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APPLICATION OF THE DRAZIN INVERSE TO THE ANALYSIS OF DESCRIPTOR FRACTIONAL DISCRETE–TIME LINEAR SYSTEMS WITH REGULAR PENCILS

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The Drazin inverse of matrices is applied to find the solutions of the state equations of descriptor fractional discrete-time systems with regular pencils. An equality defining the set of admissible initial conditions for given inputs is derived. The proposed method is illustrated by a numerical example.

Keywords: Drazin inverse, descriptor, fractional system, discrete-time system, linear system.

Dedicated to Bogumiła Adamczyk on the occasion of her birthday

1. Introduction

Descriptor (singular) linear systems were considered in many papers and books (Bru *et al.*, 2000; 2002; 2003; Campbell *et al.*, 1976; Dai, 1989; Dodig and Stosic, 2009; Fahmy and O'Reill, 1989; Kaczorek, 1992; 2004; 2007b; 2011a; 2011d; Van Dooren, 1979; Virnik, 2008). The eigenvalues and invariants assignment by state and output feedbacks were investigated by Fahmy and O'Reill (1989), Gantmacher (1960), Kaczorek (2004) as well as Kucera and Zagalak (1988), and the realization problem for singular positive continuous-time systems with delays was examined by Kaczorek (2007b). The computation of Kronecker's canonical form of the singular pencil was analyzed by Van Dooren (1979). Positive linear systems with different fractional orders were addressed by Kaczorek (2010), who also discussed selected problems of fractional linear systems theory (Kaczorek, 2011b).

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of the state of the art in positive systems theory is given by Farina and Rinaldi (2000), Commalut and Marchand (2006), as well as Kaczorek (2002; 2007a). A variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

Descriptor standard positive linear systems using the Drazin inverse were addressed by Bru *et al.* (2003; 2000; 2002) and Kaczorek (2002; 2011b). The shuffle algorithm was applied to check the positivity of descriptor linear systems by Kaczorek (2011a), while the stability of positive descriptor systems was investigated by Virnik (2008). Reduction and decomposition of descriptor fractional discrete-time linear systems were considered by Kaczorek (2011d), who also introduced a new class of descriptor fractional linear discrete-time systems (Kaczorek, 2011c).

In this paper the Drazin inverse of matrices will be applied to find the solutions of the state equations of descriptor fractional discrete-time linear systems with regular pencils. The paper is organized as follows. In Section 2 the state equation of the descriptor fractional linear discrete-time system and some basic definitions of the Drazin inverse are recalled. The solution to the state equation is given in Section 3. The proposed method is illustrated by numerical examples in Section 4. Concluding remarks are given in Section 5.

The following notation will be used: $\mathbb{R}$ is the set of real numbers, $\mathbb{R}^{n\times m}$ is the set of $n \times m$ real matrices and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$, $\mathbb{Z}_+$ is the set of nonnegative integers, $\mathbb{N}$ is the set of natural numbers, I_n is the $n \times n$ identity matrix, ker A is the kernel of the matrix, $\mathbb{C}$ is the field of complex numbers.

2. Preliminaries

Consider the descriptor fractional discrete-time linear system

$$E\Delta^\alpha x_{i+1} = Ax_i + Bu_i, \quad i \in \mathbb{Z}_+, \quad 0 < \alpha < 1, \tag{1}$$

where α is the fractional order, $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^m$ is the input vector and $E, A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m}$. It is assumed that $\det E = 0$, but the pencil (E, A) is regular, i.e.,

$$\det[Ez - A] \neq 0 \quad \text{for some } z \in \mathbb{C}. \tag{2}$$

The fractional difference of the order α is defined by (Kaczorek, 2011b)

$$\Delta^\alpha x_i = \sum_{k=0}^{i} c_k x_{i-k}, \quad n-1 < \alpha < n \in \mathbb{N}, \tag{3}$$

where

$$c_k = (-1)^k \binom{\alpha}{k}, \quad k = 0, 1, \dots, \tag{4a}$$

and

$$\binom{\alpha}{k} = \begin{cases} 1 & \text{for} \quad k = 0, \\ \dfrac{\alpha(\alpha-1)\dots(\alpha-k+1)}{k!} & \text{for} \quad k = 1, 2, \dots. \end{cases} \tag{4b}$$

Substitution of (3) into (1) yields

$$Ex_{i+1} = Fx_i - \sum_{k=1}^{i} Ec_{k+1}x_{i-k} + Bu_i, \quad i \in \mathbb{Z}_+, \tag{5a}$$

where

$$F = A - Ec_1. \tag{5b}$$

Assuming that

$$\det[Ec - F] \neq 0 \quad \text{for some } c \in \mathbb{C}, \tag{6}$$

and premultiplying (5a) by $[Ec - F]^{-1}$, we obtain

$$\bar{E}x_{i+1} = \bar{F}x_i - \sum_{k=1}^{i} \bar{E}c_{k+1}x_{i-k} + \bar{B}u_i, \tag{7a}$$

where

$$\bar{E} = [Ec - F]^{-1}E, \quad \bar{F} = [Ec - F]^{-1}F, \quad \bar{B} = [Ec - F]^{-1}B. \tag{7b}$$

Definition 1. (*Kaczorek, 1992*) The smallest nonnegative integer q satisfying

$$\operatorname{rank} \bar{E}^q = \operatorname{rank} \bar{E}^{q+1} \tag{8}$$

is called the *index* of the matrix $\bar{E} \in \mathbb{R}^{n\times n}$.

Definition 2. (*Kaczorek, 1992*) A matrix $\bar{E}^D$ is called the *Drazin inverse* of $\bar{E} \in \mathbb{R}^{n\times n}$ if it satisfies the conditions

$$\bar{E}\bar{E}^D = \bar{E}^D\bar{E}, \tag{9a}$$

$$\bar{E}^D\bar{E}\bar{E}^D = \bar{E}^D, \tag{9b}$$

$$\bar{E}^D\bar{E}^{q+1} = \bar{E}^q, \tag{9c}$$

where q is the index of $\bar{E}$ defined by (8).

The Drazin inverse $\bar{E}^D$ of a square matrix $\bar{E}$ always exists and is unique (Campbell *et al.*, 1976; Kaczorek, 1992). If $\det \bar{E} \neq 0$ then $\bar{E}^D = \bar{E}^{-1}$. Some methods for computation of the Drazin inverse are given by Kaczorek (1992).

Lemma 1. (Campbell *et al.*, 1976; Kaczorek, 1992) *The matrices $\bar{E}$ and $\bar{F}$ defined by (7b) satisfy the following equalities:*

$$\bar{F}\bar{E} = \bar{E}\bar{F}, \quad \bar{F}^D\bar{E} = \bar{E}\bar{F}^D, \quad \bar{E}^D\bar{F} = \bar{F}\bar{E}^D, \quad \bar{F}^D\bar{E}^D = \bar{E}^D\bar{F}^D, \tag{10a}$$

$$\ker \bar{F}_1 \cap \ker \bar{E} = \{0\}, \tag{10b}$$

$$\bar{E} = T\begin{bmatrix} J & 0 \\ 0 & N \end{bmatrix}T^{-1}, \quad \bar{F} = T\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}T^{-1}, \quad \bar{E}^D = T\begin{bmatrix} J^{-1} & 0 \\ 0 & 0 \end{bmatrix}T^{-1}, \tag{10c}$$

$\det T \neq 0$, $J \in \mathbb{R}^{n_1\times n_1}$ *is nonsingular*, $N \in \mathbb{R}^{n_2\times n_2}$ *is nilpotent*, $A_1 \in \mathbb{R}^{n_1\times n_1}$, $A_2 \in \mathbb{R}^{n_2\times n_2}$, $n_1 + n_2 = n$,

$$(I_n - \bar{E}\bar{E}^D)\bar{F}\bar{F}^D = I_n - \bar{E}\bar{E}^D \quad \textit{and} \quad (I_n - \bar{E}\bar{E}^D)(\bar{E}\bar{F}^D)^q = 0. \tag{10d}$$

3. Solution to the state equation

In this section the solution to the state equation (1) will be presented by the use of the Drazin inverses of the matrices $\bar{E}$ and $\bar{F}$.

Theorem 1. *The solution to Eqn. (7a) with an admissible initial condition x_0 is given by*

$$x_i = (\bar{E}^D\bar{F})^i\bar{E}^D\bar{E}x_0 + \sum_{k=0}^{i-1} \bar{E}^D(\bar{E}^D\bar{F})^{i-k-1}\Big[\bar{B}u_k - \sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j}\Big] + (\bar{E}\bar{E}^D - I_n)\sum_{k=0}^{q-1}(\bar{E}\bar{F}^D)^k\bar{F}^D\bar{B}u_{i+k}, \tag{11}$$

where q is the index of $\bar{E}$.

Proof. Using (11) and taking into account (9) and (10), we obtain

$$\begin{aligned}&\bar{E}x_{i+1}\\&=\bar{E}(\bar{E}^D\bar{F})^{i+1}\bar{E}^D\bar{E}x_0\\&\quad+\sum_{k=0}^{i}\bar{E}\bar{E}^D(\bar{E}^D\bar{F})^{i-k}\Big[\bar{B}u_k-\sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j+1}\Big]\\&\quad+\bar{E}(\bar{E}\bar{E}^D-I_n)\sum_{k=0}^{q-1}(\bar{E}\bar{F}^D)^k\bar{F}^D\bar{B}u_{i+k+1}\\&=\bar{F}(\bar{E}^D\bar{F})^i\bar{E}^D\bar{E}x_0\\&\quad+\sum_{k=0}^{i-1}(\bar{E}^D\bar{F})^{i-k}\Big[\bar{B}u_k-\sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j}\Big]+\bar{B}u_i\\&\quad+(I_n-\bar{E}\bar{E}^D)\sum_{k=0}^{q-1}(-\bar{F}^D\bar{E})^k\bar{F}\bar{F}^D\bar{B}u_{i+k}\end{aligned}\tag{12}$$

and

$$\begin{aligned}&\bar{F}x_i\\&=\bar{F}(\bar{E}^D\bar{F})^i\bar{E}^D\bar{E}x_0\\&\quad+\sum_{k=0}^{i-1}\bar{F}\bar{E}^D(\bar{E}^D\bar{F})^{i-k-1}\Big[\bar{B}u_k-\sum_{j=1}^{k}\bar{E}c_{j+1}x_{k-j}\Big]\\&\quad+\bar{F}(\bar{E}\bar{E}^D-I_n)\sum_{k=0}^{q-1}(\bar{E}\bar{F}^D)^k\bar{F}^D\bar{B}u_{i+k}.\end{aligned}\tag{13}$$

Hence

$$\bar{E}x_{i+1}-\bar{F}x_i-\sum_{k=1}^{j}\bar{E}c_{k+1}x_{i-k}=\bar{B}u_i.\tag{14}$$

Thus, the solution (11) satisfies Eqn. (7a). ∎

From (11), for $i=0$ we have

$$x_0=\bar{E}^D\bar{E}x_0+(\bar{E}\bar{E}^D-I_n)\sum_{k=0}^{q-1}(\bar{E}\bar{F}^D)^k\bar{F}^D\bar{B}u_k.\tag{15}$$

The set of admissible initial conditions x_0 for given input u_i is given by (15). In a practical case, for $u_i=0$, $i\in\mathbb{Z}_+$ we have $x_0=\bar{E}^D\bar{E}x_0$. Thus, the equation $\bar{E}x_{i+1}=Ax_i$ has a unique solution if and only if $x_0\in\mathrm{Im}\bar{E}\bar{E}^D$, where ‘Im’ denotes the image.

4. Example

Find the solution x_i to Eqn. (1) with α = 0.5 and the matrices

$$E=\begin{bmatrix}1&0\\0&0\end{bmatrix},\quad A=\begin{bmatrix}0&0\\1&-2\end{bmatrix},\quad B=\begin{bmatrix}1\\2\end{bmatrix},\tag{16}$$

and admissible initial conditions for given input u_i, $i\in\mathbb{Z}_+$. The pencil of (16) is regular since

$$\begin{aligned}\det[Ez-A]&=\begin{vmatrix}z&0\\-1&2\end{vmatrix}=2z,\\F=A-Ec_1&=A+E\alpha=\begin{bmatrix}\alpha&0\\1&-2\end{bmatrix},\\q&=1.\end{aligned}\tag{17}$$

For $c=1$ the matrices (7b) have the forms

$$\begin{aligned}\bar{E}&=[Ec-F]^{-1}E=\begin{bmatrix}1-\alpha&0\\-1&2\end{bmatrix}^{-1}\begin{bmatrix}1&0\\0&0\end{bmatrix}\\&=\frac{1}{2(1-\alpha)}\begin{bmatrix}2&0\\1&0\end{bmatrix}=\begin{bmatrix}2&0\\1&0\end{bmatrix},\\\bar{F}&=[Ec-F]^{-1}F=\begin{bmatrix}1-\alpha&0\\-1&2\end{bmatrix}^{-1}\begin{bmatrix}\alpha&0\\1&-2\end{bmatrix}\\&=\frac{1}{2(1-\alpha)}\begin{bmatrix}2\alpha&0\\1&-2(1+\alpha)\end{bmatrix}=\begin{bmatrix}-1&0\\1&-1\end{bmatrix},\\\bar{B}&=[Ec-F]^{-1}B=\begin{bmatrix}1-\alpha&0\\-1&2\end{bmatrix}^{-1}\begin{bmatrix}1\\2\end{bmatrix}\\&=\frac{1}{2(1-\alpha)}\begin{bmatrix}2\\3-2\alpha\end{bmatrix}=\begin{bmatrix}2\\2\end{bmatrix}.\end{aligned}\tag{18}$$

Using (10c) and (18), we obtain

$$\bar{E}=T^{-1}\begin{bmatrix}2&0\\0&0\end{bmatrix}T,\quad T=\begin{bmatrix}1&0\\1/2&1\end{bmatrix}\tag{19}$$

and

$$\bar{E}^D=T^{-1}\begin{bmatrix}1/2&0\\0&0\end{bmatrix}T=\begin{bmatrix}1/2&0\\-1/4&0\end{bmatrix}.\tag{20}$$

Note that

$$\det\bar{F}=\begin{vmatrix}1&0\\1&-1\end{vmatrix}=-1\neq0\tag{21}$$

and

$$\bar{F}^D=\bar{F}^{-1}=\begin{bmatrix}1&0\\1&-1\end{bmatrix}.\tag{22}$$

Taking into account that

$$\begin{aligned}\bar{E}^D\bar{F}&=\begin{bmatrix}1/2&0\\-1/4&0\end{bmatrix}\begin{bmatrix}1&0\\1&-1\end{bmatrix}=\begin{bmatrix}1/2&0\\-1/4&0\end{bmatrix},\\\bar{E}\bar{E}^D&=\begin{bmatrix}2&0\\1&0\end{bmatrix}\begin{bmatrix}1/2&0\\-1/4&0\end{bmatrix}=\begin{bmatrix}1&0\\1/2&0\end{bmatrix}\end{aligned}\tag{23}$$

and using (11), we obtain

$$
\begin{aligned}
x_i &= \begin{bmatrix} 1/2 & 0 \\ -1/4 & 0 \end{bmatrix}^i \begin{bmatrix} 1 & 0 \\ -1/2 & 0 \end{bmatrix} x_0 \\
&+ \sum_{k=0}^{i-1} \begin{bmatrix} 1/2 & 0 \\ -1/4 & 0 \end{bmatrix} \begin{bmatrix} 1/2 & 0 \\ -1/4 & 0 \end{bmatrix}^{i-k-1} \\
&\times \left\{ \begin{bmatrix} 2 \\ 2 \end{bmatrix} u_k - \sum_{j=1}^{k} \begin{bmatrix} 2 & 0 \\ 1 & 0 \end{bmatrix} c_{j+1} x_{k-j}] \right\} \\
&+ \begin{bmatrix} 0 & 0 \\ 1/2 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} u_i,
\end{aligned} \tag{24}
$$

where the coefficients c_j are defined by (4a) for $\alpha = 0.5$.

From (24), for $i = 0$ we have

$$x_0 = \begin{bmatrix} 1 & 0 \\ 1/2 & 0 \end{bmatrix} x_0 + \begin{bmatrix} 0 \\ -2 \end{bmatrix} u_0. \tag{25}$$

Hence, for given u_0, the admissible initial condition x_0 should satisfy (25).

5. Concluding remarks

The Drazin inverse of matrices has been applied to find the solutions of the state equations of the descriptor fractional discrete-time systems with regular pencils. The equality (15) defining the set of admissible initial conditions for given inputs has been derived. The proposed method has been illustrated by a numerical example.

Comparing the presented method with that based on the Weierstrass decomposition of the regular pencil (Kaczorek, 2011c), we may conclude that the proposed approach is computationally attractive since the Drazin inverse of matrices can be computed by the use of well-known numerical methods (Kaczorek, 1992). The presented method can be extended to descriptor fractional continuous-time linear systems. An open problem is the extension of the deliberations to standard and positive continuous-discrete descriptor fractional linear systems.

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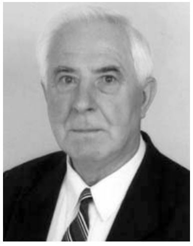

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