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REPRODUCING KERNELS AND RICCATI EQUATIONS

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The purpose of this paper is to exhibit a connection between the Hermitian solutions of matrix Riccati equations and a class of finite dimensional reproducing kernel Krein spaces. This connection is then exploited to obtain minimal factorizations of rational matrix valued functions that are J-unitary on the imaginary axis in a natural way.

Keywords: Riccati equations, Lyapunov equations, reproducing kernel spaces, de Branges spaces, J-inner matrix valued functions, J-unitary matrix valued functions

1. Introduction

The purpose of this article is to present a brief exposition of the role of Riccati equations in the theory of reproducing kernel spaces. In particular, we shall exhibit a connection between positive semidefinite solutions of matrix Riccati equations and a class of finite dimensional reproducing kernel Hilbert spaces of rational vector valued functions, and an analogous (but more general) connection between selfadjoint solutions of matrix Riccati equations and finite dimensional reproducing kernel Krein spaces of rational vector valued functions. The reproducing kernels of the former are expressed in terms of a rational J inner matrix valued function $\Theta(\lambda)$ (see formula (15) below), whereas the reproducing kernels of the latter are expressed in terms of the same formula, but now $\Theta(\lambda)$ is only J unitary on the boundary of the region of interest. A more comprehensive account of parts of this analysis will appear in (Dym, 2001).

The paper is organized as follows: In Sections 2–4 we will review a number of concepts from the theory of reproducing kernel spaces. Much of this analysis is carried out in a general notation that permits one to develop the theory simultaneously for a general region Ω_+ in the complex plane $\mathbb{C}$ that can be taken equal to either the open unit disc $\mathbb{D}$, the open upper half plane $\mathbb{C}_+$, or the open right half plane Π_+. The symbol $\rho_\omega(\lambda)$ is then defined by the rule

$$\rho_\omega(\lambda) = \begin{cases} 1-\lambda\overline{\omega} & \text{if} \quad \Omega_+ = \mathbb{D}, \\ -2\pi i(\lambda-\overline{\omega}) & \text{if} \quad \Omega_+ = \mathbb{C}_+, \\ 2\pi(\lambda+\overline{\omega}) & \text{if} \quad \Omega_+ = \Pi_+. \end{cases} \tag{1}$$

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In Section 5 we specialize in the case when $\Omega_+ = \mathbb{\Pi}_+$ and treat this case only in the rest of the paper; additional information for the case $\Omega_+ = \mathbb{D}$ is furnished in (Dym, 2001). We then show that every rational matrix valued function $\Theta(\lambda)$ that is J unitary on the boundary Ω_0 of Ω_+ corresponds to a finite dimensional reproducing kernel Krein space space $\mathcal{M}$ of vector valued rational functions, and subsequently obtain factorizations of the given $\Theta(\lambda)$ in terms of certain subspaces of $\mathcal{M}$. The innovation here is the use of the solutions of a Riccati equation to identify the relevant reproducing kernel subspaces.

The notation is fairly standard: $\mathbb{C}^{n\times m}$ denotes the set of $n \times m$ matrices, $\mathbb{C}^n$ is an abbreviated form of $\mathbb{C}^{n\times 1}$, the set of $n \times 1$ column vectors, and $\mathbb{R}$ stands for the real numbers. The symbol A^* denotes the adjoint of an operator A on a Hilbert space, with respect to the inner product of the space. If A is a finite matrix, then the adjoint will always be computed with respect to the standard inner product so that, in this case, A^* will be the Hermitian transpose, or just the complex conjugate if A is a number. However, the complex conjugate of a complex number λ will also be designated by $\overline{\lambda}$. The symbol $\sigma(A)$ denotes the spectrum of a matrix A, $\mathcal{R}_A$ denotes the range of A and J is an $m \times m$ signature matrix, i.e.,

$$J = U \begin{bmatrix} I_p & 0 \\ 0 & -I_q \end{bmatrix} U^*,$$

where U is unitary and $p + q = m$. If $F(\lambda)$ is a matrix valued function, then $\mathcal{A}_F$ denotes its domain of analyticity. The following acronyms will be used: mvf = matrix valued function, vvf = vector valued function, RKHS = reproducing kernel Hilbert space, RKKS = reproducing kernel Krein space and RK = reproducing kernel. Also, when clear from the context, an mvf that is J-unitary on Ω_0 will be referred to as a J-unitary mvf.

2. Preliminaries

In this section we shall review a number of definitions and concepts for the convenience of the reader. This will also help us to put the subsequent development into context.

2.1. Linear Independence

The n columns of an $m \times n$ meromorphic mvf $F(\lambda)$ are said to be linearly independent if they are linearly independent in the vector space of continuous $m \times 1$ vector valued functions on the domain of analyticity of F, i.e., if $F(\lambda)$ is meromorphic and $F(\lambda)u = 0$ for some $u \in \mathbb{C}^n$ and all points $\lambda \in \mathcal{A}_F$, then $u = 0$. If

$$F(\lambda) = C(I_n - \lambda A)^{-1} \quad \text{or} \quad F(\lambda) = C(\lambda I_n - A)^{-1},$$

this is easily seen to be equivalent to

$$\bigcap_{j=0}^{n-1} \ker CA^j = 0,$$

i.e., that the pair (C, A) is observable. Such a realization for F is minimal in the sense of Kalman because (in the usual terminology, see, e.g., (Zhou *et al.*, 1996)) the pair (A, B) is automatically controllable, since $B = I_n$.

2.2. Reproducing Kernel Spaces

A Hilbert space $\mathcal{H}$ of $m \times 1$ vector valued functions that are defined on some subset Δ of $\mathbb{C}$ is said to be an RKHS (reproducing kernel Hilbert space) if there exists an $m \times m$ mvf $K_\omega(\lambda)$ on $\Delta \times \Delta$ such that for every choice of $\omega \in \Delta$, $u \in \mathbb{C}^m$ and $f \in \mathcal{H}$ we have

$$K_\omega u \in \mathcal{H} \quad \text{(as a function of } \lambda) \tag{2}$$

and

$$\langle f, K_\omega u \rangle_{\mathcal{H}} = u^* f(\omega). \tag{3}$$

The RK (reproducing kernel) is unique, i.e., if $K_\omega(\lambda)$ and $L_\omega(\lambda)$ are both RK's for the same RKHS, then $K_\omega(\lambda) = L_\omega(\lambda)$ for every choice of ω and λ in Δ. Moreover,

$$K_\alpha(\beta)^* = K_\beta(\alpha) \tag{4}$$

and

$$\sum_{i,j=1}^{n} u_j^* K_i(\omega_j) u_i \geq 0 \tag{5}$$

for every choice of $\omega_1, \ldots, \omega_n$ in Δ and $u_1, \ldots, u_n$ in $\mathbb{C}^m$.

Example 1. The Hardy space $H_2^m(\Omega_+)$ is an RKHS with RK

$$K_\omega(\lambda) = I_m / \rho_\omega(\lambda) \tag{6}$$

for each of the classical choices of Ω_+, where $\rho_\omega(\lambda)$ is specified in (1). The verification of (3) is just Cauchy's theorem for $H_2(\Omega_+)$. ♦

Example 2. Let

$$\mathcal{M} = \big\{ F(\lambda) u : u \in \mathbb{C}^n \big\}, \tag{7}$$

where $F(\lambda)$ is an $m \times n$ mvf that is meromorphic in some open nonempty subset Δ of $\mathbb{C}$ and has n linearly independent columns $f_1(\lambda), \ldots, f_n(\lambda)$ (in the sense of Subsection 2.1), and let P be any $n \times n$ positive definite matrix (i.e., $P > 0$). Then the space $\mathcal{M}$ endowed with the inner product

$$\big\langle F(\lambda) u, F(\lambda) v \big\rangle_{\mathcal{M}} = v^* P u \tag{8}$$

for every choice of u and v in $\mathbb{C}^n$, is an n dimensional RKHS with RK

$$K_\omega(\lambda) = F(\lambda) P^{-1} F(\omega)^* \tag{9}$$

(at the points of analyticity). The verification is by direct computation. ♦

2.3. Reproducing Kernel Krein Spaces

Formulas (8) and (9) in Example 2 remain valid if the matrix P is Hermitian and invertible rather than positive definite. In this case, the space $\mathcal{M}$ is a reproducing kernel Krein space (RKKS) with respect to the indefinite inner product (8). That is to say, the space $\mathcal{M}$ admits a direct sum decomposition

$$\mathcal{M} = \mathcal{M}_+ + \mathcal{M}_- \quad \text{with} \quad \mathcal{M}_+ \cap \mathcal{M}_- = \{0\}$$

such that:

1. $\mathcal{M}_+$ is a Hilbert space with respect to the indefinite inner product (8),
2. $\mathcal{M}_-$ is a Hilbert space with respect to the negative of the indefinite inner product (8), and
3. $\mathcal{M}_+$ is orthogonal to $\mathcal{M}_-$ with respect to the indefinite inner product (8).

This is easily verified by setting

$$\mathcal{M}_\pm = \left\{F(\lambda)\Pi_\pm u : u \in \mathbb{C}^n\right\},$$

where $\Pi_\pm$ denotes the orthogonal projection of $\mathbb{C}^n$ onto the span of the eigenvectors of P corresponding to the eigenvalues that fall in the interval between 0 and $\pm\infty$.

For ease of future reference, we shall summarize this more general setting in the next example.

Example 3. Let $\mathcal{M}$ be the space defined in Example 2 endowed with the indefinite inner product (8) that is defined in terms of an invertible Hermitian matrix P. Then $\mathcal{M}$ is an n dimensional RKKS with RK given by (9). ♦

2.4. R_α Invariance

A major role in this subject is played by the generalized backwards shift operator R_α that acts on matrix valued meromorphic functions by the rule

$$R_\alpha F(\lambda) = \frac{F(\lambda) - F(\alpha)}{\lambda - \alpha} \tag{10}$$

for every point $\alpha \in \mathcal{A}_F$. In the next subsection we shall consider finite dimensional spaces of vector valued functions that are invariant under the action of R_α for at least one appropriately chosen point $\alpha \in \mathbb{C}$.

2.5. Some Implications of R_α Invariance

The results reported on in this subsection are taken largely from Section 3 of (Dym, 1994) and Section 4 of (Dym, 1998), where the proofs of the following statements can be found.

Theorem 1. *Let $\mathcal{M}$ be an n dimensional vector space of $m \times 1$ vvf's which are meromorphic in some open nonempty set $\Delta \subset \mathbb{C}$, and suppose further that $\mathcal{M}$ is R_α invariant for some point $\alpha \in \Delta$ in the domain of analyticity of $\mathcal{M}$. Then $\mathcal{M}$ is spanned by the columns of a rational $m \times n$ matrix valued function of the form*

$$F(\lambda) = V\{M - \lambda N\}^{-1}, \tag{11}$$

where $V \in \mathbb{C}^{m\times n}$, $M, N \in \mathbb{C}^{n\times n}$,

$$MN = NM \quad \text{and} \quad M - \alpha N = I_n. \tag{12}$$

Moreover, $\lambda \in \Delta$ is a point of analyticity of F if and only if the $n \times n$ matrix $M - \lambda N$ is invertible.

Corollary 1. *If $\det(M - \lambda N) \not\equiv 0$ and $F(\lambda) = V(M - \lambda N)^{-1}$ is a rational $m \times n$ matrix valued function with n linearly independent columns, then:*

(i) M is invertible if and only if F is analytic at zero.

(ii) N is invertible if and only if F is analytic at infinity and $F(\infty) = 0$.

Moreover, in case (i) F can be expressed in the form

$$F(\lambda) = C(I_n - \lambda A)^{-1} \tag{13}$$

whereas in case (ii) it takes the form

$$F(\lambda) = C(\lambda I_n - A)^{-1}. \tag{14}$$

Corollary 2. *Let f be an $m \times 1$ vector valued function which is meromorphic in some open nonempty set $\Delta \subset \mathbb{C}$, and let $\alpha \in \Delta$ be a point of analyticity of f. Then f is an eigenfunction of R_α if and only if it can be expressed in the form*

$$f(\lambda) = \frac{v}{\rho_\omega(\lambda)}$$

for one or more choices of $\rho_\omega(\lambda)$ in (1) with $\rho_\omega(\alpha) \neq 0$ and some nonzero constant vector $v \in \mathbb{C}^m$.

3. A Special Class of Reproducing Kernel Spaces

We shall be particularly interested in RKKS's of $m \times 1$ vector valued meromorphic functions in $\mathbb{C}$ with RK's of a special form, which will be described below in the statement of Theorem 2. The theorem is an elaboration of a fundamental result that is due to de Branges (1963). For the sake of definiteness it is formulated with respect to the right half plane. For a more comprehensive statement, see Theorem 2.3 of (Dym, 2001). The latter is a special case of the analysis in (Alpay and Dym, 1993a).

Theorem 2. *Let $\mathcal{K}$ be an RKKS of $m \times 1$ vector valued functions that are analytic in an open subset Δ of $\mathbb{C}$ that is symmetric with respect to $\Omega_0 = i\mathbb{R}$, and assume* *that* $\Delta \cap \Omega_0 \neq \emptyset$*. Then the reproducing kernel* $K_\omega(\lambda)$ *of* $\mathcal{K}$ *can be expressed in the form*

$$K_\omega(\lambda) = \frac{J - \Theta(\lambda)J\Theta(\omega)^*}{\rho_\omega(\lambda)}, \tag{15}$$

for some choice of the $m \times m$ *mvf* $\Theta(\lambda)$ *which is analytic in* Δ *and* $\rho_\omega(\lambda)$ *as in (1) for* $\Omega_+ = \Pi_+$*, if and only if the following two conditions hold:*

(i) $\mathcal{K}$ *is* R_α *invariant for every* $\alpha \in \Delta$.

(ii) *The structural identity*

$$\langle R_\alpha f, g\rangle_{\mathcal{K}} + \langle f, R_\beta g\rangle_{\mathcal{K}} + (\alpha + \beta^*)\langle R_\alpha f, R_\beta g\rangle_{\mathcal{K}} = -2\pi g(\beta)^* J f(\alpha) \tag{16}$$

holds for every choice of α, β *in* Δ *and* f, g *in* $\mathcal{K}$.

Moreover, in this case, the function $\Theta(\lambda)$ *that appears in (29) is unique up to a* J *unitary constant factor on the right hand side; it can be taken equal to*

$$\Theta(\lambda) = I_m - \rho_\mu(\lambda)K_\mu(\lambda)J \tag{17}$$

for any point $\mu \in \Delta \cap \Omega_0$.

This formulation is adapted from (Alpay and Dym, 1993a); see especially Theorems 4.1, 4.2, and 4.3. Some simplifications are possible when the region Ω_+ is restricted to be one of the three classical regions, because then the terms $r(a, b; \alpha)f$ and $r(b, a; \alpha)f$ that appear in the formulation in (Alpay and Dym, 1993a) can be reexpressed as constant multiples of $R_\alpha(af)$ and $R_\alpha(bf)$, respectively; see Section 8 of (Dym, 1998), or Theorem 2.3 of (Dym, 2001).

The restriction $\Delta \cap \Omega_0 = \emptyset$ can be relaxed at the expense of a more sophisticated formulation. However, since we shall be dealing with finite dimensional spaces and rational functions, there is no need for this extra complication. The interested reader can refer to (Alpay and Dym, 1993a) for more information.

For the other two classical choices of Ω_+, the structural identity (16) is replaced by

$$\big\langle (I + \alpha R_\alpha)f, (I + \beta R_\beta)g\big\rangle_{\mathcal{K}} - \langle R_\alpha f, R_\beta g\rangle_{\mathcal{K}} = g(\beta)^* J f(\alpha) \tag{18}$$

if $\Omega_+ = \mathbb{D}$ and

$$\langle R_\alpha f, g\rangle_{\mathcal{K}} - \langle f, R_\beta g\rangle_{\mathcal{K}} - (\alpha - \beta^*)\langle R_\alpha f, R_\beta g\rangle_{\mathcal{K}} = 2\pi i g(\beta)^* J f(\alpha) \tag{19}$$

if $\Omega_+ = \mathbb{C}_+$. Then Ω_0 is taken to be equal to the boundary of Ω_+ and $\rho_\omega(\lambda)$ is selected from formula (1).

Formula (19) appears in (de Branges, 1963); formula (18) is equivalent to a formula which appears in (Ball, 1975), where de Branges' work was adapted to the disc and an important technical improvement due to Rovnyak (1968) was incorporated. All the three references deal with the Hilbert space case only.

From time to time we shall refer to an RKKS with an RK of the form (15) as a dBK space $\mathcal{K}(\Theta)$, and to an RKHS with an RK of this form as a de Branges space $\mathcal{H}(\Theta)$.

4. An Important Conclusion

The role of the two conditions in Theorem 2 becomes particularly transparent when $\mathcal{K}$ is finite dimensional. Indeed, if the n dimensional space $\mathcal{M}$ considered in Example 3 is R_α invariant for some point α in the domain of analyticity of $F(\lambda)$, then, by Theorem 1, $F(\lambda)$ can be expressed in the form

$$F(\lambda) = V(M - \lambda N)^{-1} \tag{20}$$

with M and N satisfying (12). Thus R_α invariance forces the elements of $\mathcal{M}$ to be rational of the indicated form. Since

$$(R_\beta F)(\lambda) = F(\lambda)N(M - \beta N)^{-1}$$

for every point β at which the matrix $M - \beta N$ is invertible, i.e., for every $\beta \in \mathcal{A}_F$, the domain of analyticity of F, it is readily checked that

$$\begin{aligned} \langle R_\alpha Fu, Fv\rangle_{\mathcal{M}} &= \left\langle FN(M-\alpha N)^{-1}u, Fv\right\rangle_{\mathcal{M}} \\ &= v^* PN(M - \alpha N)^{-1}u, \end{aligned} \tag{21}$$

and similarly that

$$\langle Fu, R_\beta Fv\rangle_{\mathcal{M}} = v^*(M^* - \beta^* N^*)^{-1}N^*Pu, \tag{22}$$

and

$$\langle R_\alpha Fu, R_\beta v\rangle_{\mathcal{M}} = v^*(M^* - \beta^* N^*)^{-1}N^*PN(M - \alpha N)^{-1}u \tag{23}$$

for every choice of α, β in $\mathcal{A}_F$ and u, v in $\mathbb{C}^n$. It is now easily seen that, for each of the three special choices of Ω_+ under consideration, the associated structural identity reduces to a matrix equation for P by working out (16), (18) and (19) with the aid of (21)–(23). In other words:

In a finite dimensional R_α invariant RKKS $\mathcal{M}$ *with Gram matrix P, each of the structural identities (16), (18) and (19) is equivalent to a Lyapunov-Stein equation for P.*

This last conclusion seems to have been first established explicitly in (Dym, 1989b) by a considerably lengthier calculation. The present, more appealing argument is adapted from (Dym, 1994; Dym, 1998).

If F is analytic at zero, then we may presume that $M = I_n$ in (20) and take $\alpha = \beta = 0$ in the structural identity (18).

Theorem 3. *Let $\mathcal{M}$ denote the finite dimensional RKHS that was introduced in Example 3 and let $F(\lambda)$ be given by (20). Then the RK of $\mathcal{M}$ can be expressed in the form*

$$K_\omega(\lambda) = \frac{J - \Theta(\lambda)J\Theta(\omega)^*}{\rho_\omega(\lambda)} \tag{24}$$

with $\rho_\omega(\lambda)$ *as in (1) if and only if* P *is a solution of the equation*

$$M^*PM - N^*PN = V^*JV \qquad when \quad \Omega_+ = \mathbb{D}, \tag{25}$$

$$M^*PN - N^*PM = 2\pi i V^*JV \qquad when \quad \Omega_+ = \mathbb{C}_+, \tag{26}$$

$$M^*PN + N^*PM = -2\pi V^*JV \qquad when \quad \Omega_+ = \Pi_+. \tag{27}$$

Moreover, in each of these cases $\Theta(\lambda)$ *is uniquely specified up to a* J *unitary constant multiplier on the right hand side by the formula*

$$\Theta(\lambda) = I_m - \rho_\mu(\lambda)F(\lambda)P^{-1}F(\mu)^*J \tag{28}$$

for any choice of the point $\mu \in \Omega_0 \cap \mathcal{A}_F$.

Note that (28) is a realization formula for $\Theta(\lambda)$, and that in the usual notation of (13) and (14) it depends only upon A, C and P. It can be reexpressed in one of the standard A, B, C, D forms by elementary manipulations. A very general class of realization formulas of the form (28) and extensions thereof can be found in (Alpay and Dym, 1996).

5. Specialization of the Setting

From now on we shall restrict our attention to the case where the fundamental region of interest is the right half plane. Accordingly, we shall set

$$\mathcal{M} = \{F(\lambda)u : u \in \mathbb{C}^n\}, \tag{29}$$

where

$$F(\lambda) = C(\lambda I_n - A)^{-1}, \tag{30}$$

(C, A) is an observable pair and the space $\mathcal{M}$ is endowed with the indefinite inner product

$$\langle F(\lambda)u, F(\lambda)v\rangle_{\mathcal{M}} = v^*Pu \tag{31}$$

for every choice of u and v in $\mathbb{C}^n$, and P is an $n \times n$ invertible Hermitian matrix solution of the Lyapunov equation

$$A^*P + PA = -2\pi C^*JC. \tag{32}$$

Thus, in view of the preceding discussion, $\mathcal{M}$ is a dBK space based on the $m \times m$ mvf

$$\Theta(\lambda) = I_m - 2\pi C(\lambda I_n - A)^{-1}P^{-1}C^*J. \tag{33}$$

Now, let $\widetilde{\mathcal{M}}$ be a k dimensional subspace of the vector space $\mathcal{M}$. Then there exists an $m \times k$ matrix B such that

$$\widetilde{\mathcal{M}} = \{F(\lambda)Bu : u \in \mathbb{C}^k\}.$$

However, since $\mathcal{R}_B = \mathcal{R}_{BB^*}$, we may assume that

$$\widetilde{\mathcal{M}} = \{F(\lambda)Xu : u \in \mathbb{C}^n\},$$

for some $n \times n$ Hermitian matrix X. There are clearly many choices of X that generate the same vector space $\widetilde{\mathcal{M}}$, some of which are positive semidefinite. Our next objective is to characterize those X for which $\widetilde{\mathcal{M}}$ is a dBK space that is isometrically embedded into $\mathcal{M}$. This will involve three conditions that are not independent of one another, i.e., (38), (42) and (44).

6. The Spaces Underlying the Riccati Equation

Let

$$\mathcal{M}_X = \{F(\lambda)Xu : u \in \mathbb{C}^n\}, \tag{34}$$

denote the vector space that is constructed from the columns of the $m \times n$ mvf

$$F(\lambda) = C(\lambda I_n - A)^{-1}, \tag{35}$$

where $C \in \mathbb{C}^{m\times n}$, $A, X \in \mathbb{C}^{n\times n}$, X is Hermitian and it is assumed throughout that the pair (C, A) is observable.

The next result is a special case of Theorem 2.1 of (Dym, 2001). We repeat the proof for the convenience of the reader, because it is central to what follows.

Theorem 4. *Let the pair (C, A) be observable and let $X \in \mathbb{C}^{n\times n}$ be a nonzero Hermitian matrix. Then the linear space $\mathcal{M}_X$ defined by (34) and (35) is an RKKS with respect to the indefinite inner product*

$$\langle FXu, FXv\rangle_{\mathcal{M}_X} = v^*Xu. \tag{36}$$

The RK $K_\omega(\lambda)$ of this RKKS is given by the formula

$$K_\omega(\lambda) = F(\lambda)XF(\omega)^*. \tag{37}$$

Moreover, $\mathcal{M}_X$ is included isometrically inside $\mathcal{M}$ if and only if

$$XPX = X. \tag{38}$$

Proof. The first order of business is to check that the indicated inner product is well defined. But if

$$FXu_1 = FXu_2 \quad \text{and} \quad FXv_1 = FXv_2$$

for some choice of u_1, u_2, v_1, v_2 in $\mathbb{C}^n$, then the presumed observability guarantees that

$$Xu_1 = Xu_2 \quad \text{and} \quad Xv_1 = Xv_2.$$

Thus,

$$\langle FXu_1, FXv_1\rangle_{\mathcal{M}_X} = \langle FXu_2, FXv_2\rangle_{\mathcal{M}_X}$$

as needed.

Next, setting

$$\mathcal{R}_X^+ = \text{span}\{\text{eigenvectors of } X \text{ with positive eigenvalues}\},$$

$$\mathcal{R}_X^- = \text{span}\{\text{eigenvectors of } X \text{ with negative eigenvalues}\}$$

and

$$\mathcal{M}_X^\pm = \{F(\lambda)Xu : u \in \mathcal{R}_X^\pm\},$$

it is easily verified that:

1. $\mathcal{M}_X = \mathcal{M}_X^+ + \mathcal{M}_X^-$.
2. $\mathcal{M}_X^+ \cap \mathcal{M}_X^- = \{0\}$.
3. $\mathcal{M}_X^+$ is a Hilbert space with respect to the inner product defined by the restriction of (36) to u, v in $\mathcal{R}_X^+$.
4. $\mathcal{M}_X^-$ is a Hilbert space with respect to the inner product defined by the negative of the restriction of (36) to u, v in $\mathcal{R}_X^-$.
5. $\mathcal{M}_X^+$ is orthogonal to $\mathcal{M}_X^-$ with respect to the indefinite inner product (36).

Thus, $\mathcal{M}_X$ is a Krein space. Next, it is readily checked that if $K_\omega(\lambda)$ is defined by (37), then the conditions (2) and (3) are met. Therefore, $\mathcal{M}_X$ is an RKKS and its RK is given by (37). Finally, the asserted condition for isometric inclusion is follows immediately from (8) and (36). ∎

In the sequel, we shall be particularly interested in the case when the RK $K_\omega(\lambda)$ of the RKKS $\mathcal{M}_X$ can be expressed in the form (15). If such a representation for $K_\omega(\lambda)$ exists, then the mvf $\Theta(\lambda)$ which appears in (15) is unique up to a right J-unitary constant factor. It is a rational $m \times m$ mvf, and both $K_\omega(\lambda)$ and $\Theta(\lambda)$ depend upon X. We do not always indicate this dependence in the notation in order to keep the typography simple. However, it should be noted that:

1. The number of negative squares of the kernel $K_\omega(\lambda)$ is equal to the number of negative eigenvalues of X, counting multiplicities.
2. The mvf $\Theta(\lambda)$ in (15) will be J-inner if and only if $X \geq 0$.
3. The McMillan degree of $\Theta(\lambda)$ is equal to the rank of X.

Thus, we see that if the RK $K_\omega(\lambda)$ of the RKKS $\mathcal{M}_X$ can be expressed in the form (15), then:

$\mathcal{M}_X$ is a de Branges space $\mathcal{H}(\Theta)$ if $X \geq 0$.

$\mathcal{M}_X$ is a dBK space $\mathcal{K}(\Theta)$ if X is only Hermitian.

7. R_α Invariance Again

In this section we characterize those Hermitian matrices X for which the RKKS's $\mathcal{M}_X$ are R_α invariant for any one point α (and hence, in fact, every point α) at which $G(\alpha) = \alpha I_n - A$ is invertible.

Let $X^\dagger$ denote the Moore-Penrose inverse of X. Then, since X is Hermitian and hence admits a representation of the form

$$X = U \begin{bmatrix} X_1 & 0 \\ 0 & 0 \end{bmatrix} U^* \tag{39}$$

with U unitary and X_1 both Hermitian and invertible, it follows that

$$X^\dagger = U \begin{bmatrix} X_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{40}$$

Thus, $X^\dagger$ commutes with X and

$$X^\dagger X = XX^\dagger \tag{41}$$

is an orthogonal projection: $XX^\dagger = \Pi_X$, the orthogonal projection of $\mathbb{C}^n$ onto $\mathcal{R}_X$.

Lemma 1. *Assume that $G(\alpha) = \alpha I_n - A$ is invertible. Then the space $\mathcal{M}_X$ is R_α invariant if and only if*

$$AX = XX^\dagger AX, \tag{42}$$

i.e., if and only if $\mathcal{R}_X$ is invariant under A.

Proof. By direct calculation, we have

$$(R_\alpha F)(\lambda)X = -F(\lambda)G(\alpha)^{-1}X.$$

Therefore, $\mathcal{M}_X$ will be R_α invariant if and only if there exists a matrix $Q_\alpha \in \mathbb{C}^{n\times n}$ such that

$$-G(\alpha)^{-1}X = XQ_\alpha.$$

If Q_α is invertible, then this is the same as to require

$$AX = X(I_n + \alpha Q_\alpha)(Q_\alpha)^{-1}. \tag{43}$$

We thus get

$$XX^\dagger AX = XX^\dagger X(I_n + \alpha Q_\alpha)(Q_\alpha)^{-1} = X(I_n + \alpha Q_\alpha)(Q_\alpha)^{-1} = AX.$$

This yields the desired result when Q_α is invertible. The same conclusion holds even if Q_α is not invertible, but the proof is more elaborate; see Lemma 3.1 of (Dym, 2001).

The proof of the sufficiency of (42) for R_α invariance is a straightforward calculation that is left for the reader. ∎

Remark 1. At first glance the conclusions of the last lemma seem to contradict the conditions for R_α invariance that were established in Theorem 1. Notice, however, that if A is subject to (42), then

$$C(\lambda I_n - A)^{-1}X = CX(\lambda I_n - X^\dagger AX)^{-1},$$

which is of the requisite form.

8. The Riccati Equation

Theorem 5. *Let $F(\lambda)$ be given by (30) with (C, A) observable. Then the RKKS $\mathcal{M}_X$ is a dBK space $\mathcal{K}(\Theta)$ if and only if the Hermitian matrix X is a solution of the Riccati equation*

$$AX + XA^* = -2\pi XC^*JCX. \tag{44}$$

Moreover, in this case the mvf $\Theta(\lambda) = \Theta_X(\lambda)$ is uniquely determined by the formula

$$\Theta_X(\lambda) = I_m - 2\pi C(\lambda I_n - A)^{-1}XC^*J \tag{45}$$

up to a J-unitary constant multiplier on the right, and the following identity holds:

$$\frac{J - \Theta_X(\lambda)J\Theta_X(\omega)^*}{\rho_\omega(\lambda)} = F(\lambda)XF(\omega)^*. \tag{46}$$

Thus, the mvf $\Theta_X(\lambda)$ is J-inner iff $X \geq 0$; it can be expressed in terms of

$$\widehat{A} = -(A^* + 2\pi C^*JCX) \tag{47}$$

as

$$\Theta_X(\lambda) = I_m - 2\pi CX(\lambda I_n - \widehat{A})^{-1}C^*J. \tag{48}$$

Proof. Suppose first that $\mathcal{M}_X$ is a dBK space. Then, in view of Theorem 2 and Lemma 1,

$$AX = XX^\dagger AX \tag{49}$$

and the structural identity (16) holds.

Let A_1 be any matrix in $\mathbb{C}^{n\times n}$ which meets the equality

$$AX = XA_1. \tag{50}$$

(The existence of at least one such matrix is guaranteed by (49).) Then

$$(\alpha I_n - A)X = X(\alpha I_n - A_1)$$

and thus,

$$(\alpha I_n - A)^{-1}X = X(\alpha I_n - A_1)^{-1}$$

for every point $\alpha \in \mathbb{C}$ for which the two inverses exist. Let

$$f(\lambda) = F(\lambda)Xu \quad \text{and} \quad g(\lambda) = F(\lambda)Xv$$

for any choice of u, v in $\mathbb{C}^n$, and suppose that $\alpha, \beta \notin \sigma(A) \cup \sigma(A_1)$. Then it is readily checked that

$$\begin{aligned}
(R_\alpha f)(\lambda) &= -F(\lambda)(\alpha I_n - A)^{-1}Xu = -F(\lambda)X(\alpha I_n - A_1)^{-1}u, \\
(R_\beta g)(\lambda) &= -F(\lambda)(\beta I_n - A)^{-1}Xv = -F(\lambda)X(\beta I_n - A_1)^{-1}v, \\
f(\alpha) &= C(\alpha I_n - A)^{-1}Xu = CX(\alpha I_n - A_1)^{-1}u, \\
g(\beta) &= C(\beta I_n - A)^{-1}Xv = CX(\beta I_n - A_1)^{-1}v.
\end{aligned}$$

Next, upon substituting these formulas into the structural identity (16) and invoking the inner product rule (36), we see that

$$\begin{aligned}
v^*\Big\{X(\alpha I_n - A_1)^{-1} + (\overline{\beta} I_n - A_1^*)^{-1}X - (\alpha + \overline{\beta})(\overline{\beta} I_n - A_1^*)^{-1}X(\alpha I_n - A_1)^{-1}\Big\}u \\
= 2\pi v^*\Big\{(\overline{\beta} I_n - A_1^*)^{-1}XC^*JCX(\alpha I_n - A_1)^{-1}\Big\}u.
\end{aligned}$$

However, this last equality holds for every choice of $u, v \in \mathbb{C}^n$ if and only if

$$(\overline{\beta} I_n - A_1^*)X + X(\alpha I_n - A_1) - (\alpha + \overline{\beta})X = 2\pi XBJCX,$$

that is, if and only if

$$-A_1^*X - XA_1 = 2\pi XC^*JCX.$$

But, in view of (50), this last identity implies that X is a solution of the Riccati equation (44) and thus serves to complete the proof of the assertion that if $\mathcal{M}_X$ is a dBK space, then X is a solution of (44).

Conversely, if X is a solution of (44), then it follows easily that (49) and the structural identity (16) hold and therefore, by Lemma 1 and Theorem 2, that $\mathcal{M}_X$ is a dBK space. Formula (45) for $\Theta(\lambda) = \Theta_X(\lambda)$ is obtained by letting $\mu \longrightarrow \infty$ along the imaginary axis in the general formula (17). The fact that $\Theta_X(\lambda)$ is J-inner if and only if $X \geq 0$ follows from (46). One direction is easy. The other exploits the fact that if (C, A) is an observable pair, then there exist a set of points $\omega_1, \ldots, \omega_n$ in the domain of analyticity of $F(\lambda)$ and a set of vectors $u_1, \ldots, u_n$ in $\mathbb{C}^m$ such that the $n \times n$ matrix

$$Y = \left[F(\omega_1)^*u_1 \cdots F(\omega_n)^*u_n\right]$$

is invertible. Now, if $\Theta_X(\lambda)$ is J inner, then the $n \times n$ matrix

$$Y^*XY = \left[u_i^*K_{\omega_j}(\omega_i)u_j\right]$$

is positive semidefinite. Therefore $X \geq 0$, since Y is invertible.

Finally, when X is a solution of the Riccati equation (44), we may define $\widehat{A}$ by (47) and verify that

$$(\lambda I_n - A)^{-1}X = X(\lambda I_n - \widehat{A})^{-1}$$

for all points $\lambda \notin \sigma(A) \cup \sigma(\widehat{A})$. This leads easily to (48). ∎

9. Factorization

Let X and Y be nonzero Hermitian matrices such that

$$XPX = X \quad \text{and} \quad YPY = Y. \tag{51}$$

Then $\mathcal{M}_X$ and $\mathcal{M}_Y$ are both included 'isometrically' inside the finite dimensional Krein space $\mathcal{M}$. Moreover, these two spaces will be orthogonal inside $\mathcal{M}$ iff

$$YPX = XPY = 0. \tag{52}$$

In this case,

$$(X+Y)P(X+Y) = X+Y. \tag{53}$$

Moreover, if these two spaces are also complementary, then $X+Y$ is invertible and hence

$$(X+Y)P = I_n. \tag{54}$$

In general, $\mathcal{M}_Y$ is not R_α invariant and hence cannot be a dBK space. However, it turns out that $\Theta_X^{-1}\mathcal{M}_Y$ is R_α invariant; see e.g., (Alpay and Dym, 1986; Alpay and Dym, 1993b). Indeed, by a direct calculation that exploits the Lyapunov equation (32), we get

$$\Theta_X(\lambda)^{-1}C(\lambda I_n - A)^{-1}Y = C(I_n - XP)(\lambda I_n - A)^{-1}Y + CX(\lambda I_n + A^*)^{-1}PY.$$

But the second term on the right hand side is equal to zero, since

$$X(\lambda I_n + A^*)^{-1}PY = (\lambda I_n + \widehat{A}^*)^{-1}XPY = 0.$$

Thus, we are left with the formula

$$\Theta_X(\lambda)^{-1}C(\lambda I_n - A)^{-1}Y = CYP(\lambda I_n - A)^{-1}Y \tag{55}$$

which can also be reexpressed as

$$\Theta_X(\lambda)^{-1}C(\lambda I_n - A)^{-1}Y = CYP(\lambda I_n - A)^{-1}P^{-1}, \tag{56}$$

since $P^{-1} = X+Y$. This last form serves to clarify the asserted R_α invariance of the space $\Theta_X^{-1}\mathcal{M}_Y$.

A similar set of calculations leads to the supplementary identity

$$\Theta_X(\lambda)^{-1}\Theta(\lambda) = I_m - 2\pi CYP(\lambda I_n - A)^{-1}P^{-1}C^*J, \tag{57}$$

or equivalently,

$$\Theta_X(\lambda)^{-1}\Theta(\lambda) = I_m - 2\pi CYP(\lambda I_n - A)^{-1}YC^*J, \tag{58}$$

just as before.

Thus, as

$$\Theta(\lambda) = \Theta_X(\lambda)\cdot(\Theta_X^{-1}\Theta)(\lambda), \tag{59}$$

where the first factor has McMillan degree equal to the rank of X and the second factor has McMillan degree equal to the rank of Y, we are led to the following conclusion:

Theorem 6. *In the setting of Section 5, let X be a nonzero Hermitian solution of the Riccati equation (44) such that $XPX = X$ and let*

$$Y = P^{-1} - X.$$

Then the mvf $\Theta(\lambda)$ defined by (33) admits the factorization

$$\Theta(\lambda) = \left\{I_m - 2\pi C(\lambda I_n - A)^{-1} XC^* J\right\}\left\{I_m - 2\pi CYP(\lambda I_n - A)^{-1} YC^* J\right\}. \quad (60)$$

This factorization is minimal. Moreover, the first term on the right hand side is J-inner if and only if $X \geq 0$.

Remark 2. In the setting of Section 5, every Hermitian solution X of the Riccati equation (44) is also a solution of the equation

$$(XPX - X)X^\dagger AX + XA^* X^\dagger (XPX - X) = 0.$$

Therefore, the condition $XPX = X$ is automatically met provided that $\sigma(X^\dagger AX) \cap \sigma(-XA^* X^\dagger) = \emptyset$. However, since $\sigma(X^\dagger AX) \subset \sigma(A)$ and $\sigma(-XA^* X^\dagger) \subset \sigma(-A^*)$, it is enough to have $\sigma(A) \cap \sigma(-A^*) = \emptyset$.

10. Conclusions and Comparisons

The basic facts underlying the preceding calculations can be summarized as follows:

Theorem 7. *Let $X \in \mathbb{C}^{n\times n}$ be a nonzero Hermitian matrix and suppose that $\sigma(A) \cap \sigma(-A^*) = \emptyset$. Then, in the setting of Section 5, the following statements are equivalent:*

- *(i) $\mathcal{M}_X$ is an R_α invariant subspace of $\mathcal{M}$ (with isometric inclusion, i.e., $XPX = X$),*
- *(ii) $AX = XX^\dagger AX$ and $XPX = X$,*
- *(iii) $A\Pi_X = \Pi_X A \Pi_X$ and $XPX = X$,*
- *(iv) X is a solution of the Riccati equation (44).*

If any one (and hence all) of the preceding four conditions are met, then the mapping from X to $\mathcal{M}_X$ is one-to-one and $\Theta_X(\lambda) = I_m - 2\pi C(\lambda I_n - A)^{-1} XC^* J$ is a left J-unitary divisor of $\Theta(\lambda)$ such that the factorization (59) is minimal. Every minimal factorization is obtained in this way.

Proof. (i) implies (ii) by Theorem 4 and Lemma 1. To obtain (iv) from (ii), multiply the Lyapunov equation (32) by X and then invoke the two conditions in (ii) to get

$$\begin{aligned} -2\pi XC^* JCX &= XPAX + XA^* PX = XPXX^\dagger AX + XA^* X^\dagger XPX \\ &= XX^\dagger AX + XA^* X^\dagger X = AX + XA^*. \end{aligned}$$

Next, if (iv) holds, then $\mathcal{M}_X$ is a dBK space that is included isometrically inside $\mathcal{M}$, owing to Theorem 5 and Remark 2. Thus, (iv) implies (i). The equivalence of (ii) and (iii) is easy.

Suppose now that $X_1 \in \mathbb{C}^{n\times n}$ and $X_2 \in \mathbb{C}^{n\times n}$ are any two nonzero Hermitian matrices that meet any one (and hence all) of the conditions (i)–(iv), and that $\mathcal{M}_{X_1} = \mathcal{M}_{X_2}$. Then $\mathcal{K}(\Theta_{X_1}) = \mathcal{K}(\Theta_{X_2})$ and hence, as $\Theta_{X_1}(\lambda) = \Theta_{X_2}(\lambda)$ for $\lambda = \infty$, the equality prevails for all $\lambda \in \mathbb{C}$ and thus, in view of the presumed observability, $X_1C^* = X_2C^*$. Therefore,

$$A(X_2 - X_1) + (X_2 - X_1)A^* = 2\pi X_1C^*JCX_1 - 2\pi X_2C^*JCX_2 = 0.$$

Consequently, $X_2 = X_1$.

Finally, Theorem 6 guarantees that the factorization (59) is minimal. Conversely, if $\Theta = \Theta_1\Theta_2$ is a minimal factorization of $\Theta(\lambda)$ with J-unitary factors, then, as follows from either the construction in (Alpay and Dym, 1993b) that is discussed below or by adapting the proof of Theorem 5.7 in (Alpay and Dym, 1986), the dBK space $\mathcal{K}(\Theta_1)$ is embedded isometrically into $\mathcal{M}$, i.e., $\mathcal{K}(\Theta_1) = \mathcal{M}_X = \mathcal{K}(\Theta_X)$ for some Hermitian solution X of the Riccati equation (44). Therefore, assuming $\Theta_1(\infty) = I_m$, as we may, it follows that $\Theta_1(\lambda) = \Theta_X(\lambda)$ as claimed. ■

The connection between invariant subspaces of the principle operator A in the realization of a J-inner matrix (and even operator) valued function $\Theta(\lambda)$ and the factorization of $\Theta(\lambda)$ was already observed in the early work of Livsic and Brodskii, see e.g. (Brodskii, 1971) and the notes and references therein. The equivalence of (i), (iv) and the factorization of $\Theta(\lambda)$ were established for the definite case ($J = I_m$) with $P > 0$ and $X \geq 0$ in Theorem 4.3 of (Fuhrmann, 1995). A generalization to the indefinite J case was announced by Gombani and Weiland (2000) in their lecture at the MTNS meeting in Perpignan.

The role of the R_α invariant subspaces of a dBK space $\mathcal{K}(\Theta)$ in the factorization of $\Theta(\lambda)$, i.e., the connection between (i) and (ii), was explored in assorted degrees of generality in (Alpay and Dym, 1986; 1993b; 1996), using the structural identity and/or Lyapunov equations. The connection of these reproducing kernel spaces with Riccati equations that was exhibited here and in (Dym, 2001) seems to be new.

The next theorem, which is adapted from Theorem 4.2 in (Alpay and Dym, 1993b), serves to exhibit the connection (in the setting of Section 5) between:

1. R_α-invariant subspaces of the finite dimensional RKKS $\mathcal{M}$,
2. subblocks of the invertible structured Hermitian matrix P that serves to define its indefinite inner product via (8), and
3. factors of the mvf $\Theta(\lambda)$.

Theorem 8. *Let (C, A) be observable and P be an $n \times n$ invertible solution of the Lyapunov equation (32). Assume that*

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix} \quad \textit{and} \quad A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}$$

are conformable block decompositions, where the upper left hand block in each of these two matrices is $k \times k$*, and suppose that* P_{11} *is invertible and*

$$A_{21} = 0. \tag{61}$$

Write

$$\Pi_1 = \begin{bmatrix} I_k \\ 0 \end{bmatrix}, \quad \Pi_2 = \begin{bmatrix} -P_{11}^{-1}P_{12} \\ I_{n-k} \end{bmatrix} \tag{62}$$

and let

$$Q = \Pi_2^* P \Pi_2 = P_{22} - P_{21}P_{11}^{-1}P_{12} \tag{63}$$

be the Schur complement of P_{11} *with respect to* P*. Then the mvf*

$$\Theta(\lambda) = I_m - 2\pi C(\lambda I_n - A)^{-1}P^{-1}C^*J \tag{64}$$

admits a factorization of the form

$$\Theta(\lambda) = \Theta_1(\lambda)\Theta_2(\lambda), \tag{65}$$

where

$$\Theta_1(\lambda) = I_m - 2\pi C(\lambda I_n - A)^{-1})\Pi_1 P_{11}^{-1}\Pi_1^* C^* J, \tag{66}$$

$$\Theta_2(\lambda) = I_m - 2\pi C\Pi_2(\lambda I_{n-k} - A_{22})^{-1}Q^{-1}\Pi_2^* C^* J. \tag{67}$$

Moreover,

$$A_{22}^* Q + QA_{22} = -2\pi \Pi_2^* C^* JC\Pi_2. \tag{68}$$

These formulas are obtained from the factorization formulas that are presented in Theorem 4.2 of (Alpay and Dym, 1993b) for the case $\Omega_+ = \Pi_+$ upon (in the notation of that paper) letting μ tend to ∞ along the $i\mathbb{R}$ axis. Comparing with the present set of formulas for the factors of Θ, we expect that

$$X = \Pi_1 P_{11}^{-1}\Pi_1^*$$

is a solution of the Riccati equation (44). It is readily checked that this is truly the case. Indeed, since $A_{21} = 0$, we have

$$A\Pi_1 P_{11}^{-1}\Pi_1^* + \Pi_1 P_{11}^{-1}\Pi_1^* A^* = \begin{bmatrix} A_{11}P_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} P_{11}^{-1}A_{11}^* & 0 \\ 0 & 0 \end{bmatrix},$$

whereas

$$-2\pi\Pi_1 P_{11}^{-1}\Pi_1^* C^* JC\Pi_1 P_{11}^{-1}\Pi_1^* = -2\pi \begin{bmatrix} P_{11}^{-1}\Pi_1^* C^* JC\Pi_1 P_{11}^{-1} & 0 \\ 0 & 0 \end{bmatrix}.$$

Thus, $X = \Pi_1 P_{11}^{-1} \Pi_1^*$ will be a solution of the Riccati equation (44) if and only if P_{11} is a solution of the Lyapunov equation

$$P_{11} A_{11} + A_{11}^* P_{11} = -2\pi \Pi_1^* C^* J C \Pi_1.$$

But this is just the 11 block of the Lyapunov equation (32). Furthermore, by invoking the well known formulas for the inverse of a matrix in terms of its Schur complement (see e.g., Chapter 0 of (Dym, 1989a)), we see that

$$Y = P^{-1} - X = \Pi_2 Q^{-1} \Pi_2^*.$$

Substituting this choice of Y into (58), we obtain

$$\Theta_X(\lambda)^{-1} \Theta(\lambda) = I_m - 2\pi C \Pi_2 Q^{-1} \Pi_2^* P (\lambda I_n - A)^{-1} \Pi_2 Q^{-1} \Pi_2^* C^* J.$$

But now, as

$$Q^{-1} \Pi_2^* P = [0 \;\; I_k],$$

it is readily checked that

$$Q^{-1} \Pi_2^* P (\lambda I_n - A)^{-1} \Pi_2 = (\lambda I_{n-k} - A_{22})^{-1}$$

and hence (58) coincides with (67) for $\Theta_2(\lambda)$. That is to say, (67) exhibits a minimum realization for the second factor on the right hand side in (60). It is readily checked that a minimal realization for the first factor on the right hand side in (60) (i.e., the mvf defined by (66)) is given by the expression

$$\Theta_1(\lambda) = I_m - 2\pi C \Pi_1 (\lambda I_k - A_{11})^{-1} P_{11}^{-1} \Pi_1^* C^* J.$$

Note added in the proof: Some of the factorization formulas established here appear to be closely related to results that were obtained earlier in (Lerer and Ran, 1997) by other methods; see also the preprint (Karelin *et al.*, 2001) for further developments. I am indebted to Andrei Ran for calling my attention to these references.

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