dots, N-1, \\ \Psi_0^f = 0, \end{cases} \tag{18}$$

where

$$u_f(\theta) = R^{-1}B_{i-1}^*(\theta)z_i^f, \quad \theta \in [t_{i-1}, t_i[, \quad i = 1, 2, \dots, N.$$

Remark 1. We can easily see that

$$\Psi_k^f = \sum_{j=1}^{k}\Phi^{k-j}\Big[\int_{t_{j-1}}^{t_j} B_{j-1}(\theta)R^{-1}B_{j-1}^*(\theta)\,\mathrm{d}\theta\Big]z_j^f, \quad k = 1, \dots, N.$$

Define the operator Λ by

$$\Lambda : \begin{cases} \mathcal{F} & \rightarrow \quad \mathcal{F}, \\ f & \rightarrow \quad f + (M^{\frac{1}{2}}\Psi_1^f, \dots, M^{\frac{1}{2}}\Psi_{N-1}^f, G^{\frac{1}{2}}\Psi_N^f). \end{cases}$$

Lemma 2. *The operator Λ is bounded and self-adjoint, and we have*

$$\langle \Lambda f, f\rangle = |||f|||^2.$$

Proof. Setting $g = (y_1, y_2, \dots, y_N)$, we have

$$\begin{aligned}\langle \Lambda f, g\rangle &= \langle f + (M^{\frac{1}{2}}\Psi_1^f, \dots, M^{\frac{1}{2}}\Psi_{N-1}^f, G^{\frac{1}{2}}\Psi_N^f), g\rangle \\ &= \langle f, g\rangle + \sum_{i=1}^{N-1} \langle M^{\frac{1}{2}}\Psi_i^f, y_i\rangle + \langle G^{\frac{1}{2}}\Psi_N^f, y_N\rangle.\end{aligned}$$

If we define $P_i = M^{\frac{1}{2}}$ for all $i \in \{1, \dots, N-1\}$ and $P_N = G^{\frac{1}{2}}$, then

$$\begin{aligned}\langle \Lambda f, g\rangle &= \langle f, g\rangle + \sum_{i=1}^{N} \langle \Psi_i^f, P_i y_i\rangle \\ &= \langle f, g\rangle + \sum_{i=1}^{N} \Big\langle \sum_{j=1}^{i} \Phi^{i-j} \\ &\quad \times \Big(\int_{t_{j-1}}^{t_j} B_{j-1}(\theta) R^{-1} B_{j-1}^*(\theta)\, \mathrm{d}\theta \Big) z_j^f, P_i y_i \Big\rangle \\ &= \langle f, g\rangle + \sum_{i=1}^{N} \Big(\sum_{j=1}^{i} \int_{t_{j-1}}^{t_j} \Big\langle z_j^f, B_{j-1}(\theta) R^{-1} \\ &\quad \times B_{j-1}^*(\theta) \Phi^{*\,i-j} P_i y_i \Big\rangle \, \mathrm{d}\theta \Big) \\ &= \langle f, g\rangle + \sum_{j=1}^{N}\sum_{i=j}^{N} \Big\langle z_j^f, \\ &\quad \times \Big(\int_{t_{j-1}}^{t_j} B_{j-1}(\theta) R^{-1} B_{j-1}^*(\theta)\, \mathrm{d}\theta \Big) \Phi^{*\,i-j} P_i y_i \Big\rangle \\ &= \langle f, g\rangle + \sum_{j=1}^{N-1} \Big\langle z_j^f, \Big(\int_{t_{j-1}}^{t_j} B_{j-1}(\theta) R^{-1} B_{j-1}^*(\theta)\, \mathrm{d}\theta \Big) \\ &\quad \times \Big(\sum_{k=j}^{N-1} \Phi^{*\,i-j} M^{\frac{1}{2}} y_i + \Phi^{*\,N-j} G^{\frac{1}{2}} y_N \Big) \Big\rangle \\ &\quad + \Big\langle z_N^f, \Big(\int_{t_{N-1}}^{t_N} B_{N-1}(\theta) R^{-1} B_{N-1}^*(\theta)\, \mathrm{d}\theta \Big) G^{\frac{1}{2}} y_N \Big\rangle \\ &= \langle f, g\rangle + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \langle B_{j-1}^*(\theta) z_j^f, R^{-1} B_{j-1}^*(\theta) z_j^g \rangle \, \mathrm{d}\theta \\ &= \langle f, \Lambda g\rangle,\end{aligned}$$

and

$$\begin{aligned}\langle \Lambda f, f\rangle &= \|f\|^2 + \sum_{j=1}^{N} \int_{t_{j-1}}^{t_j} \langle B_{j-1}^*(\theta) z_j^f, R^{-1} B_{j-1}^*(\theta) z_j^f \rangle \, \mathrm{d}\theta \\ &= |||f|||^2.\end{aligned}$$

∎

Remark 2. As a consequence of Lemma 2, we easily deduce that Λ is an isomorphism.

Finally, we state our fundamental result of this section.

Theorem 1. *The optimal control u^* minimizing the functional (13) in $L^2(0, T; \mathcal{U})$ is*

$$u^*(\theta) = R^{-1} B_{i-1}^*(\theta) z_i^f, \quad \theta \in [t_{i-1}, t_i[, \quad i = 1, 2, \dots, N, \tag{19}$$

where z_i^f is the solution of the difference equation

$$\begin{cases} z_i^f = \Phi^{*\,N-i} G^{\frac{1}{2}} f_N + \displaystyle\sum_{k=i}^{N-1} \Phi^{*\,k-i} M^{\frac{1}{2}} f_k, \\ \qquad\qquad i = 1, 2, \dots, N-1, \\ z_N^f = G^{\frac{1}{2}} f_N, \end{cases} \tag{20}$$

and $f = (f_1, \dots, f_{N-1}, f_N)$ is the unique solution of the algebraic equation

$$\begin{aligned}\Lambda f = -\big(& M^{\frac{1}{2}}(\Phi x_0 - r_1), \dots, M^{\frac{1}{2}}(\Phi^{N-1} x_0 - r_{N-1}), \\ & G^{\frac{1}{2}}(\Phi^N x_0 - x_d) \big).\end{aligned} \tag{21}$$

Moreover, the optimal cost is

$$J(u^*) = |||f|||^2. \tag{22}$$

Proof. Since the operator $\Lambda \in \mathcal{L}(\mathcal{F})$ constitutes an isomorphism, Eqn. (21) possesses a unique solution $f = (f_1, \dots, f_{N-1}, f_N)$. Using the optimality system and the definition

$$\begin{cases} z_i^f = \Phi^{*\,N-i} G^{\frac{1}{2}} f_N + \displaystyle\sum_{k=i}^{N-1} \Phi^{*\,k-i} M^{\frac{1}{2}} f_k, \\ \qquad\qquad i = 1, 2, \dots, N-1, \\ z_N^f = G^{\frac{1}{2}} f_N, \end{cases}$$

it is sufficient to establish that

$$\begin{cases} f_i = -M^{\frac{1}{2}}(x_i^u - r_i), \quad i = 1, \dots, N-1, \\ f_N = -G^{\frac{1}{2}}(x_N^u - x_d), \end{cases}$$

where

$$x_k^u = \Phi^k x_0 + \sum_{j=1}^{k} \Phi^{k-j} \int_{t_{j-1}}^{t_j} B_{j-1}(\theta) u(\theta)\,\mathrm{d}\theta.$$

But

$$\Lambda f = -\big(M^{\frac{1}{2}}(\Phi x_0 - r_1), \dots, M^{\frac{1}{2}}(\Phi^{N-1} x_0 - r_{N-1}), G^{\frac{1}{2}}(\Phi^N x_0 - x_d)\big),$$

which implies

$$\begin{cases} f_k = -M^{\frac{1}{2}}(\Psi_k^f + \Phi^k x_0 - r_k), \quad k = 1, \dots, N-1, \\ f_N = -G^{\frac{1}{2}}(\Psi_N^f + \Phi^N x_0 - x_d). \end{cases}$$

If we replace Ψ^f by its value given by (18), we get

$$\begin{cases} f_k = -M^{\frac{1}{2}}\Big(\Phi^k x_0 + \sum_{j=1}^{k} \Phi^{k-j} \\ \qquad \times \int_{t_{j-1}}^{t_j} B_{j-1}(\theta) \underbrace{R^{-1} B_{j-1}^*(\theta) z_j^f}_{u^*(\theta)} \,\mathrm{d}\theta - r_k\Big), \\ \qquad k = 1, \dots, N-1, \\ f_N = -G^{\frac{1}{2}}\Big(\Phi^N x_0 + \sum_{j=1}^{N} \Phi^{N-j} \\ \qquad \times \int_{t_{j-1}}^{t_j} B_{j-1}(\theta) \underbrace{R^{-1} B_{j-1}^*(\theta) z_j^f}_{u^*(\theta)} \,\mathrm{d}\theta - x_d\Big). \end{cases}$$

That gives

$$\begin{cases} f_k = -M^{\frac{1}{2}}(x_k^{u^*} - r_k), \quad k = 1, \dots, N-1, \\ f_N = -G^{\frac{1}{2}}(x_N^{u^*} - x_d). \end{cases}$$

So u^* is the optimum of $\overline{J}$. Moreover,

$$\begin{aligned} \overline{J}(u) &= \langle (x_N^{u^*} - x_d), G(x_N^{u^*} - x_d) \rangle \\ &\quad + \sum_{k=1}^{N-1} \langle (x_k^{u^*} - r_k), M(x_k^{u^*} - r_k) \rangle \\ &\quad + \|u^*\|^2_{L^2(0,T;\mathcal{U})} \\ &= \|G^{\frac{1}{2}}(x_N^{u^*} - x_d)\|^2 \\ &\quad + \sum_{k=1}^{N-1} \|M^{\frac{1}{2}}(x_k^{u^*} - r_k)\|^2 + \|u^*\|^2 \\ &= \|f_N\|^2 + \sum_{k=1}^{N-1} \|f_k\|^2 + \int_0^T \langle u^*(t), R u^*(t) \rangle \,\mathrm{d}t \\ &= \|f\|^2 + \sum_{i=1}^{N} \int_{t_{i-1}}^{t_i} \langle R^{-1} B_{i-1}^*(\theta) z_i^f, B_{i-1}^*(\theta) z_i^f \rangle \,\mathrm{d}t \\ &= |||f|||^2. \end{aligned}$$

■

Remark 3. (i) In order to obtain the minimizing control u^*, one has to solve the infinite dimensional algebraic equation (21). However, in general, we do not know an explicit form of the operator Λ^{-1}. Since the bilinear continuous form

$$\begin{aligned} \mathcal{F} \times \mathcal{F} &\to \mathbb{R}, \\ (x, y) &\mapsto \langle x, \Lambda y \rangle_{\mathcal{F}} \end{aligned}$$

is coercive, the Galerkin method can be applied to approximate the solution f of (21) and, consequently, the optimal control u^*.

(ii) If in the functional J we have $M = 0$, then setting $\mathcal{F} = \mathcal{X}$ suffices to consider Γ instead of Λ, where $\Gamma \in \mathcal{L}(\mathcal{X})$ is the operator defined by

$$\Gamma f = f + G^{\frac{1}{2}} \Psi_N^f$$

with

$$\Psi_N^f = \sum_{j=1}^{N} \Phi^{N-j} \Big[\int_{t_{j-1}}^{t_j} B_{j-1}(\theta) R^{-1} B_{j-1}^*(\theta) \,\mathrm{d}\theta \Big] z_j^f,$$

$$z_j^f = \Phi^{*N-j} G^{\frac{1}{2}} f, \quad j = 1, 2, \dots, N.$$

Then, for $M = 0$, from Theorem 1 it follows that the minimizing control u^* is

$$u^*(\theta) = R^{-1} B_{i-1}^*(\theta) z_i^f$$

for $\theta \in [t_{i-1}, t_i[$ and $i = 1, 2, \dots, N$, where

$$z_i^f = \Phi^{*N-i} G^{\frac{1}{2}} f, \quad i = 1, 2, \dots, N,$$

and f constitutes the unique solution of the algebraic equation

$$\Gamma f = -G^{\frac{1}{2}}(\Phi^N x_0 - x_d).$$

Moreover, using the Galerkin method, an approximate control sequence is given by

$$\begin{cases} u_n^*(\theta) = R^{-1} B_{i-1}^*(\theta) z_i^{f_n}, \quad \theta \in [t_{i-1}, t_i[, \\ \qquad\qquad i = 1, 2, \dots, N, \\ z_i^{f_n} = \Phi^{*N-i} G^{\frac{1}{2}} f_n, \quad i = 1, 2, \dots, N, \\ f_n = \sum_{i=1}^{n} f_{n,i} \varphi_i, \end{cases}$$

where $(\varphi_n)_n$ is an orthonormal basis of $\mathcal{X}$, and the vector

$$\begin{pmatrix} f_{n,1} \\ f_{n,2} \\ \vdots \\ f_{n,n} \end{pmatrix}$$

is the unique solution of the matrix equation

$$\Gamma_n \begin{pmatrix} f_{n,1} \\ f_{n,2} \\ \vdots \\ f_{n,n} \end{pmatrix} = \begin{pmatrix} \langle -G^{\frac{1}{2}}(\Phi^N x_0 - x_d), \varphi_1 \rangle \\ \langle -G^{\frac{1}{2}}(\Phi^N x_0 - x_d), \varphi_2 \rangle \\ \vdots \\ \langle -G^{\frac{1}{2}}(\Phi^N x_0 - x_d), \varphi_n \rangle \end{pmatrix},$$

with $\Gamma_n = (\langle \Gamma\varphi_i, \varphi_j \rangle)_{1 \le i,j \le n}$.

Example 1. Consider the system (7) defined in Section 1, and the cost functional

$$J(u) = \|x_N - x_d\|^2 + \int_0^T |u(\theta)|^2 \, \mathrm{d}\theta,$$

where $\mathcal{U} = \mathbb{R}$, $\mathcal{X} = L^2(0,1)$, $R = 1$, $M = 0$, $G = I$, $x_0 = 0$, $x_d = \gamma\varphi_1$ and $B_i u = u \cdot \mathcal{X}_{w_i}$, with $w_i = [a_i, b_i] \subset [0,1]$. This means that we suppose that the activity of the control u on the parabolic system (7) is restricted to the zone $[a_i, b_i]$ (the action support of the control u changes at each moment i), where

$$\varphi_n(\theta) = \sqrt{2} \sin n\pi\theta$$

is an orthonormal basis of $\mathcal{X}$.

Since

$$S(t)x = \sum_{n=1}^{\infty} e^{-n^2\pi^2 t} \langle x, \varphi_n \rangle_{\mathcal{X}} \varphi_n,$$

the operator Φ is such that

$$\Phi x = \sum_{n=1}^{\infty} e^{-n^2\pi^2 \delta} \langle x, \varphi_n \rangle_{\mathcal{X}} \varphi_n,$$

and

$$B_i(\theta) = S\big((i+1)\delta - \theta\big) B_i.$$

Here we have

$$\begin{aligned} B_i(\theta)u &= S(t_{i+1} - \theta) B_i u \\ &= \sum_{n=1}^{\infty} e^{-n^2\pi^2(t_{i+1}-\theta)} \langle B_i u, \varphi_n \rangle \varphi_n \\ &= u \cdot \sum_{n=1}^{\infty} e^{-n^2\pi^2(t_{i+1}-\theta)} \Big(\int_{w_i} \varphi_n(x) \, \mathrm{d}x \Big) \varphi_n. \end{aligned}$$

Consequently, setting

$$\begin{aligned} \alpha_i(n) &= \int_{w_i} \varphi_n(x) \, \mathrm{d}x = \sqrt{2} \int_{a_i}^{b_i} \sin n\pi x \, \mathrm{d}x \\ &= \frac{\sqrt{2}}{n\pi} \Big[\cos n\pi x \Big]_{a_i}^{b_i}, \end{aligned}$$

we have

$$B_i(\theta)u = u \cdot \Big[\sum_{n=1}^{\infty} \alpha_i(n) e^{-n^2\pi^2(t_{i+1}-\theta)} \varphi_n \Big].$$

Therefore

$$\begin{aligned} \langle B_i(\theta)u, x \rangle &= \Big\langle u \cdot \Big[\sum_{n=1}^{\infty} \alpha_i(n) e^{-n^2\pi^2(t_{i+1}-\theta)} \varphi_n \Big], x \Big\rangle \\ &= u \cdot \Big[\sum_{n=1}^{\infty} \alpha_i(n) e^{-n^2\pi^2(t_{i+1}-\theta)} \langle x, \varphi_n \rangle \Big], \end{aligned}$$

and hence

$$B_i^*(\theta)x = \sum_{n=1}^{\infty} \alpha_i(n) e^{-n^2\pi^2(t_{i+1}-\theta)} \langle x, \varphi_n \rangle.$$

According to Theorem 1, the solution to the optimal control problem is as follows:

$$\begin{aligned} u^*(\theta) &= B_{i-1}^*(\theta) z_i^f \\ &= \sum_{n=1}^{\infty} \alpha_{i-1}(n) e^{-n^2\pi^2(t_i - \theta)} \langle z_i^f, \varphi_n \rangle \end{aligned}$$

for $\theta \in [t_{i-1}, t_i[$ and $i = 1, 2, \dots, N$, where

$$\begin{aligned} z_i^f &= \Phi^{N-i} f = \big[S(\delta)\big]^{(N-i)} f = S\big((N-i)\delta\big) f \\ &= S(t_{N-i}) f = \sum_{k=1}^{\infty} e^{-k^2\pi^2 t_{N-i}} \langle f, \varphi_k \rangle \varphi_k. \end{aligned}$$

Then

$$\langle z_i^f, \varphi_n \rangle = e^{-n^2\pi^2 t_{N-i}} \langle f, \varphi_n \rangle,$$

and hence

$$u^*(\theta) = \sum_{n=1}^{\infty} \alpha_{i-1}(n) e^{-n^2\pi^2(T-\theta)} \langle f, \varphi_n \rangle \tag{23}$$

for $\theta \in [t_{i-1}, t_i[$ and $i = 1, 2, \dots, N$, where f is the unique solution of the algebraic equation

$$\Gamma f = x_d,$$

which is equivalent to the infinite linear system

$$A \begin{pmatrix} f_1 \\ f_2 \\ \vdots \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \\ \vdots \end{pmatrix}, \tag{24}$$

A being an infinite matrix,

$$A = (\langle \Gamma\varphi_i, \varphi_j \rangle)_{1 \le i,j \le \infty},$$

with

$$\langle \Gamma\varphi_i, \varphi_j\rangle = \begin{cases} 1 + e^{-2i^2\pi^2 T}\displaystyle\sum_{k=1}^{N} \alpha_{k-1}^2(i)\left(\frac{e^{2i^2\pi^2 t_k} - e^{2i^2\pi^2 t_{k-1}}}{2i^2\pi^2}\right) \\ \hfill \text{if } \ i = j, \\ e^{-i^2\pi^2 T} e^{-j^2\pi^2 T}\displaystyle\sum_{k=1}^{N} \alpha_{k-1}(i)\alpha_{k-1}(j) \\ \quad \times \left(\frac{e^{(i^2+j^2)\pi^2 t_k} - e^{(i^2+j^2)\pi^2 t_{k-1}}}{(i^2+j^2)\pi^2}\right) \ \text{if } \ i \neq j. \end{cases}$$

As was mentioned in Remark 3, $f = \sum_{i=1}^{\infty} f_i\varphi_i$ is such that

$$f = \lim_{n\to\infty} f^n,$$

where $f^n = \sum_{i=1}^{n} f_i\varphi_i$, with

$$\begin{pmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_n^n \end{pmatrix}$$

being the unique solution of the algebraic equation

$$A_n \begin{pmatrix} f_1^n \\ f_2^n \\ \vdots \\ f_n^n \end{pmatrix} = \begin{pmatrix} \gamma \\ 0 \\ \vdots \\ 0 \end{pmatrix}.$$

A_n is the symmetric and positive definite matrix given by

$$A = (\langle \Gamma\varphi_i, \varphi_j\rangle)_{1\le i,j\le n}.$$

Concluding, in accordance with (23), the optimal control u^* can be approximated by the sequence $(u_n^*)_{n\geq 1}$ defined as follows:

$$u_n^*(\theta) = \sum_{k=1}^{n} \alpha_{i-1}(k) e^{-k^2\pi^2(T-\theta)} f_k^n \tag{25}$$

for $\theta \in [t_{i-1}, t_i[$ and $i = 1, 2, \ldots, N$. ♦

Remark 4. Consider the above example with $Bu = u \cdot \mathcal{X}_w$, where $w = [a, b] \subset [0, 1]$. (Here, in contrast to Example 1, we suppose that the action support of the control u is independent of i.) Then we have

$$B_i(\theta)u = u \cdot \left[\sum_{n=1}^{\infty} \alpha(n) e^{-n^2\pi^2(t_{i+1}-\theta)}\varphi_n\right],$$

with

$$\alpha(n) = \int_w \varphi_n(x)\,\mathrm{d}x = \sqrt{2}\int_a^b \sin n\pi x\,\mathrm{d}x.$$

On the other hand,

$$B_i^*(\theta)x = \sum_{n=1}^{\infty} \alpha(n) e^{-n^2\pi^2(t_{i+1}-\theta)}\langle x, \varphi_n\rangle, \quad \theta \in [0, T].$$

Hence

$$u^*(\theta) = \sum_{n=1}^{\infty} \alpha(n) e^{-n^2\pi^2(T-\theta)}\langle f, \varphi_n\rangle, \quad \theta \in [0, T], \tag{26}$$

where f is the unique solution of the algebraic equation

$$\Gamma f = x_d,$$

with

$$\langle \Gamma\varphi_i, \varphi_j\rangle = \begin{cases} 1 + \alpha_i^2\left(\dfrac{1 - e^{-2i^2\pi^2 T}}{2i^2\pi^2}\right) & \text{if } \quad i = j, \\ \alpha_i\alpha_j\left(\dfrac{1 - e^{-(i^2+j^2)\pi^2 T}}{(i^2+j^2)\pi^2}\right) & \text{if } \quad i \neq j. \end{cases}$$

As in the previous example, using the Galerkin method, the optimal control u^* can be approximated by the sequence $(u_n^*)_{n\geq 1}$ given by

$$u_n^*(\theta) = \sum_{k=1}^{n} \alpha(k) e^{-k^2\pi^2(T-\theta)} f_k^n, \quad \theta \in [0, T],$$

5. Conclusion

The passage from the continuous version of a linear system

$$\dot{x}(t) = Ax(t) + Bu(t) \tag{27}$$

to its discrete counterpart

$$x_{i+1} = \Phi x_i + \Psi u_i \tag{28}$$

is, generally, based on the assumption that

$$u(s) = u(t_i), \quad \forall s \in [t_i, t_{i+1}[, \tag{29}$$

where t_i and t_{i+1} are two consecutive sampling instants. The approximation of the continuous system (27) by the difference equation (28) is often justified by the choice of a rather small sampling period.

In this paper, we have studied the quadratic linear control problem associated with a linear system having a discrete state variable and a continuous control variable. Such a system can be regarded as a sampled version of the continuous system (27) in the absence of the assumption (29) (when the time interval $[t_i, t_{i+1}[$ is rather large or when variations in the control variable $u(\cdot)$ are very fast, it makes no sense to adopt the hypothesis (29)). To solve the problem, we introduced an adequate Hilbertian structure and proved that the optimum and optimal cost stem from an algebraic linear infinite dimensional equation which is easily solvable by the classical Galerkin method. As a natural continuation of this work, while being inspired by (Rachik *et al.*, 2003), we are going to investigate the linear quadratic control problem considered for an infinite time horizon.

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