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ON FAST STATE-SPACE ALGORITHMS FOR PREDICTIVE CONTROL†

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A Riccati-equation-based solution to a class of receding-horizon predictive control problems for an explicit-delay state-space model of an ARMAX system is found and the corresponding vector Chandrasekhar-type equations are derived for both filter and controller gains to improve the computational efficiency.

Keywords: Riccati equation, Chandrasekhar equation, LQG control, predictive control.

1. Introduction

A quarter of a century has passed since Åström (1970) started a direction of control theory based on the input-output description of discrete-time systems working under stochastic disturbances, the aim being the design of controllers which optimise a receding horizon quadratic performance index. One of the best known and widely used algorithms of this class is the so-called Generalised Predictive Control (GPC) (Clarke and Mohtadi, 1989; Clarke *et al.*, 1985; 1987). Just from the beginning, this approach attracted an immense attention of control practitioners, which resulted in a large number of papers presenting further theoretical developments and applications.

In the classical literature (Clarke and Mohtadi, 1989; Clarke *et al.*, 1987), the GPC controller has been usually presented as a solution to a control problem with a receding horizon quadratic performance index. Both the performance index and the process model have been formulated in the input-output implicit delay terms and the solution has been based on free predictions of the output variable calculated from an ARMAX or ARIMAX model. As a result, calculation of control requires inversion of a symmetric matrix whose dimensions equal the control horizon N_u. For numerical reasons, this horizon should be limited, which affects the quality of control, particularly when the sampling rate is high.

Along with this, for about two decades efforts have been made to perform the state-space synthesis of minimum-variance and predictive controllers (Błachuta, 1987; 1996a; 1996b; Błachuta and Ordys, 1987; Byun *et al.*, 1990; Caines, 1972; Gambier

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and Unbehauen, 1993; Grimble, 1994; Krauss *et al.*, 1994; Kwon *et al.*, 1992a; 1992b; Lee *et al.*, 1994; Matko, 1990; Ordys and Clarke, 1993; Warwick, 1987; 1990; Warwick and Peterka, 1991).

State-space solutions to a control problem can be obtained by using either the OLF (Open Loop Feedback) (Błachuta, 1996b; Gambier and Unbehauen, 1993; Krauss *et al.*, 1994; Kwon *et al.*, 1992a; 1992b; Kwon and Byun, 1989; Lee *et al.*, 1994; Ordys and Clarke, 1993; Warwick and Peterka, 1991) or the CL (Closed Loop) approach (Bitmead *at el.*, 1990; Błachuta, 1996a; 1996b; Gambier and Unbehauen, 1993; Lee *et al.*, 1994; Ordys and Clarke, 1993).

The computational complexity connected with the controllers that base on Riccati equations depends on the system order and not on the control horizon. Unlike the OLF solution, an important feature of the CL solution is that an infinite-horizon problem, i.e. a problem where $N \to \infty$, can be stated and successfully solved (Błachuta, 1996a; 1996b).

However, the solutions can suffer from implicit delay singularities when the control costing $\lambda = 0$, for an explanation see (Błachuta, 1987; Kowalczuk and Suchomski, 1997), and a nonstationary noise model. Very often, (Bitmead *et al.*, 1990; Lee *et al.*, 1994) the state-space models employed do not match any state-space realization of an ARMAX or ARIMAX model and thus cannot be compared with classical solutions. An approach where the solution is based on an 'innovation-like' representation of an explicit-delay ARMAX (or ARIMAX) model, yielding a controller which is functionally equivalent to a classical GMV, GPC or LQG controller in the steady state and better in transient states, was proposed in (Błachuta, 1996a; 1996b). This paper provides a detailed derivation of that controller.

As is well-known (Morf *et al.*, 1974), for some special filtration problems the matrix Riccati equations can be replaced by the so-called vector Chandrasekhar-type equations. In the paper, fast Chandrasekhar-type algorithms for both the filter and controller gain vectors are derived to minimize the number of operations required to perform the necessary computations.

2. State-Space Formulation of Predictive Control

It is assumed that the system to be controlled is described by the following state-space model:

$$\boldsymbol{x}_{i+1} = \boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{b}u_i + \boldsymbol{c}v_i \tag{1}$$

$$y_i = \boldsymbol{d}'\boldsymbol{x}_i + v_i \tag{2}$$

in which $\boldsymbol{A}$ is an $n \times n$ matrix, $\boldsymbol{b}$, $\boldsymbol{c}$ and $\boldsymbol{d}$ are n-vectors, v_i is a white noisewith covariance $\mathrm{E}\, v_i^2 = \sigma^2$, and the initial condition, $\boldsymbol{x}_0$, is a normal random vector independent of the disturbances, i.e. $\mathrm{E}\,(\boldsymbol{x}_0 v_i) = 0$, $i = 0, 1, \ldots$ with $\mathrm{E}\,\{\boldsymbol{x}_0\} = \boldsymbol{m}_0$, $\mathrm{cov}\,\{(\boldsymbol{x}_0 - \boldsymbol{m}_0)\} = \sigma^2 \boldsymbol{Q}_0$. The model (1)–(2) is misleadingly called 'innovations model' in the literature. Indeed, it has a constant vector $\boldsymbol{c}$, constant variance σ^2 and a random initial condition $\boldsymbol{x}_0$, while in the innovations model the initial condition $\boldsymbol{x}_0$

is deterministic and both the Kalman gain $\boldsymbol{c}_i$ and innovations variance σ_i^2 do depend on time i. This is explained in detail in Section 4. For a relationship of (1)–(2) to other stochastic state-space models see (Błachuta and Polański, 1987). Model (1)–(2) is input-output equivalent to an ARMAX model

$$\tilde{A}(z^{-1})y_i = \tilde{B}(z^{-1})u_{i-k} + \tilde{C}(z^{-1})v_i \tag{3}$$

where $z^n\tilde{A}(z^{-1})$, $z^n\tilde{C}(z^{-1})$ are monic n-th degree polynomials in the shift operator z and $z^m\tilde{B}(z^{-1})$, with $m = n - k$, is an m-th degree polynomial with leading coefficient b_0. Conversely, given an ARMA model (3), a state-space representation (1)–(2) can be constructed using canonical forms. The system (1)–(2) can also be considered by as a state-space representation of a Box-Jenkins, Ljung-Söderström or step-response model with an ARMA disturbance (Błachuta, 1996a).

A receding-horizon explicit delay performance index I_i is of the form

$$I_i = \mathrm{E}\{J_i\} = \mathrm{E}\left\{\sum_{j=i}^{i+N-1} y_{j+k}^2 + \lambda \sum_{j=i}^{i+N_u-1} u_j^2\right\} \tag{4}$$

where $N_u \leq N$, and additionally it is assumed that

$$u_j = 0 \quad \text{for} \quad j > N_u \tag{5}$$

Here N is called the cost horizon, N_u a control horizon, and k is a discrete-time delay in the control path.

Denote by $\vec{\boldsymbol{y}}_i = [y_0, y_1, \dots, y_i, u_0, u_1, \dots, u_{i-1}]'$ the vector containing the information available at time i. The control problem consists in finding $u_i = f_i(\vec{\boldsymbol{y}}_i)$ which minimizes I_i of (4).

The above problem statement is flexible enough to encompass both LQG and GPC control problems. Usually, a different problem statement can be found in the literature (Clarke and Mohtadi, 1989) where a problem is defined using the increments of the control signal $\Delta u_i = u_i - u_{i-1}$, rather than the actual control input u_i. In this way, an integral action in the control loop is guaranteed but, unfortunately, the implicit disturbance model becomes nonstationary:

$$\tilde{A}(z^{-1})y_i = \tilde{B}(z^{-1})u_{i-k} + \frac{\tilde{C}(z^{-1})}{\Delta}v_i \tag{6}$$

Note that realistic disturbances are usually modelled by stationary stochastic processes characterized by their spectral density or correlation function.

Methods of imposing the integral loop action without resorting to nonrealistic nonstationary noise models are discussed in (Błachuta, 1996a).

3. Riccati Equation Solution to the Control Problem

Denote by $\hat{\boldsymbol{x}}_{i|i} = E(\boldsymbol{x}_i|\vec{\boldsymbol{y}}_i)$ the optimal estimate of $\boldsymbol{x}_i$ based on the information available at time i and supplied by the Kalman filter. Then we have

Theorem 1. *The optimal control law has the form*

$$u_i = -\boldsymbol{k}_c'\big[\boldsymbol{F}\hat{\boldsymbol{x}}_{i|i} + \boldsymbol{c}y_i\big] \tag{7}$$

$$\boldsymbol{k}_c = \frac{\boldsymbol{P}_0\boldsymbol{b}}{\lambda + \boldsymbol{b}'\boldsymbol{P}_0\boldsymbol{b}} \tag{8}$$

where $\boldsymbol{P}_0$ *is calculated from the following set of recursive equations:*

i. *(Lyapunov)*

$$\boldsymbol{P}_j = \boldsymbol{A}'\boldsymbol{P}_{j+1}\boldsymbol{A} + \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}', \quad \boldsymbol{P}_N = \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}' \tag{9}$$

for $j = N-1, \dots, N_u$ *and*

ii. *(Riccati)*

$$\boldsymbol{P}_j = \boldsymbol{A}'\left(\boldsymbol{P}_{j+1} - \frac{\boldsymbol{P}_{j+1}\boldsymbol{b}\boldsymbol{b}'\boldsymbol{P}_{j+1}}{\lambda + \boldsymbol{b}'\boldsymbol{P}_{j+1}\boldsymbol{b}}\right)\boldsymbol{A} + \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}' \tag{10}$$

for $j = N_u - 1, \dots, 0$, *where the vector* $\boldsymbol{d}_{k-1}$ *results from the recursion*

$$\boldsymbol{d}_0 = \boldsymbol{d}, \ \ \boldsymbol{d}_j = \boldsymbol{A}'\boldsymbol{d}_{j-1}, \quad j = 1, 2, \dots, k-1 \tag{11}$$

Remark 1. Notice that due to the special form of system equations (1)–(2), the control law in eqn. (7), which is a function not only of the state estimate but also of the current reading y_i, is somewhat different from the usual linear state feedback. This issue is further discussed in Remark 2 following the proof.

Proof. From (1)–(2) and (11) it follows that

$$y_{j+k} = \boldsymbol{d}_{k-1}'(\boldsymbol{A}\boldsymbol{x}_j + \boldsymbol{c}v_j) + g_k u_j + \sum_{l=0}^{k-1} e_l v_{j+k-l} \tag{12}$$

where g_i and e_i are the corresponding Markov parameters:

$$g_0 = 0, \quad g_j = \boldsymbol{d}_{j-1}'\boldsymbol{b} = \boldsymbol{d}'\boldsymbol{A}^{j-1}\boldsymbol{b}, \quad j > 0 \tag{13}$$

$$e_0 = 0, \quad e_j = \boldsymbol{d}_{j-1}'\boldsymbol{g} = \boldsymbol{d}'\boldsymbol{A}^{j-1}\boldsymbol{g}, \quad j > 0 \tag{14}$$

and for a k-step time delay in the control channel we have

$$g_0 = 0, \quad g_1 = 0, \quad \dots, \quad g_{k-1} = 0, \quad g_k = b_0 \neq 0 \tag{15}$$

The performance index J_i in (4) with (5) is equivalent to the following:

$$I_i = \mathrm{E}\{J_i\} = \mathrm{E}\left\{\sum_{j=i}^{i+N-1} y_{j+k}^2 + \lambda_j u_j^2\right\} \tag{16}$$

where

$$\lambda_i = \lambda \quad \text{for} \quad i = 0, 1, \ldots, N_u - 1 \tag{17}$$

and

$$\lambda_i \to \infty \quad \text{for} \quad i = N_u, \ldots, N - 1 \tag{18}$$

Substituting (12) into the above performance index and averaging the terms containing noise which will appear later with respect to any time instant j, we get

$$I_i = \mathrm{E}\{J_i'\} + N\sigma^2 \sum_{l=1}^{k-1} e_l^2 \tag{19}$$

and

$$J_i' = \sum_{j=i}^{i+N-1} \left\{ \left[\boldsymbol{d}_{k-1}'(\boldsymbol{A}\boldsymbol{x}_j + \boldsymbol{c}v_j) + b_0 u_j \right]^2 + \lambda_j u_j^2 \right\} \tag{20}$$

A solution to the deterministic problem (1) with the performance index (20) can be found based on the Hamiltonian:

$$H_j = \left[\boldsymbol{d}_{k-1}'(\boldsymbol{A}\boldsymbol{x}_j + \boldsymbol{c}v_j) + b_0 u_j \right]^2 + \lambda_j u_j^2 + 2\boldsymbol{p}_{j+1}'(\boldsymbol{A}\boldsymbol{x}_j + \boldsymbol{b}u_j + \boldsymbol{c}v_j) \tag{21}$$

Assume that the adjoint variable $\boldsymbol{p}_j$ is of the form

$$\boldsymbol{p}_j = \frac{\partial H}{\partial \boldsymbol{x}_j} = (\boldsymbol{P}_j - \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}')\boldsymbol{x}_j + \boldsymbol{f}_j \tag{22}$$

with $\boldsymbol{p}_{i+N} = \boldsymbol{0}$. The optimal control minimizes the Hamiltonian, i.e. it can be calculated from

$$u_j = -\boldsymbol{k}_j^{c\prime}(\boldsymbol{A}\boldsymbol{x}_j + \boldsymbol{c}v_j) - \frac{\boldsymbol{b}'\boldsymbol{f}_{j+1}}{\lambda_j + \boldsymbol{b}'\boldsymbol{P}_{j+1}\boldsymbol{b}} \tag{23}$$

$$\boldsymbol{k}_j^c = \frac{\boldsymbol{P}_{j+1}\boldsymbol{b}}{\lambda_j + \boldsymbol{b}'\boldsymbol{P}_{j+1}\boldsymbol{b}} \tag{24}$$

$$\boldsymbol{f}_j = \boldsymbol{A}'\left[\boldsymbol{I} - \frac{\boldsymbol{P}_{j+1}\boldsymbol{b}\boldsymbol{b}'}{\lambda_j + \boldsymbol{b}'\boldsymbol{P}_{j+1}\boldsymbol{b}}\right](\boldsymbol{f}_{j+1} + \boldsymbol{P}_{j+1}\boldsymbol{c}v_j) \tag{25}$$

$$\boldsymbol{P}_j = \boldsymbol{A}'\left(\boldsymbol{P}_{j+1} - \frac{\boldsymbol{P}_{j+1}\boldsymbol{b}\boldsymbol{b}'\boldsymbol{P}_{j+1}}{\lambda_j + \boldsymbol{b}'\boldsymbol{P}_{j+1}\boldsymbol{b}}\right)\boldsymbol{A} + \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}' \tag{26}$$

with $\boldsymbol{f}_{i+N} = \boldsymbol{0}$ and $\boldsymbol{P}_{i+N} = \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}'$, for $j = i + N - 1, \ldots, i$. As a result, for the current time i we have

$$u_i = -\boldsymbol{k}_i^c(\boldsymbol{A}\boldsymbol{x}_i + \boldsymbol{c}v_i) + \sum_{k=1}^{N-1} \theta_{i,k} v_{i+k} \tag{27}$$

where $\theta_{i,k}$ depends on neither the state nor the noise. Applying the Certainty Equivalence Principle (Uchida and Shimemura, 1976), i.e. replacing stochastic variables by their estimates based on the information available at time i, eqns. (7)–(10). ■

Remark 2. The way the problem is solved in certain references is (a) to solve a deterministic LQ problem with $v_i = 0$ in (1)–(2), and (b) to combine it with a Kalman filter for (1)–(2). The resulting control law $u_i = -\boldsymbol{k}_i^c \boldsymbol{A}\hat{\boldsymbol{x}}_{i|i}$, linear in $\hat{\boldsymbol{x}}_{i|i}$, is then incorrect.

4. State Filtration and Prediction

The Kalman filter equations (Anderson and Moore, 1979) for the system (1)–(2) have the form

$$\hat{\boldsymbol{x}}_{i|i} = \hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{k}_i^f [y_i - \boldsymbol{d}'\hat{\boldsymbol{x}}_{i|i-1}] \tag{28}$$

$$\hat{\boldsymbol{x}}_{i+1|i} = \boldsymbol{F}\hat{\boldsymbol{x}}_{i|i} + \boldsymbol{b}u_i + \boldsymbol{c}y_i \tag{29}$$

with

$$\boldsymbol{F} = \boldsymbol{A} - \boldsymbol{c}\boldsymbol{d}' \tag{30}$$

$\hat{\boldsymbol{x}}_{0|-1} = \boldsymbol{m}_0$ and the Kalman filter gain, $\boldsymbol{k}_i^f$, given by the formula

$$\boldsymbol{k}_i^f = \frac{\boldsymbol{\Sigma}_i \boldsymbol{d}}{1 + \boldsymbol{d}'\boldsymbol{\Sigma}_i \boldsymbol{d}} \tag{31}$$

where $\boldsymbol{\Sigma}_i = \operatorname{cov}(\tilde{\boldsymbol{x}}_{i|i-1})/\sigma^2$, with $\tilde{\boldsymbol{x}}_{i|i-1} = \boldsymbol{x}_{i|i-1} - \hat{\boldsymbol{x}}_{i|i-1}$, is calculated from the recursive Riccati equation

$$\boldsymbol{\Sigma}_{i+1} = \boldsymbol{F}\left(\boldsymbol{\Sigma}_i - \frac{\boldsymbol{\Sigma}_i \boldsymbol{d}\boldsymbol{d}'\boldsymbol{\Sigma}_i}{1 + \boldsymbol{d}'\boldsymbol{\Sigma}_i \boldsymbol{d}}\right)\boldsymbol{F}', \quad \boldsymbol{\Sigma}_0 = \boldsymbol{Q}_0 \tag{32}$$

The Kalman filter (28)–(29) can be transformed to the following innnovations form:

$$\hat{\boldsymbol{x}}_{i+1|i} = \boldsymbol{A}\hat{\boldsymbol{x}}_{i|i-1} + \boldsymbol{b}u_i + \boldsymbol{c}_i e_i \tag{33}$$

$$y_i = \boldsymbol{d}'\hat{\boldsymbol{x}}_{i|i-1} + e_i \tag{34}$$

where the initial condition is deterministic $\hat{\boldsymbol{x}}_{0|-1} = \boldsymbol{m}_0$, and both the noise covariance and gain are time-varying:

$$\mathrm{E}\{e_i^2\} = \sigma^2(1 + \boldsymbol{d}'\boldsymbol{\Sigma}_i \boldsymbol{d}) \tag{35}$$

$$\boldsymbol{c}_i = \boldsymbol{c} + \boldsymbol{F}\boldsymbol{k}_i^f \tag{36}$$

If the Box-Jenkins model is invertible, i.e. $C(z) = \det(z\boldsymbol{I} - \boldsymbol{F})$ is a stable polynomial, then $\lim_{i\to\infty} \boldsymbol{\Sigma}_i = \boldsymbol{0}$ and, as a result, both the predicted, $\hat{\boldsymbol{x}}_{i|i-1}$, and the filtered,

$\hat{\boldsymbol{x}}_{i|i}$, values of the state vector become equal as time i tends to infinity (Caines, 1972) and are given by

$$\hat{\boldsymbol{x}}_{i+1} = \boldsymbol{F}\hat{\boldsymbol{x}}_i + \boldsymbol{b}u_i + \boldsymbol{c}y_i \tag{37}$$

In (Błachuta, 1996b) it was shown that the filter (37) together with the control law (7) are equivalent with respective predictive algorithms derived in the input-output framework, with the polynomial $T(z^{-1}) = \tilde{C}(z^{-1})$.

5. Chandrasekhar-Type Equations for the Controller

If one's aim is only to find a series of gain vectors $\boldsymbol{k}_i^c$ instead of the matrices $\boldsymbol{P}_i$, $i = 0, 1, \ldots, N-1$, it can be found from a set of vector Chandrasekhar equations.

Theorem 2. *Assume that $N_u = N$ and $\lambda > 0$. Then the controller gain, $\boldsymbol{k}_i^c$, reads as follows:*

$$\boldsymbol{k}_i^c = \frac{\boldsymbol{q}_i}{\lambda_i} \tag{38}$$

where

$$\lambda_i = \lambda_{i+1}(1+\beta_{i+1}^2), \qquad \lambda_N = \lambda + b_0^2 \tag{39}$$

$$\boldsymbol{q}_i = \boldsymbol{q}_{i+1} + \beta_{i+1}\boldsymbol{A}'\boldsymbol{p}_{i+1}, \qquad \boldsymbol{q}_N = b_0\boldsymbol{d}_k \tag{40}$$

$$\boldsymbol{p}_i = \boldsymbol{A}'\boldsymbol{p}_{i+1} - \beta_{i+1}\boldsymbol{q}_{i+1}, \qquad \boldsymbol{p}_N = \sqrt{\lambda}\boldsymbol{d}_k \tag{41}$$

and $\beta_i = (\boldsymbol{b}'\boldsymbol{p}_i)/\lambda_i$. The Riccati matrix $\boldsymbol{P}_i$ can then be calculated from

$$\boldsymbol{P}_i = \boldsymbol{P}_{i+1} + \frac{(\boldsymbol{p}_i\boldsymbol{p}_i')}{\lambda_i}, \qquad \boldsymbol{P}_N = \boldsymbol{d}_{k-1}\boldsymbol{d}_{k-1}' \tag{42}$$

Proof. Let us introduce the differences $\delta\boldsymbol{P}_i = \boldsymbol{P}_{i+1} - \boldsymbol{P}_i$ of the Riccati matrix $\boldsymbol{P}_i$ and set $\boldsymbol{A}_{i+1}^* = \boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_{i+1}^c$. Then λ_i, $\boldsymbol{k}_i^c$ and $\delta\boldsymbol{P}_i$ can be expressed by the differences $\delta\boldsymbol{P}_{i+1}$ as follows:

$$\lambda_i = \lambda_{i+1} - \boldsymbol{b}'\delta\boldsymbol{P}_{i+1}\boldsymbol{b}, \tag{43}$$

$$\boldsymbol{k}_i^c = \boldsymbol{k}_{i+1}^c - \frac{\boldsymbol{A}_{i+1}^{*\prime}\delta\boldsymbol{P}_{i+1}\boldsymbol{b}}{\lambda_i} \tag{44}$$

$$\delta\boldsymbol{P}_i = \boldsymbol{A}_{i+1}^{*\prime}\left[\delta\boldsymbol{P}_{i+1} + \frac{\delta\boldsymbol{P}_{i+1}\boldsymbol{b}\boldsymbol{b}'\delta\boldsymbol{P}_{i+1}}{\lambda_i}\right]\boldsymbol{A}_{i+1}^* \tag{45}$$

for $i = N-1, \ldots, 0$ with the terminal condition for eqn. (45):

$$\delta\boldsymbol{P}_{N-1} = \boldsymbol{P}_N - \boldsymbol{P}_{N-1} = \boldsymbol{d}_k\left(-\frac{\lambda}{\lambda + b_0^2}\right)\boldsymbol{d}_k' \tag{46}$$

From (46) it follows that for all $i = N-1, \ldots, 0$ we have $\delta \boldsymbol{P}_i = \boldsymbol{w}_i(-\phi_i)\boldsymbol{w}_i'$ and

$$\boldsymbol{w}_i \phi_i \boldsymbol{w}_i' = \boldsymbol{A}_{i+1}^{*\prime} \boldsymbol{w}_{i+1} \left[\phi_{i+1} - \frac{(\boldsymbol{w}_{i+1}' \boldsymbol{b})^2}{\lambda_i} \phi_{i+1}^2 \right] \boldsymbol{w}_{i+1}' \boldsymbol{A}_{i+1}^* \tag{47}$$

The matrix equation (47) is then factorized yielding the following system of equations:

$$\boldsymbol{w}_i = \left(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_{i+1}^{c\prime}\right)' \boldsymbol{w}_{i+1} \tag{48}$$

$$\phi_i = \phi_{i+1} - \frac{(\boldsymbol{w}_{i+1}' \boldsymbol{b})^2}{\lambda_i} \phi_{i+1}^2 \tag{49}$$

$$\lambda_i = \lambda_{i+1} + (\boldsymbol{w}_{i+1}' \boldsymbol{b})^2 \phi_{i+1} \tag{50}$$

A transformation of eqn. (44) gives

$$\boldsymbol{k}_i^c = \boldsymbol{k}_{i+1}^c + \frac{(\boldsymbol{A} - \boldsymbol{b}\boldsymbol{k}_{i+1}^{c\prime})' \boldsymbol{w}_{i+1} \phi_{i+1} (\boldsymbol{w}_{i+1}' \boldsymbol{b})}{\lambda_i}. \tag{51}$$

Introduce new variables, $\boldsymbol{q}_i$ and $\boldsymbol{p}_i$, where

$$\boldsymbol{q}_i = \lambda_i \boldsymbol{k}_i^c, \quad \boldsymbol{p}_i = \boldsymbol{w}_i (\phi_i \lambda_i)^{1/2} \tag{52}$$

with the terminal conditions $\boldsymbol{q}_N = b_0 \boldsymbol{d}_k$ and $\boldsymbol{p}_N = \sqrt{\lambda} \boldsymbol{d}_k$. From (51), when expressing $\boldsymbol{k}_i^c$ by $\boldsymbol{k}_i^c = \boldsymbol{q}_i / \lambda_i$, we have

$$\boldsymbol{q}_i = \boldsymbol{q}_{i+1} + \phi_{i+1} (\boldsymbol{w}_{i+1}' \boldsymbol{b}) \boldsymbol{A}' \boldsymbol{w}_{i+1} \tag{53}$$

Finally, expressing $\boldsymbol{w}_{i+1}$ as

$$\boldsymbol{w}_{i+1} = \boldsymbol{p}_{i+1} (\phi_{i+1} \lambda_{i+1})^{-1/2} \tag{54}$$

gives

$$\boldsymbol{q}_i = \boldsymbol{q}_{i+1} + \frac{\boldsymbol{b}' \boldsymbol{p}_{i+1}}{\lambda_{i+1}} \boldsymbol{A}' \boldsymbol{p}_{i+1} \tag{55}$$

$$\lambda_i = \lambda_{i+1} + \frac{\boldsymbol{b}' \boldsymbol{p}_{i+1}}{\lambda_{i+1}} \tag{56}$$

Proceeding in the same way, the first two equations in (50) become

$$\boldsymbol{p}_i = \left(\boldsymbol{A}' \boldsymbol{p}_{i+1} - \frac{\boldsymbol{b}' \boldsymbol{p}_{i+1}}{\lambda_{i+1}} \boldsymbol{q}_{i+1} \right) \left(\frac{\phi_i \lambda_i}{\phi_{i+1} \lambda_{i+1}} \right)^{1/2} \tag{57}$$

$$\phi_i = \phi_{i+1} \left[1 - \frac{(\boldsymbol{b}' \boldsymbol{p}_{i+1})^2}{\lambda_i \lambda_{i+1}} \right] \tag{58}$$

From (58) and (56) it follows, however, that $\phi_i \lambda_i = \phi_{i+1} \lambda_{i+1}$ and

$$\boldsymbol{p}_i = \boldsymbol{A}' \boldsymbol{p}_{i+1} - \frac{\boldsymbol{b}' \boldsymbol{p}_{i+1}}{\lambda_{i+1}} \boldsymbol{q}_{i+1} \tag{59}$$

Finally, from (47) it results that $\delta \boldsymbol{P}_i = -\boldsymbol{p}_i \boldsymbol{p}_i' / \lambda_i$. ∎

6. Chandrasekhar-Type Equations for the Filter

It is now assumed that the process defined by (1)–(2) is stationary, i.e. that the covariance matrix $\sigma^2 \boldsymbol{Q}_0$ is based on a solution to

$$\boldsymbol{Q}_0 = \boldsymbol{A}\boldsymbol{Q}_0\boldsymbol{A}' + \boldsymbol{c}\boldsymbol{c}' \tag{60}$$

In order for such $\boldsymbol{Q}_0 \geq 0$ to exist, the subsystem controllable from v_i must be stable.

Theorem 3. *Assume that* $\boldsymbol{Q}_0 \geq 0$ *fulfils (60). Then the vector* $\boldsymbol{k}_i^f$ *is defined as*

$$\boldsymbol{k}_i^f = \frac{\boldsymbol{h}_i}{r_i} \tag{61}$$

where

$$r_{i+1} = r_i(1-\alpha_i), \qquad r_0 = 1 + \boldsymbol{d}'\boldsymbol{Q}_0\boldsymbol{d} \tag{62}$$

$$\boldsymbol{h}_{i+1} = \boldsymbol{h}_i - \alpha_i \boldsymbol{l}_i, \qquad \boldsymbol{h}_0 = \boldsymbol{Q}_0\boldsymbol{d} \tag{63}$$

$$\boldsymbol{l}_{i+1} = \boldsymbol{F}(\boldsymbol{l}_i - \alpha_i \boldsymbol{h}_i), \quad \boldsymbol{l}_0 = r_0(\boldsymbol{F}\boldsymbol{Q}_0\boldsymbol{d} + \boldsymbol{c}) \tag{64}$$

with $\alpha_i = (\boldsymbol{d}'\boldsymbol{l}_i)/r_i$. *The matrix* $\boldsymbol{\Sigma}_i$ *is given by*

$$\boldsymbol{\Sigma}_{i+1} = \boldsymbol{\Sigma}_i - \frac{(\boldsymbol{l}_i\boldsymbol{l}_i')}{r_i}, \quad \boldsymbol{\Sigma}_0 = \boldsymbol{Q}_0 \tag{65}$$

Proof. If we define $\delta\boldsymbol{\Sigma}_i = \boldsymbol{\Sigma}_{i+1} - \boldsymbol{\Sigma}_i$ and $r_i = 1 + \boldsymbol{d}'\boldsymbol{\Sigma}_i\boldsymbol{d}$, then the following equations hold:

$$r_{i+1} = r_i + \boldsymbol{d}'\delta\boldsymbol{\Sigma}_i\boldsymbol{d} \tag{66}$$

$$\boldsymbol{k}_{i+1}^f = \boldsymbol{k}_i^f + \frac{(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')\delta\boldsymbol{\Sigma}_i\boldsymbol{d}}{r_{i+1}} \tag{67}$$

$$\delta\boldsymbol{\Sigma}_{i+1} = \boldsymbol{F}(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')\left[\delta\boldsymbol{\Sigma}_i - \frac{\delta\boldsymbol{\Sigma}_i\boldsymbol{d}\boldsymbol{d}'\delta\boldsymbol{\Sigma}_i}{r_{i+1}}\right](\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')'\boldsymbol{F}' \tag{68}$$

We also have

$$\delta\boldsymbol{\Sigma}_0 = -r_0(\boldsymbol{F}\boldsymbol{k}_0^f + \boldsymbol{c})(\boldsymbol{F}\boldsymbol{k}_0^f + \boldsymbol{c})' \tag{69}$$

The above formula can be rewritten in the form $\delta\boldsymbol{\Sigma}_0 = \boldsymbol{w}_0\varphi_0\boldsymbol{w}_0'$ with $\boldsymbol{w}_0 = \boldsymbol{F}\boldsymbol{k}_0^f + \boldsymbol{c}$, $\varphi_0 = -r_0$, which leads to the factorization $\delta\boldsymbol{\Sigma}_i = \boldsymbol{w}_i\varphi_i\boldsymbol{w}_i'$. Now, eqn. (68) takes the form

$$\boldsymbol{w}_{i+1}\varphi_{i+1}\boldsymbol{w}_{i+1}' = \boldsymbol{F}(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')\boldsymbol{w}_i\left[\varphi_i - \frac{(\varphi_i\boldsymbol{w}_i'\boldsymbol{d})^2}{r_{i+1}}\right]\boldsymbol{w}_i'(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')'\boldsymbol{F}' \tag{70}$$

which implies

$$\boldsymbol{w}_{i+1} = \boldsymbol{F}(\boldsymbol{I} - \boldsymbol{k}_i^f\boldsymbol{d}')\boldsymbol{w}_i \tag{71}$$

$$\varphi_{i+1} = \varphi_i - \frac{(\varphi_i\boldsymbol{w}_i'\boldsymbol{d})^2}{r_{i+1}} \tag{72}$$

The remaining equations are

$$\Sigma_{i+1} = \Sigma_i + \varphi_i w_i w_i' \tag{73}$$

$$r_{i+1} = r_i + \varphi_i (d' w_i)^2 \tag{74}$$

$$k_{i+1}^f = k_i^f + \frac{(I - k_i^f d')\varphi w_i w_i' d}{r_{i+1}} \tag{75}$$

In the next step, we introduce vectors h_i and l_i as follows:

$$h_i = r_i k_i^f, \quad l_i = w_i(-\varphi_i r_i)^{1/2} \tag{76}$$

We are now able to eliminate the variable φ_i and to transform eqns. (74)–(75) to the form defined by eqns. (62)–(64). Inserting $w_i = l_i(-\varphi_i r_i)^{-1/2}$ to (74) gives

$$r_{i+1} = r_i - \frac{(d' l_i)^2}{r_i} \tag{77}$$

From (75) we have

$$h_{i+1} = h_i + \varphi_i w_i w_i' d \tag{78}$$

Hence from (78) eqn. (63) is obtained. Similarly, from (72) and (74), we have

$$\varphi_{i+1} r_{i+1} = \varphi_i r_i \tag{79}$$

while from (72) and (76) it follows that:

$$l_{i+1} = F\left(I - \frac{1}{r_i} h_i d'\right) l_i \left[\frac{\varphi_{i+1} r_{i+1}}{\varphi_i r_i}\right]^{1/2} \tag{80}$$

As a result, combining (80) and (79) gives (64). ■

7. Conclusion

In this paper, a class of predictive control problems has been solved based on an explicit-delay 'innovations-type' state-space process model and a receding-horizon quadratic performance index. The solution consists of two parts. The first one, which consists in finding the optimal controller gain, can be found as a solution to some LQG problem. The computational complexity of the solution that bases on a Riccati equation depends both on the cost horizon N and the system order n, and not on the control horizon N_u. The other part consists in finding the filtered state variable, which can be accomplished either optimally by using a full Kalman filter (28)–(32) or only asymptotically optimally by using the time invariant filter (37).

It has been shown that the Chandrasekhar equations can improve the computational efficiency in comparison with the Riccati equations, because instead of updating n^2 entries of the Riccati matrix only $2n$ entries of two vectors plus one scalar variable are to be updated. For $n \geq 3$, this reduces the number of calculations. The above savings are particularly important for systems with a large value of the delay/sampling period ratio, and for higher-order step-response models.

Finally, vector Chandrasekhar-type equations have been derived for both the controller and filter gain vectors.

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