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REGIONAL DETECTION AND RECONSTRUCTION OF UNKNOWN INTERNAL OR BOUNDARY SOURCES

LARBI AFIFI*, ABDELHAQ EL JAI**
MALIKA MERRY*

The purpose of this paper is to study the problem of regional detection, to characterize internal or boundary regionally detectable sources and regionally spy sensors, and to establish a relationship between these sensors and regionally strategic sensors. It is shown how to reconstruct a regionally detectable internal or a boundary source from a given output, with an extension to the case when the output is affected by an observation error. Numerical results are given in the case of a diffusion system.

Keywords: sources, detection, observation, region, sensors

1. Introduction

This work concerns the regional analysis of distributed-parameter systems introduced and developed for continuous (El Jai *et al.*, 1993; 1995) or discrete (Afifi, 1994) systems, with special emphasis on controllability and observability. It constitutes an extension of previous works on detection and reconstruction of unknown internal sources (Afifi and El Jai, 1994; Afifi *et al.*, 2000).

Other works in this area were devoted to the study of inverse or identification problems (Isakov, 1998; Rafajłowicz, 1984a; 1984b). The problem considered here is different, and the approach developed seems general enough to be extented to other types of problems.

We study the existence of an output operator (sensors) ensuring a unique regional detection and reconstruction of any internal or boundary disturbance in the system, even if the observation is not exact. The regional aspect is motivated by the fact that we may be interested in the detection and reconstruction of a source only in a subregion ω of the geometrical support Ω of the considered system and, as it will be shown, by the fact that a source can be regionally detectable without being detectable in the whole domain Ω. Even if we have a possibility of detection on all Ω, it is easier to detect a source in a subregion ω than to do so in the whole domain Ω.

* University Hassan II Ain Chock, Department of Mathematics and Computer Science, B.P. 5366-Maârif, Casablanca, Morocco, e-mail: afifi@facsc-achok.ac.ma

** University of Perpignan, Systems Theory Laboratory, 52 Avenue de Villeneuve, 66860 Perpignan Cedex, France, e-mail: Abdelhaq.El.Jai@univ.perp.fr

First, we characterize regionally detectable sources and regional spy sensors ensuring a regional detection. Then we show how to reconstruct a regionally detectable source in the cases where the output is exact or affected by an unknown error, with extension of the approach to boundary sources which have not been considered in previous works. Applications and numerical results are also given.

The work is organized as follows: In Section 2, we recall notions of sources, define regional detection and regionally spy sensors, and characterize regionally detectable sources. In Section 3, concerning internal sources, we characterize ω-spy sensors (regionally spies with respect to the subregion $\omega \subset \Omega$) and we show how to reconstruct an unknown ω-detectable source in ω from the output in the cases where the observation is exact or affected by an error. In the latter case, we study the reconstruction error with respect to the observation one. Then we demonstrate an application to a diffusion system, as well as examples and numerical results. Finally, in Section 4, we extend the approaches and characterizations developed for internal sources to the case of boundary sources. We also give examples and numerical results.

2. Sources and Regional Detection

In this section, we recall the notions of sources (Afifi, 1994; Afifi and El Jai, 1994) and define the regional detection as well as sensors ensuring it. We consider a system (S) with a geometrical support Ω. We suppose that (S) is disturbed by an unknown source denoted by s, the corresponding state being $x(s) \in L^2(0,T;V)$, where Ω is an open and bounded subset of $\mathbb{R}^n$ with a sufficiently regular boundary Γ. Here V is a Hilbert space such that $V \subset L^2(\Omega) \subset V'$ with continuous injection, and V' constitutes the dual space of V.

2.1. Sources

The definition of a source disturbing (S) is as follows:

Definition 1. A source s is a triplet (Σ, g, J), with

1. $\Sigma(\cdot) : t \in J \to \Sigma(t) \subset \bar{\Omega}$ $(\bar{\Omega} = \Omega \cup \Gamma)$ defining the geometrical support of the source at time t,
2. $g(\cdot,\cdot) : \xi \in \Sigma(t) \to g(t,\xi) \in W$ (W is a Hilbert space) defining the intensity of the excitation at ξ at time t, and
3. $J = \{t \mid g(t,\cdot) \neq 0$ on $\Sigma(t)\}$ being the support of g.

The set of all sources will be denoted by $\mathcal{E}$. It is a Hilbert space (Afifi *et al.*, 2000).

Remark 1.

- A source $s = (\Sigma, g, J)$ is said to be internal (respectively boundary) if $\Sigma(t) \subset \Omega$ (resp. $\Sigma(t) \subset \Gamma$) $\forall t \in J$.
- If $\mu(J) > 0$, the source is persistent. It is instantaneous if $\mu(J) = 0$, where μ is the Lebesgue measure.

- A source $s = (\Sigma, g, J)$ is said to be pointwise (resp. zonal) if its support $\Sigma(t)$ is reduced to a single point (resp. to a region) of $\bar{\Omega}$ for all t in J.

Remark 2. A source $s = (\Sigma, g, J)$ can be identified with g because Σ and J can be determined by the knowledge of g. In this case, we have $\mathcal{E} \subset \mathcal{F}(]0,T[\times\Omega; W)$, where W is a Hilbert space and $\mathcal{F}(]0,T[\times\Omega; W)$ is the space of functions $f :]0,T[\times\Omega \to W$ (W is a subspace of $\mathbb{R}^N$, $N \in \mathbb{N}^*$, in a general case we have $N = 1$).

2.2. Regional Detection

Let ω be a non-empty subregion of Ω or Γ, where ω is not necessarily connected, and $\mathcal{E}_\omega$ be the set of sources located in ω:

$$\mathcal{E}_\omega \big\{ s = (\Sigma, g, J) \in \mathcal{E} \mid \Sigma(t) \in \omega \big\}.$$

We suppose that (S) excited by a source $s \in \mathcal{E}_\omega$ is augmented by the output equation

$$(E) \quad y = Cx,$$

where x is the state of (S), $C : V \to Y$ is a linear operator, $y \in L^2(0,T;Y)$, and Y is a Hilbert space (observation space).

Definition 2. If we can reconstruct a source s located in ω based on the system description (S) and the output equation (E), we say that s is regionally detectable in ω or ω-detectable on $]0,T[$.

Let Q_ω be the operator defined by

$$Q_\omega : s_\omega \in \mathcal{E}_\omega \longrightarrow y_{s_\omega} \in L^2(0,T;Y), \tag{1}$$

where y_{s_ω} is the observation corresponding to a source s_ω. Then every source is ω-detectable on $]0,T[$ if the operator Q_ω is injective.

Remark 3. If the nature of the source to be detected is known, we may consider only the restriction of Q_ω to the corresponding subset Z_ω of $\mathcal{E}_\omega$, i.e.

$Z_\omega \equiv \mathcal{E}_\omega^{z,pe}$ being the set of zone persistent sources, and

$Z_\omega \equiv \mathcal{E}_\omega^{z,i}$ being the set of zone instantaneous sources.

In this case, any source $s \in Z_\omega$ is ω-detectable on $]0,T[$ if $Q_\omega : Z_\omega \to L^2(0,T;Y)$ is injective.

Let us note that this work can be extended to the case of sources which are not necessarily located in the subregion ω ($\Sigma \cap \omega \neq \emptyset$ and $\Sigma \cap \omega^c \neq \emptyset$, where $\omega^c = \Omega \backslash \omega$). Indeed, we consider the operator

$$Q : s \in \mathcal{E} \longrightarrow y_s \in L^2(0,T;Y) \tag{2}$$

and the set E_ω

$$E_\omega = \big\{ s_\omega = P_\omega s \mid s \in \mathcal{E} \big\},$$

where P_ω is the operator defined by

$$P_\omega : \begin{array}{lcl} \mathcal{E} & \longrightarrow & E_\omega, \\ s = (\Sigma, g, J) & \longmapsto & s_\omega = (\Sigma_\omega, g_\omega, J_\omega), \end{array} \tag{3}$$

with $\Sigma_\omega = \Sigma \cap \omega$, $g_\omega = p_\omega g$, J_ω being the support of g_ω and p_ω the restriction operator to ω, defined by

$$p_\omega : \begin{array}{lcl} \mathcal{F}(]0, T[\times \bar{\Omega}, W) & \longrightarrow & \mathcal{F}(]0, T[\times \omega, W), \\ g & \longmapsto & g_\omega = g_{|\omega}. \end{array} \tag{4}$$

The adjoint operator P_ω^* of P_ω, denoted by I_ω, is given by

$$I_\omega : \begin{array}{lcl} E_\omega & \longrightarrow & \mathcal{E}, \\ s_\omega = (\Sigma_\omega, g_\omega, J_\omega) & \longmapsto & I_\omega s_\omega = s = (\Sigma, g, J), \end{array} \tag{5}$$

with $\Sigma = \Sigma_\omega$, $J = J_\omega$ and $g = i_\omega g_\omega$, where $i_\omega = p_\omega^*$ is given by

$$i_\omega : \begin{array}{lcl} \mathcal{F}(]0, T[\times \omega, W) & \longrightarrow & \mathcal{F}(]0, T[\times \bar{\Omega}, W), \\ g_\omega & \longmapsto & i_\omega g_\omega = \begin{cases} g_\omega & \text{in } \omega, \\ 0 & \text{otherwise.} \end{cases} \end{array} \tag{6}$$

For (S) augmented by the regional output[1]

$$(E) \;\; ; \quad y_\omega(t) = Cx_\omega(t), \quad t \in]0, T[,$$

where x_ω is the state corresponding to the source $s_\omega = I_\omega P_\omega s$, if the operator QI_ω is injective and if any source $s = (\Sigma, g, J)$ such that $\Sigma \cap \omega \neq \emptyset$ can be reconstructed from (S) and (E), then s is said ω-detectable. In this case, the approach and results developed in this paper are the same.

Let us note that for a source s located in ω, we have $s \equiv I_\omega P_\omega s$ and so the two notations can be used.

Definition 3. Sensors ensuring the regional detection of any source in ω are called the ω-spies.

Sensors can be ω-spies, but not spies on the whole domain (Ω-spies). The following example illustrates this phenomenon.

Example 1. Consider the following one-dimensional diffusion system defined in $\Omega =]0, 1[$:

$$\begin{cases} \dfrac{\partial x}{\partial t}(\xi, t) = \dfrac{\partial^2 x}{\partial \xi^2}(t, \xi) + f(t)\delta_b(t), \\ x(0, t) = x(1, t) = 0, \\ x(\cdot, 0) = 0, \end{cases}$$

[1] In the case of systems where it is possible to extract regional observation y_ω corresponding to $s_\omega = I_\omega P_\omega s$.

where δ_b is the Dirac delta function concentrated at b. We assume that the measurements are given by means of a pointwise sensor (c, δ_c) located at $c \in]0,1[$. Hence the output equation is

$$y(t) = x(c,t), \quad t \in]0,T[.$$

The system state is given by

$$x(t,\xi) = \sum_{n\geq 1} \int_0^t e^{\lambda_n(t-\tau)} \Phi_n(b) f(\tau)\, \mathrm{d}\tau \Phi_n(c)$$

with $\lambda_n = -n^2\pi^2$ and $\Phi_n(\xi) = \sqrt{2}\sin(n\pi\xi)$. If $c = 1/2$, the sensor (c,δ_c) is not an Ω-spy. (Afifi *et al.*, 2000), but it is a regional spy on $\omega =]0,1/2[$ (Q_ω injective). ♦

2.3. ω-Spy Sensors and ω-Strategic Sensors

In this part, we recall the notions of ω-observability in the case where it is desired to reconstruct regionally an initial state x_0 in $\omega \subset \Omega$ (internal case), or on $\omega \subset \Gamma$ (boundary case), as well as the relationship between the sensors ensuring the ω-observability and those being ω-spies.

2.3.1. Internal Case

We consider the autonomous system

$$\begin{cases} \dot{x}(t,\xi) = Ax(t,\xi) & \text{in }]0,T[\times\Omega, \\ x(t,\xi) = 0 & \text{on }]0,T[\times\Gamma, \\ x(0,\xi) = x_0(\xi) \in L^2(\Omega) & \text{in } \Omega, \end{cases} \tag{7}$$

where x_0 is supposed to be unknown. We assume that (7) is augmented by the output equation

$$y(t) = Cx(t,\cdot), \quad t \in]0,T[. \tag{8}$$

If K is the operator defined by $K : z \in L^2(\Omega) \to Kz = C\,S\,z \in L^2(0,T;Y)$, where S_t is the strongly continuous semi group given by

$$S_t x = \sum_{n\geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle x, \Phi_{nj}\rangle_{L^2(\Omega)} \Phi_{nj},$$

then the weak regional observability can be defined as follows (Zerrik, 1993):

Definition 4. The system (7), augmented by (8), is weakly observable in ω (or ω-weakly observable) if $\operatorname{Ker} K i_\omega = \{0\}$.

2.3.2. Boundary Case

Without loss of generality, consider the following autonomous system:

$$\begin{cases} \dot{x}(t,\xi) = Ax(t,\xi) & \text{in }]0,T[\times\Omega, \\ \dfrac{\partial x}{\partial \nu}(t,\xi) = 0 & \text{on }]0,T[\times\Gamma, \\ x(0,\xi) = x_0(\xi) & \text{in } \bar{\Omega}, \end{cases} \tag{9}$$

where A generates the strongly continuous semigroup defined by

$$S_t x = \sum_{n\geq 0} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle x, \Psi_{nj}\rangle_{L^2(\Omega)} \Psi_{nj}, \quad x \in L^2(\Omega).$$

We assume that the system (9) is augmented with the output equation

$$y(t) = CS_t x_0. \tag{10}$$

Set

$$K : z \in H^1(\Omega) \longrightarrow CSz \in L^2(0,T;Y)$$

and

$$\gamma : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma) \quad \text{(the trace operator).}$$

If $i_\omega : z \in H^{1/2}(\omega) \longrightarrow i_\omega z \in H^{1/2}(\Gamma)$, the definition of ω-weak boundary observability is as follows (Badraoui *et al.*, 1998):

Definition 5. The system (9), augmented by the output equation (10), is said to be ω-weakly observable if

$$\operatorname{Ker} K\gamma^* i_\omega = \{0\}.$$

Definition 6. Sensors ensuring ω-weak observability are called ω-strategic.

Proposition 1. (Afifi and El Jai, 1994; Merry, 2000) *ω-strategic sensors are ω-spy sensors.*

The converse is not true. This will be illustrated by examples in the case of internal or boundary pointwise sources.

2.4. Regionally Detectable Sources

To study the regional detection of a source $\tilde{s} = (\tilde{\Sigma}, \tilde{g}, \tilde{J}) \in Z_\omega$ located in ω, we consider the function

$$F_\omega(s) = \|y_s - y_{\tilde{s}}\|^2_{L^2(0,T;Y)}, \quad s \in Z_\omega.$$

The problem of regional detection is then equivalent to the minimization problem

$$\begin{cases} \inf F_\omega(s) \\ s \in Z_\omega \end{cases}$$

We have $\tilde{s} \in Z_\omega$ and $F(\tilde{s}) = 0$. Hence the set

$$S^\omega_{\tilde{s}} = \left\{ \bar{s} \in Z_\omega \mid F_\omega(\bar{s}) = \inf_{s \in Z_\omega} F_\omega(s) \right\} = \left\{ \bar{s} \in Z_\omega \mid Q_\omega(\bar{s}) = Q_\omega(\tilde{s}) \right\}$$

is not empty.

Proposition 2. *A source $\tilde{s}$ is ω-detectable if and only if $S^\omega_{\tilde{s}} = \{\tilde{s}\}$.*

3. Case of Internal Sources

This section concerns the regional detection of internal sources. We give a characterization of ω-spy sensors as well as their relationship with ω-strategic sensors, and we show how to reconstruct a source located in ω from observations.

3.1. System under Consideration

Let Ω be an open and bounded subset of $\mathbb{R}^n$ with a sufficiently regular boundary $\Gamma = \partial\Omega$. We consider the following system:

$$\begin{cases} \dot{x}(t) = Ax(t) + g(t), \quad t \in]0, T[, \\ x(0) = x_0 \in X, \end{cases} \tag{11}$$

where $X = L^2(\Omega)$, $g \in L^2(0, T; V')$ stand for the intensity of the source supposed to be unknown and located in ω, V is a Hilbert space such that $V \subset X \subset V'$ with continuous injections, and A is a linear operator generating a strongly continuous semigroup $(S_t)_{t \geq 0} \in \mathcal{L}(V, X)$. In this case, the state of (11) is given by

$$x(t) = S_t x_0 + \int_0^t S_{t-\tau} g(\tau) \, \mathrm{d}\tau = S_t x_0 + \int_0^t S_{t-\tau} i_\omega g(\tau) \, \mathrm{d}\tau, \quad t \in]0, T[$$

with $x \in L^2(0, T; V)$ (Curtain and Pritchard, 1978). We suppose that the system (11) is augmented by the output equation

$$y(t) = Cx(t), \quad t \in]0, T[. \tag{12}$$

Then any source $s = (\Sigma, g, J)$ is ω-detectable on $]0, T[$ if the operator

$$Q_\omega : \begin{array}{ccl} \mathcal{E}_\omega & \longrightarrow & L^2(0, T; Y) \\ s & \longmapsto & y(t) = CS_t x_0 + C \int_0^t S_{t-\tau} i_\omega g(\tau) \, \mathrm{d}\tau \end{array} \tag{13}$$

is injective.

3.2. Characterization of Regionally Spy Sensors

Without loss of generality, we consider the case where the strongly continuous semigroup $(S_t)_{t\geq 0}$ is defined by

$$S_t x = \sum_{n\geq 1} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle x, \Phi_{nj}\rangle_{L^2(\Omega)} \Phi_{nj}, \tag{14}$$

$(\Phi_{nj})_{\{j=1\ldots r_n, n\geq 1\}}$ being an orthonormal basis of eigenfunctions of A, associated with the eigenvalues λ_n of multiplicities r_n such that $\sup_{n\geq 1} r_n < \infty$. We suppose that the initial state $x_0 = 0$ (the results obtained are also valid if $x_0 \neq 0$) and that the output is given by q zone sensors $(D_k, h_k)_{1\leq k\leq q}$, where D_k is the geometrical support of the sensor (D_k, h_k) and h_k its spatial distribution (El Jai and Pritchard, 1988; Uciński, 1992). The observation y corresponding to the source $s = (\Sigma, g, J)$ is given by

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{pmatrix} \in Y = \mathbb{R}^q$$

with

$$y_k(t) = \sum_{n\geq 1} \sum_{j=1}^{r_n} \left(\int_0^t e^{\lambda_n (t-\tau)} \langle g(\tau), \Phi_{nj}\rangle_{L^2(\omega)} \, \mathrm{d}\tau \right) \langle h_k, \Phi_{nj}\rangle_{L^2(D_k)}. \tag{15}$$

For $n \geq 1$, we consider the matrix

$$M_n = \begin{pmatrix} \langle h_1, \Phi_{n1}\rangle_{L^2(D_1)} & \cdots & \langle h_1, \Phi_{nr_n}\rangle_{L^2(D_1)} \\ \vdots & \ddots & \vdots \\ \langle h_q, \Phi_{n1}\rangle_{L^2(D_q)} & \cdots & \langle h_q, \Phi_{nr_n}\rangle_{L^2(D_q)} \end{pmatrix}$$

and the function

$$f_n : \xi \in \omega \to \begin{pmatrix} \Phi_{n1}(\xi) \\ \vdots \\ \Phi_{nr_n}(\xi) \end{pmatrix} \in \mathbb{R}^{r_n}.$$

In what follows, we give hereafter the characterization of ω-spy sensors, first in the case of sources $s \in \mathcal{E}$ such that s is persistent pointwise, and then in the case where s is zone persistent.

3.2.1. Case of Persistent Pointwise Sources

For pointwise and persistent sources $s \in \mathcal{E}_\omega$, we have the following result (Afifi and El Jai, 1994):

Proposition 3. *Sensors* $(D_k, h_k)_{1 \le k \le q}$ *are* ω*-spies if and only if*

$$\left. \begin{array}{l} \alpha f_n(\xi) - \beta f_n(\mu) \in \operatorname{Ker} M_n, \quad \forall n \ge 1 \\ \alpha, \beta \in \mathbb{R}, \quad \xi, \mu \in \omega \end{array} \right\} \Rightarrow \alpha = \beta \text{ and } \xi = \mu.$$

In this case, sensors may be ω-spies without being ω-strategic.

Example 2. For $\Omega =]0,1[$, we consider the system disturbed by a pointwise source located at a point b of a subregion $\omega \subset \Omega$ and with intensity e:

$$\left\{ \begin{array}{rcll} \dfrac{\partial x}{\partial t}(t,\xi) &=& \dfrac{\partial^2 x}{\partial \xi^2}(t,\xi) + e(t)\delta_b(\xi) & \text{in }]0,T[\times\Omega, \\ x(t,0) &=& x(t,1) = 0 & \text{in }]0,T[, \\ x(0,\cdot) &=& 0 & \text{in } \Omega. \end{array} \right.$$

We suppose that the output is given by a zone sensor (D,h) with $D =]1/2-c, 1/2+c[$, $0 < c < 1/2$, and h is symmetrical with respect to $1/2$. Then

$$y(t) = \langle x(t), h \rangle_{L^2(D)}, \quad t \in]0,T[.$$

If $\omega =]\alpha, \beta[$ is such that $0 < \alpha < \beta < 1/2$ and $(1/2-\alpha)/(\beta-\alpha) \in \mathbb{Q}$, then the sensor (D,h) is not ω-strategic (Zerrik, 1993), but it is an ω-spy. ♦

3.2.2. Case of Persistent Zone Sources

In the case of persistent zone sources, set

$$Z_\omega = \mathcal{E}_\omega^{z,pe} = \Big\{ s = (\Sigma, g, J) \in \mathcal{E}_\omega \mid g \in L^2\big(\mathbb{R}_+^*; L^2(\omega)\big) \Big\}.$$

The operator Q_ω is defined by

$$Q_\omega s = \begin{pmatrix} (Q_\omega s)_1 \\ \vdots \\ (Q_\omega s)_q \end{pmatrix}, \quad s \in Z_\omega, \tag{16}$$

with $s = (\Sigma, g, J)$. For $k \in \{1, \dots, q\}$, we have

$$\begin{aligned} (Q_\omega s)_k(t) &= \sum_{n \ge 1} \sum_{j=1}^{r_n} \int_{]0,t[\cap J} e^{\lambda_n(t-\tau)} \langle i_\omega g(\tau), \Phi_{nj} \rangle_{L^2(\Sigma\cap\Omega)} \langle h_k, \Phi_{nj} \rangle_{L^2(D_k)} \, \mathrm{d}\tau \\ &= \sum_{n \ge 1} \sum_{j=1}^{r_n} \int_0^t e^{\lambda_n(t-\tau)} \langle g(\tau), \Phi_{nj} \rangle_{L^2(\omega)} \langle h_k, \Phi_{nj} \rangle_{L^2(D_k)} \, \mathrm{d}\tau \end{aligned} \tag{17}$$

by identifying Z_ω with $L^2(]0,T[; L^2(\omega))$ (cf. Remark 2).

Proposition 4. *Sensors* $(D_k, h_k)_{1 \le k \le q}$ *are* ω*-spies if and only if they are* ω*-strategic.*

Proof. If the sensors $(D_k, h_k)_{1 \le k \le q}$ are ω-strategic, then they are ω-spies, according to Proposition 1. Conversely, if $(D_k, h_k)_{1 \le k \le q}$ are not ω-strategic, then (Zerrik, 1993) there exists $z_* \in L^2(\omega) \setminus \{0\}$ such that

$$CS_t i_\omega z_* = 0, \quad \forall t \in]0, T[, \tag{18}$$

with

$$i_\omega : \begin{array}{lll} L^2(\omega) & \longrightarrow & L^2(\Omega) \\ g & \longmapsto & i_\omega g = \begin{cases} g \text{ in } \omega, \\ 0 \text{ otherwise,} \end{cases} \end{array}$$

that is to say,

$$\sum_{n \ge 1} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z_*, \Phi_{nj} \rangle_{L^2(\omega)} \langle h_k, \Phi_{nj} \rangle_{L^2(D_k)} = 0, \quad \forall t \in]0, T[, \quad \forall k \in \{1, \dots, q\}.$$

Consequently, using (17), we have

$$Q_\omega z_* = 0.$$

Therefore, for a given source $s = (\Sigma, g, J) \in Z_\omega$ and

$$\bar{g} = g + z_*,$$

we have

$$Q_\omega s = Q_\omega \bar{s},$$

where $\bar{s}$ is the source having $\bar{g}$ as its intensity. According to Proposition 1, the sensors $(D_k, h_k)_{1 \le k \le q}$ are not ω-spies. ■

In general, for the detection of any persistent source located in ω (zone or pointwise), i.e. in the case when $Z_\omega = \mathcal{E}_\omega$, we have the following result:

Corollary 1. *Sensors* $(D_k, h_k)_{1 \le k \le q}$ *are* ω*-spies if and only if they are* ω*-strategic.*

3.3. Reconstruction of a Regionally Detectable Source

In this section, under a regional detection hypothesis, we show how to reconstruct a source $s \in \mathcal{E}_\omega$, first in the case of an observation without errors, and then in the case with errors.

3.3.1. Case of Observations without Errors

Consider the system (11) augmented by the output (12) and suppose that the operator Q_ω is injective. The semi-norm defined by

$$\|s\|_{F_\omega} = \|Q_\omega s\|_{L^2(0,T;Y)}, \quad s \in \mathcal{E}_\omega$$

is a norm. If $F_\omega = \bar{\mathcal{E}}_\omega^{\|\cdot\|_{F_\omega}}$, then F_ω is a Hilbert space with the inner product

$$\langle s_1, s_2 \rangle_{F_\omega} = \langle Q_\omega s_1, Q_\omega s_2 \rangle_{L^2(0,T;Y)}.$$

Consider the operator $\Lambda_\omega : \mathcal{E}_\omega \longrightarrow \mathcal{E}_\omega$ defined by

$$\begin{aligned} \Lambda_\omega s &= Q_\omega^* Q_\omega s \\ &= \int_{\cdot}^{T} S_{r-\cdot}^* C^* C \int_0^r S_{r-\tau} g(\tau)\, \mathrm{d}\tau\, \mathrm{d}r, \quad s = (\Sigma, g, J) \in \mathcal{E}_\omega. \end{aligned} \tag{19}$$

Λ_ω has a unique extension as an isomorphism from F_ω into its dual F_ω', such that

$$\begin{cases} \langle \Lambda_\omega s_1, s_2 \rangle_{\mathcal{E}_\omega} = \langle s_1, s_2 \rangle_{F_\omega}, & \forall s_1, s_2 \in F_\omega, \\ \|\Lambda_\omega s_1\|_{F_\omega'} = \|s_1\|_{F_\omega}, & \forall s_1 \in F_\omega, \end{cases} \tag{20}$$

where, for $s_1 = (\Sigma_1, g_1, I_1)$, $s_2 = (\Sigma_2, g_2, I_2) \in \mathcal{E}_\omega$, we get

$$\begin{aligned} \langle s_1, s_2 \rangle_{\mathcal{E}_\omega} &= \int_{(I_1 \cap I_2)^z} \int_{(\Sigma_1(\tau) \cap \Sigma_2(\tau))^z} g_1(\tau, \xi)\, g_2(\tau, \xi)\, \mathrm{d}\tau\, \mathrm{d}\xi \\ &+ \sum_{t_i \in (I_1 \cap I_2)^p} \int_{(\Sigma_1(t_i) \cap \Sigma_2(t_i))^z} g_1(t_i, \xi)\, g_2(t_i, \xi)\, \mathrm{d}\xi \\ &+ \int_{(I_1 \cap I_2)^z} \sum_{x_j \in (\Sigma_1(\tau) \cap \Sigma_2(\tau))^p} g_1(\tau, x_j)\, g_2(\tau, x_j)\, \mathrm{d}\tau \\ &+ \sum_{t_i \in (I_1 \cap I_2)^p} \sum_{x_j \in (\Sigma_1(t_i) \cap \Sigma_2(t_i))^p} g_1(t_i, x_j)\, g_2(t_i, x_j). \end{aligned} \tag{21}$$

Here $(I_1 \cap I_2)^z$ and $(I_1 \cap I_2)^p$ are respectively the zone and pointwise parts of $(I_1 \cap I_2)$. Similarly, $(\Sigma_1 \cap \Sigma_2)^z$ and $(\Sigma_1 \cap \Sigma_2)^p$ are respectively the zone and pointwise parts of $(\Sigma_1 \cap \Sigma_2)$ (Afifi *et al.*, 2000).

Proposition 5. *If Q_ω is injective, the source s is obtained from the corresponding observation y as the unique solution of the equation*

$$\Lambda_\omega s = Q_\omega^* y.$$

3.3.2. Case of Observations with Errors

In this case, the system (11) is augmented by the output equation

$$z(t) = y(t) + e_\omega(t), \quad t \in]0, T[, \tag{22}$$

where y is given by (12) and e_ω is an error in the observation, which is usually unknown.

Write

$$K_{e_\omega}(s) = \|Q_\omega s - z\|^2_{L^2(0,T;Y)}, \quad s \in \mathcal{E}_\omega. \tag{23}$$

Proposition 6. *If $Q^*_\omega z \in F'_\omega$, then K_{e_ω} possesses a unique extension to F_ω and a unique minimum s_{e_ω} in F_ω, given by*

$$\Lambda_\omega s_{e_\omega} = Q^*_\omega z.$$

Proof. If $Q^* z \in F'_\omega$, there exists a unique $f^e_* \in F_\omega$ such that $Q^* z = \Lambda_\omega f^e_*$. Then for $s \in \mathcal{E}_\omega$ we have

$$\begin{aligned} K_{e_\omega}(s) &= \langle Q_\omega s, Q_\omega s\rangle_{L^2(0,T;Y)} - 2\langle Q_\omega s, z\rangle_{L^2(0,T;Y)} + \langle z, z\rangle_{L^2(0,T;Y)} \\ &= \langle \Lambda_\omega s, s\rangle_{\mathcal{E}_\omega} - 2\langle s, \Lambda_\omega f_{e_\omega}\rangle_{\mathcal{E}_\omega} + \|z\|^2_{L^2(0,T;Y)} \\ &= \|s\|^2_{F_\omega} - 2\langle s, f_{e_\omega}\rangle_{F_\omega} + \|z\|^2_{L^2(0,T;Y)}. \end{aligned}$$

Therefore, by density it is easy to show that K_{e_ω} has a unique extension to F_ω and then a unique minimum $s_{e_\omega} = f_{e_\omega}$. ■

The following result gives an estimate of the reconstruction error for the source s, with respect to the observation error.

Proposition 7. *We have*

(i) $\quad \|s_{e_\omega} - s\|_{F_\omega} = \|Q^*_\omega e_\omega\|_{F'_\omega}$,

(ii) $\quad \|s_{e_\omega} - s\|_{F_\omega} \leq \sqrt{2}\|e_\omega\|_{L^2(0,T;Y)}$.

Proof. (i) According to (20), we have

$$\|s_{e_\omega} - s\|_{F_\omega} = \|\Lambda_\omega(s_{e_\omega} - s)\|_{F'_\omega}$$

and therefore

$$\|s_{e_\omega} - s\|_{F_\omega} = \|Q^*_\omega Q_\omega s_{e_\omega} - Q^*_\omega Q_\omega s\|_{F'_\omega} = \|Q^*_\omega e_\omega\|_{F'_\omega}.$$

(ii) Using (20), we have

$$\begin{aligned} \|s_{e_\omega} - s\|^2_{F_\omega} &= \langle \Lambda_\omega(s_{e_\omega} - s), s_{e_\omega} - s\rangle_{\mathcal{E}_\omega} = \langle Q^*_\omega e_\omega, s_{e_\omega} - s\rangle_{\mathcal{E}_\omega} \\ &= \langle e_\omega, Q_\omega s_{e_\omega} - z\rangle_{L^2(0,T;Y)} + \langle e_\omega, z - y\rangle_{L^2(0,T;Y)} \\ &\leq \|e_\omega\|_{L^2(0,T;Y)}\|(\min_{s \in F_\omega} K_{e_\omega}(s))^{1/2} + \|e_\omega\|^2_{L^2(0,T;Y)} \\ &\leq 2\|e_\omega\|^2_{L^2(0,T;Y)}. \end{aligned}$$

Then

$$\|s_{e_\omega} - s\|_{F_\omega} \leq \sqrt{2}\|e_\omega\|_{L^2(0,T;Y)}.$$

■

As $\|e_\omega\|_{L^2(0,T;Y)} \to 0$, we get the result obtained in the case of an observation without error.

3.4. Simulation Results

Let $\Omega =]0,1[$ and $\omega \subset \Omega$. We consier the system described by the equation.

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \Delta(t,\xi) + g(t,\xi) & \text{in }]0,T[\times\Omega, \\ x(0,\xi) = 0 & \text{in } \Omega, \\ x(t,\xi) = 0 & \text{on }]0,T[\times\Gamma, \end{cases} \tag{24}$$

where g is the intensity of the source $s = (\Sigma, g, J)$ exciting the system and Δ is the Laplacian operator. Δ generates on $X = L^2(\Omega)$ a strongly continuous semigroup $(S_t)_{t\geq 0}$ defined by

$$S_t x = \sum_{n\geq 1} e^{\lambda_n t}\langle x, \Phi_n\rangle_{L^2(\Omega)}\Phi_n. \tag{25}$$

$(\Phi_n)_{n\geq 1}$ is the orthonormal basis of the eigenfunctions of Δ associated with the eigenvalues λ_n,

$$\begin{cases} \Phi_n(\xi) = \sqrt{2}\sin(n\pi\xi), \\ \lambda_n = -n^2\pi^2, \quad r_n = 1. \end{cases}$$

We suppose that the source s is zonal and independent of time (constant). We then have $g(t,\xi) \equiv g(\xi)$, $\forall \xi \in \omega$, $\forall t \in]0,T[$. Therefore g can be rewritten as

$$g(\cdot) = \sum_{n\geq 1} \alpha_n \Phi_n(\cdot), \quad \forall t \in]0,T[\tag{26}$$

with $\alpha_n = \langle g, \Phi_n\rangle_{L^2(\omega)}$.

3.4.1. Observations without Errors

In this part, we suppose that the system (24) is augmented by the output equation

$$y(t) = Cx(t), \quad t \in]0,T[, \tag{27}$$

given by an ω-spy zone sensor (D,h) located in Ω. In this case, the operator Q_ω is injective and given by

$$(Q_\omega s)(t) = \sum_{n\geq 1}\left(\int_0^t e^{\lambda_n(t-\tau)}\langle g, \Phi_n\rangle_{L^2(\omega)}\,\mathrm{d}\tau\right)\langle \Phi_n, h\rangle_{L^2(D)}, \quad s \in \mathcal{E}_\omega. \tag{28}$$

Its adjoint operator is defined by

$$(Q_\omega^* y)(t) = \sum_{m\geq 1}\left(\int_t^T e^{\lambda_m(r-t)} y(r)\,\mathrm{d}r\right)\langle \Phi_m, h\rangle_{L^2(D)} p_\omega \Phi_m. \tag{29}$$

Then

$$\begin{aligned}\Lambda_\omega s(t) \;=\; & \sum_{m\geq 1}\sum_{n\geq 1}\sum_{j\geq 1}\left(\int_t^T \int_0^\tau e^{\lambda_m(\tau-t)} e^{\lambda_n(\tau-r)}\langle g_\omega(r), \Phi_j\rangle_{L^2(\omega)}\,\mathrm{d}r\,\mathrm{d}\tau\right)\\ & \times\langle \Phi_j, \Phi_n\rangle_{L^2(\omega)}\langle \Phi_m, h\rangle_{L^2(D)}\langle \Phi_n, h\rangle_{L^2(D)} p_\omega \Phi_m.\end{aligned} \tag{30}$$

Since the source s is given by

$$\Lambda_\omega s = Q_\omega^* y,$$

we have, in accordance with (26),

$$\sum_{n\geq 1}\alpha_n \Lambda_\omega p_\omega \Phi_n = Q_\omega^* y. \tag{31}$$

Multiplying (31) by Φ_m, we get

$$\sum_{n\geq 1}\alpha_n \langle \Lambda_\omega p_\omega \Phi_n, \Phi_m\rangle_{L^2(\omega)} = \langle Q_\omega^* y, \Phi_m\rangle_{L^2(\omega)},$$

i.e.

$$\sum_{n\geq 1}\alpha_n a_{mn} = b_m, \quad \forall m\geq 1, \tag{32}$$

with

$$\begin{cases} a_{mn} \;=\; \langle \Lambda_\omega p_\omega \Phi_n, \Phi_m\rangle_{L^2(\omega)},\\ \;\;b_m \;=\; \langle Q_\omega^* y, \Phi_m\rangle_{L^2(\omega)}.\end{cases}$$

Therefore, for a sufficiently large M, we have

$$\sum_{n=1}^{M}\alpha_n a_{mn} \simeq b_m, \quad m\in\{1,\dots,M\}, \tag{33}$$

According to (26), to have an approximation of s, we have to calculate the coefficients α_n for $n\in\{1,\dots,M\}$. An approximation of s is then obtained as the solution of (33) whose matrix is symmetric and positive deinite.

Example 3. We consider the case when $\omega =]2/5, 7/8[$ and a constant source with respect to time $(g(t,\cdot)\equiv g(\cdot))$ with the intensity

$$g(t,x) = g(x) = \begin{cases} -144x^2+168x-48 & \text{on }]1/2, 7/12[,\\ 1 & \text{on }]7/12, 2/3[\;\; \forall t\in]0,T[,\\ -144x^2+192x-63 & \text{on }]2/3, 3/4[,\\ 0 & \text{otherwise.}\end{cases}$$

We suppose that the output is given by an ω-spy sensor (D, h) with $D =]5/12, 7/12[$ and $h(\xi) = 1$. Figure 1 shows the correspondimg results for $M = 20$. ♦

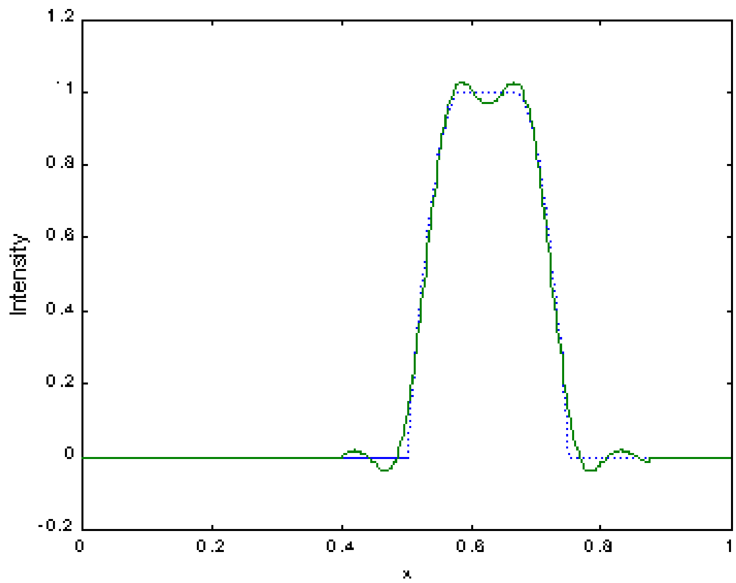

Fig. 1. Exact (dotted line) and reconstructed (solid line) source intensities of Example 3.

3.4.2. Observations with Errors

We consider the system (24) augmented by the output

$$z(t) = y(t) + e_\omega(t), \quad t \in]0, T[, \tag{34}$$

where $y(t) = Cx(t)$ and e_ω is an observation error. We suppose that the system is excited by a zone source s independent of time (i.e. constant). In this case, to have an approximation of s, it is sufficient to solve the system of linear equations

$$\sum_{n=1}^{M} \alpha_n a_{mn} = b_m, \quad m \in \{1, \dots, M\}, \tag{35}$$

with

$$\begin{cases} a_{mn} = \langle \Lambda_\omega p_\omega \Phi_n, \Phi_m \rangle, \\ b_m = \langle Q_\omega^* y, \Phi_m \rangle + \langle Q_\omega^* e_\omega, \Phi_m \rangle. \end{cases} \tag{36}$$

Example 4. We consider the case of the region $\omega =]0, 1/2[$ and the zone sensor $D = (]5/12, 7/12[, 1)$. If g denotes the exact intensity of the source s and g_ω^i the estimated one corresponding to the error e_i, $i = 1, 4$, then for $M = 10$, $e_1 = 0$, $e_2 = 10^{-4}$, $e_3 = 10^{-3}$, $e_4 = 10^{-2}$ and $g(x) = (500x^3 - 405x^2 + 82x)\mathbf{1}_{]0,2/5[}$, we obtain

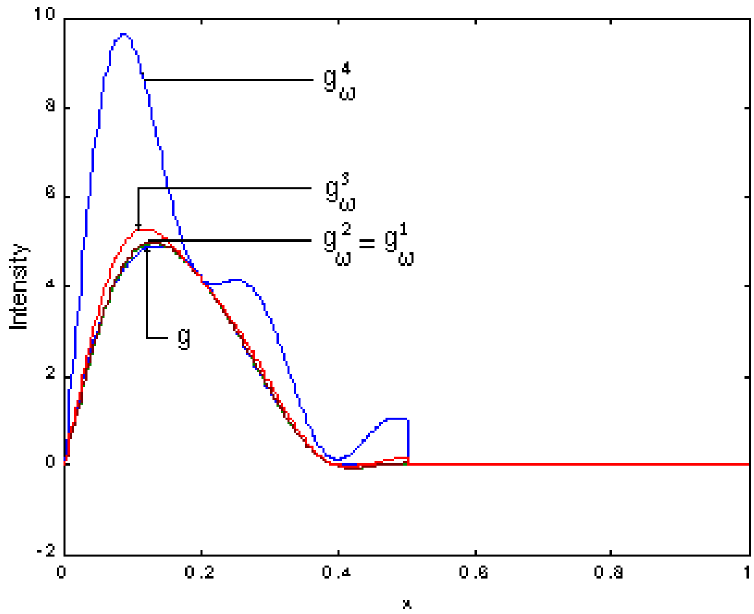

Fig. 2. Exact (dotted line) and estimated (solid line) intensities of Example 4.

the numerical results given in Fig. 2. As can be seen, these numerical results then conform to those obtained in the theoretical part. ♦

4. Case of Boundary Sources

In this part, we characterize regional spy sensors in the case of boundary sources, and we show how to reconstruct regionally such sources from the observation only, with an extension to the case when the output is affected by an error. Then we present an application and numerical results.

4.1. System under Consideration

Let Ω be an open and bounded subset of $\mathbb{R}^n$ with a sufficiently regular boundary Γ, and let Ω stand for a subregion of Γ. We consider the system described by

$$\begin{cases} \dot{x}(t,\xi) = \mathcal{A}x(t,\xi) & \text{in }]0,T[\times\Omega, \\ \mathcal{B}x(t,\xi) = g(t,\xi) & \text{on }]0,T[\times\Gamma, \\ x(0,\xi) = 0 & \text{in } \Omega, \end{cases} \tag{37}$$

where $g \in L^2(0,T;Z)$ is the unknown excitation of a source located in ω, and Z is a separable Hilbert space. Furthermore, $\mathcal{A} : D(\mathcal{A}) \subset V \to V$ is a linear operator, V

is a Hilbert space such that $V \subset X = L^2(\Omega) \subset V'$ with continuous injections, and $\mathcal{B} : D(\mathcal{B}) \subset V \to Z$ is a boundary operator such that $D(\mathcal{A}) \subset D(\mathcal{B})$.

The system (37) is augmented by the output equation

$$y(t) = Cx(t,\cdot), \quad t \in]0,T[, \tag{38}$$

where $C \in \mathcal{L}(V,Y)$, $y \in L^2(0,T;Y)$ and Y is a Hilbert space.

Next, we consider the space (also denoted by $\mathcal{E}_\omega$) of boundary sources located in ω. We suppose that the output is given by q zone sensors $(D_k, h_k)_{1\le k\le q}$ with $D_k \subset \Omega$ for $k = 1, q$.

4.2. Characterization of Regional Spy Sensors

In order to characterize regional spy sensors, without loss of generality we consider the system (37) with $\mathcal{A} = \Delta$ and $\mathcal{B}(\cdot) = \partial(\cdot)/\partial\nu$ (ν being the outward unit normal). The eigenfunctions $(\Psi_{nj})_{j=1,r_n;n\ge 0}$ of Δ with respect to the considered Neumann boundary condition and the associated eigenvalues $(\lambda_n)_{n\ge 0}$ are respectively defined by

$$\begin{cases} \Delta\Psi_{nj} = \lambda_n \Psi_{nj} & \text{in} \quad \Omega, \\ \dfrac{\partial \Psi_{nj}}{\partial\nu} = 0 & \text{on} \quad \Gamma, \quad 1 \le j \le r_n, \end{cases} \tag{39}$$

where r_n is the multiplicity of λ_n. For $n \ge 0$, we consider the matrix

$$M_n = \begin{pmatrix} \langle h_1, \Psi_{n1}\rangle_{L^2(D_1)} & \cdots & \langle h_1, \Psi_{nr_n}\rangle_{L^2(D_1)} \\ \vdots & \ddots & \vdots \\ \langle h_q, \Psi_{n1}\rangle_{L^2(D_q)} & \cdots & \langle h_q, \Psi_{nr_n}\rangle_{L^2(D_q)} \end{pmatrix}$$

and the function

$$f_n : \xi \in \omega \to \begin{pmatrix} \Psi_{n1}(\xi) \\ \vdots \\ \Psi_{nr_n}(\xi) \end{pmatrix} \in \mathbb{R}^{r_n}.$$

If we know the nature of the source $s \in \mathcal{E}_\omega$ to be detected, and if Z_ω is the corresponding set (cf. Remark 3), then the sensors $(D_k, h_k)_{1\le k\le q}$ are ω-spies if and only if the operator $Q_\omega : Z_\omega \to L^2(0,T;Y)$ is injective. In the sequel, we characterize ω-spy sensors for pointwise or zone persistent boundary sources.

4.2.1. Case of Regionally Persistent Pointwise Sources

A regionally persistent pointwise boundary source is ω-detectable if the operator

$$Q_\omega : s \in Z_\omega = \mathcal{E}_\omega^{p,pe} \to y_s \in L^2(0,T;Y)$$

is injective, where $\mathcal{E}_\omega^{p,pe}$ is the set of persistent pointwise sources located in ω.

Proposition 8. *The sensors* $(D_k, h_k)_{1\le k\le q}$ *are* ω*-spies if and only if*

$$\left.\begin{array}{l}\alpha f_n(\xi) - \beta f_n(\mu) \in \mathrm{Ker}\ M_n, \quad \forall n \ge 0\\ \alpha, \beta \in \mathbb{R}, \quad \xi, \mu \in \omega\end{array}\right\} \Rightarrow \alpha = \beta \text{ and } \xi = \mu. \tag{40}$$

Proof. The sensors $(D_k, h_k)_{1\le k\le q}$ are ω-spies if and only if Q_ω is injective. Therefore, if $s_1 = (\{b_1\}, \delta_{b_1}\varrho_1, J_1)$ and $s_2 = (\{b_2\}, \delta_{b_2}\varrho_2, J_2)$ are two elements of $\mathcal{E}^{\omega}_{p,pe}$ with

$$\delta_{b_i}(\xi) = \begin{cases} 1 & \text{for } \xi = b_i,\\ 0 & \text{otherwise,}\end{cases}$$

and

$$\varrho_1(t) = \sum_{j=1}^{N} \alpha_j \mathbf{1}_{[t_j, t_{j+1}[}(t), \quad \varrho_2(t) = \sum_{j=1}^{N} \beta_j \mathbf{1}_{[t_j, t_{j+1}[}(t)$$

for N large enough, where $t_1 = 0 < t_2 < \cdots < t_{N+1} = T$, α_j, $\beta_j \in \mathbb{R}$ and

$$\mathbf{1}_{[t_j, t_{j+1}[}(t) = \begin{cases} 1 & \text{if } \ t \in [t_j, t_{j+1}[,\\ 0 & \text{otherwise,}\end{cases}$$

then the sensors are ω-spies if and only if

$$Q_\omega s_1 = Q_\omega s_2 \Rightarrow b_1 = b_2 \text{ and } \alpha_j = \beta_j, \ \forall j \in \{1, \dots, N\}.$$

But, using the same method as in (El Jai and Berrahmoune, 1984, pp.179; El Jai and Pritchard, 1988, pp.95–96), the solution of (37) excited by a source $s = (\{b\}, \delta_b\varrho, I)$ is given by

$$x(t) = \sum_{n\ge 0}\sum_{k=1}^{r_n} \Psi_{nk}(b) \int_0^t e^{\lambda_{nk}(t-\tau)} \varrho(\tau)\, \mathrm{d}\tau \ \Psi_{nk}. \tag{41}$$

Then

$$Q s_1 = Q s_2 \iff \sum_{n\ge 0} \int_0^t e^{\lambda_n(t-\tau)} \sum_{k=1}^{r_n} \Big(\Psi_{nk}(b_1)\varrho_1(\tau) - \Psi_{nk}(b_2)\varrho_2(\tau)\Big)\, \mathrm{d}\tau$$
$$\times \langle \Psi_{nk}, h_i\rangle_{D_i} = 0, \quad \forall t \in]0, T[, \quad 1 \le i \le q. \tag{42}$$

Consequently, in $]t_1, t_2[$ we have

$$\sum_{n\ge 0} \int_{t_1}^t e^{\lambda_n(t-\tau)}\, \mathrm{d}\tau \sum_{k=1}^{r_n} \Big(\Psi_{nk}(b_1)\alpha_1 - \Psi_{nk}(b_2)\beta_1\Big)$$
$$\times \langle \Psi_{nk}, h_i\rangle_{L^2(D_i)} = 0, \quad 1 \le i \le q. \tag{43}$$

Then

$$\sum_{k=1}^{r_n} \Big(\Psi_{nk}(b_1)\alpha_1 - \Psi_{nk}(b_2)\beta_1\Big) \langle \Psi_{nk}, h_i\rangle_{L^2(D_i)} = 0, \quad \forall n \ge 0, \ 1 \le i \le q$$

in $]t_2, t_3[$, and according to (43), we get

$$\sum_{n\geq 0}\int_{t_2}^{t} e^{\lambda_n(t-\tau)}\,\mathrm{d}\tau \sum_{k=1}^{r_n}\Big(\Psi_{nk}(b_1)\alpha_2 - \Psi_{nk}(b_2)\beta_2\Big) \times\langle \Psi_{nk}, h_i\rangle_{L^2(D_i)} = 0, \quad 1\leq i\leq q. \tag{44}$$

Thus

$$\sum_{k=1}^{r_n}\Big(\Psi_{nk}(b_1)\alpha_2 - \Psi_{nk}(b_2)\beta_2\Big)\langle \Psi_{nk}, h_i\rangle_{L^2(D_i)} = 0, \quad \forall n\geq 0, \quad 1\leq i\leq q.$$

In much the same way, we show that

$$\begin{cases} \displaystyle\sum_{k=1}^{r_n}\Big(\Psi_{nk}(b_1)\alpha_l - \Psi_{nk}(b_2)\beta_l\Big)\langle \Psi_{nk}, h_i\rangle_{L^2(D_i)} = 0, \\ \forall n\geq 0, \quad 1\leq i\leq q, \quad 1\leq l\leq N. \end{cases} \tag{45}$$

Therefore

$$Q_\omega s_1 = Q_\omega s_2 \Longleftrightarrow \begin{cases} \displaystyle\sum_{k=1}^{r_n}\Big(\alpha_j\Psi_{nk}(b_1) - \beta_j\Phi_{nk}(b_2)\Big)\langle h_i, \Psi_{nk}\rangle_{L^2(D_i)} = 0, \\ \forall n\geq 0, \quad j\in\{0,\dots,N\}, \quad i\in\{1,\dots,q\}, \end{cases} \tag{46}$$

and consequently, the sensors are ω-spies if and only if (40) is satisfied. ■

Note that also in this case, sensors may be ω-spies without being ω-strategic.

Example 5. For $\Omega =]0, a_1[\times]0, a_2[$ such that $a_1^2/a_2^2 \notin \mathbb{Q}$, and $\omega =]0, a_1[\times\{0\}$, we consider the system

$$\begin{cases} \dfrac{\partial x}{\partial t}(t,\xi) = \Delta x(t,\xi) & \text{in }]0,T[\times\Omega, \\ \dfrac{\partial x}{\partial \nu}(t,\xi) = f(t)\delta_b(\xi) & \text{on }]0,T[\times\Gamma, \\ x(0,\xi) = 0 & \text{in } \Omega, \end{cases}$$

where f is the intensity of a pointwise source exciting the system and located in the subregion ω. We consider the case where the output is given by a pointwise sensor (c, δ_c) with $c = (a_1/4, a_2/4)$. The state $\phi(x_1, x_2) = \mathbf{1}_\omega(x_1, x_2)\cos(2\pi x_1/a_1)$ is not ω-observable (Badraoui *et al*, 1998). Hence the sensor (c, δ_c) is not ω-strategic, but it is an ω-spy. ♦

4.2.2. Case of Regionally Persistent Zone Sources

If we have to regionally detect a persistent zone source, we consider the set

$$Z_\omega = \big\{ s = (\Sigma, g, I) \in \mathcal{E}_\omega \mid g \in L^2\big(\mathbb{R}_+^*; L^2(\omega)\big) \big\} = \mathcal{E}_\omega^{z,pe}.$$

In this case, the state of the system is given by (El Jai and Pritchard, 1988; Curtain and Zwart, 1995)

$$x(t) = -\int_0^t AS_{t-\tau} Gg(\tau)\,\mathrm{d}\tau + \int_0^t S_{t-\tau} Gg(\tau)\,\mathrm{d}\tau,$$

where

$$G : g \in L^2(\Gamma) \longrightarrow h = Gg \in L^2(\Omega)$$

with

$$\begin{aligned} h - \Delta h &= 0 \quad \text{in } \Omega, \\ \frac{\partial h}{\partial \nu} &= g \quad \text{on } \Gamma, \end{aligned} \tag{47}$$

and

$$Az = \Delta z, \quad \forall z \in D(A) = H^2(\Omega).$$

Since the observation is given by q zone sensors $(D_k, h_k)_k$, the output (38) becomes

$$y(t) = \begin{pmatrix} y_1(t) \\ \vdots \\ y_q(t) \end{pmatrix} \in Y = \mathbb{R}^q$$

with

$$\begin{aligned} y_k(t) = \langle x(t), h_k \rangle_{L^2(D_k)} \;&=\; -\sum_{n\geq 0}\sum_{j=1}^{r_n} \int_0^t e^{\lambda_n(t-\tau)}\,\mathrm{d}\tau \Big(\langle Gg(\tau), A\Psi_{nj}\rangle_{L^2(\Omega)} \\ &\quad - \langle Gg(\tau), \Psi_{nj}\rangle_{L^2(\Omega)} \Big) \langle h_k, \Psi_{nj}\rangle_{L^2(D_k)}. \end{aligned} \tag{48}$$

From (47) and the Green formula, we have

$$\langle Gg(\tau), A\Psi_{nj}\rangle_{L^2(\Omega)} - \langle Gg(\tau), \Psi_{nj}\rangle_{L^2(\Omega)} = -\langle g(\tau), \Psi_{nj}\rangle_{L^2(\omega)}.$$

Therefore

$$y_k(t) = \sum_{n\geq 0}\sum_{j=1}^{r_n} \int_0^t e^{\lambda_n(t-\tau)} \langle g(\tau), \Psi_{nj}\rangle_{L^2(\omega)}\,\mathrm{d}\tau \langle h_k, \Psi_{nj}\rangle_{L^2(D_k)}, \quad 1 \leq k \leq q. \tag{49}$$

Proposition 9. $(D_k, h_k)_{1 \le k \le q}$ *are ω-spy sensors if and only if they are ω-strategic.*

Proof. Since ω-strategic sensors are ω-spies, it is sufficient to show the converse. If the sensors $(D_k, h_k)_{1 \le k \le q}$ are not ω-strategic, then $\exists z_* \neq 0 \in H^{1/2}(\omega)$ (Badraoui *et al.*, 1998), such that

$$CS_t \gamma_0^* i_\omega z_* = 0, \quad \forall t \in]0, T[, \tag{50}$$

where γ_0^* is the adjoint of the trace operator $\gamma_0 : H^1(\Omega) \longrightarrow H^{1/2}(\Gamma)$. Then we have

$$\sum_{n \ge 0} e^{\lambda_n t} \sum_{j=1}^{r_n} \langle z_*, \Psi_{nj} \rangle_{L^2(\omega)} \langle h_k, \Psi_{nj} \rangle_{L^2(D_k)} = 0, \ \ \forall t \in]0, T[, \quad \forall k \in \{1, \dots, q\}. \tag{51}$$

Hence

$$\sum_{n \ge 0} \int_0^t e^{\lambda_n (t-\tau)} \sum_{j=1}^{r_n} \langle z_*, \Psi_{nj} \rangle_{L^2(\omega)} \, \mathrm{d}\tau$$
$$\times \langle h_k, \Psi_{nj} \rangle_{L^2(D_k)} = 0, \quad \forall t \in]0, T[, \quad \forall k \in \{1, \dots, q\}, \tag{52}$$

i.e.

$$Q_\omega z_* = 0. \tag{53}$$

Thus, for a given source $s = (\Sigma, g, I) \in \mathcal{E}_\omega$, we have

$$Q_\omega s = Q_\omega \bar{s}, \tag{54}$$

where $\bar{s}$ is the source having as its intensity the function

$$\bar{g} = g + z_*. \tag{55}$$

Consequently, according to Proposition 2, $(D_k, h_k)_{1 \le k \le q}$ are not ω-spy sensors. ■

Generally, for $Z_\omega = \mathcal{E}_\omega$, we have the following result:

Corollary 2. $(D_k, h_k)_{1 \le k \le q}$ *are ω-spy sensors if and only if they are ω-strategic.*

It is easy to show that for regional reconstruction of boundary sources, the results and approches are similar to those obtained for internal sources.

4.3. Simulation Results

Let $\Omega =]0, \alpha[\times]0, \beta[$ with $\alpha^2/\beta^2 \notin \mathbb{Q}$. Consider the system disturbed by a source $s = (\Sigma, g, I)$ located in ω:

$$\begin{cases} \dfrac{\partial x}{\partial t}(t, \xi) = \Delta x(t, \xi) & \text{on }]0, T[\times \Omega, \\ \dfrac{\partial x}{\partial \nu}(t, \xi) = g(t, \xi) & \text{on }]0, T[\times \Gamma, \\ x(0, \xi) = 0 & \text{in } \Omega. \end{cases} \tag{56}$$

In this case, the eigenvalues and the associated eigenfunctions are given by

$$\lambda_{m,n} = -\left(\frac{m^2}{\alpha^2} + \frac{n^2}{\beta^2}\right)\pi^2, \quad m,n \geq 0,$$

$$\Psi_{mn}(\xi,\zeta) = \varphi_m^\alpha(\xi)\varphi_n^\beta(\zeta),$$

respectively, with

$$\varphi_m^\alpha(\xi) = \begin{cases} 1/\sqrt{\alpha} & \text{if } \ m=0, \\ \sqrt{2/\alpha}\cos(m\pi\xi/\alpha) & \text{if } \ m \geq 1, \end{cases}$$

$$\varphi_n^\beta(\zeta) = \begin{cases} 1/\sqrt{\beta} & \text{if } \ n=0, \\ \sqrt{2/\beta}\cos(n\pi\zeta/\beta) & \text{if } \ n \geq 1. \end{cases}$$

Since $\alpha^2/\beta^2 \notin \mathbb{Q}$, we have $r_{mn} = 1, \ \forall\, m,n \geq 0$.

We suppose that the boundary source exciting the system is zonal and constant so that $g(t,\xi) \equiv g(\xi) \in L^2(\omega)$ for $\xi = (\xi_1,\xi_2) \in \omega$. Consequently, we have

$$g(\cdot,0) \in L^2(\omega_1), \quad g(\alpha,\cdot) \in L^2(\omega_2),$$

$$g(\cdot,\beta) \in L^2(\omega_3), \quad g(0,\cdot) \in L^2(\omega_4),$$

with

$$\begin{cases} \omega_1 = \Gamma_1 \cap \omega, & \omega_2 = \Gamma_2 \cap \omega, & \omega_3 = \Gamma_3 \cap \omega, & \omega_4 = \Gamma_4 \cap \omega, \\ \Gamma_1 =]0,\alpha[\times\{0\}, & \Gamma_2 = \{\alpha\}\times]0,\beta[, & \Gamma_3 =]0,\alpha[\times\{\beta\}, & \Gamma_4 = \{0\}\times]0,\beta[, \end{cases} \tag{57}$$

and g can be written down as

$$\begin{aligned} g(\xi_1,\xi_2) \ = \ \sum_{l\geq 0} \Big(& \mathbf{1}_{\Gamma_1}(\xi_1,\xi_2)\langle g(\cdot,0),\varphi_l^\alpha\rangle_{L^2(\omega_1)}\varphi_l^\alpha(\xi_1) \\ & + \mathbf{1}_{\Gamma_2}(\xi_1,\xi_2)\langle g(\alpha,\cdot),\varphi_l^\beta\rangle_{L^2(\omega_2)}\varphi_l^\beta(\xi_2) \\ & + \mathbf{1}_{\Gamma_3}(\xi_1,\xi_2)\langle g(\cdot,\beta),\varphi_l^\alpha\rangle_{L^2(\omega_3)}\varphi_l^\alpha(\xi_1) \\ & + \mathbf{1}_{\Gamma_4}(\xi_1,\xi_2)\langle g(0,\cdot)\varphi_l^\beta\rangle_{L^2(\omega_4)}\varphi_l^\beta(\xi_2)\Big), \end{aligned} \tag{58}$$

where

$$\mathbf{1}_{\Gamma_i}(\xi_1,\xi_2) = \begin{cases} 1 & \text{if } \ (\xi_1,\xi_2) \in \Gamma_i, \\ 0 & \text{otherwise.} \end{cases}$$

If the output is given by a zone sensor (D,h), then

$$(Q_\omega s)(t) = \sum_{k,l\geq 0} \langle h, \Psi_{kl}\rangle_{L^2(D)} \int_0^r e^{\lambda_{kl}(r-\tau)} \langle g(\tau), \Psi_{kl}\rangle_{L^2(\omega)} \,\mathrm{d}\tau \tag{59}$$

$$(Q_\omega^* z)(t) = \sum_{m,n\geq 0} \langle h, \Psi_{mn}\rangle_{L^2(D)} \int_t^T e^{\lambda_{mn}(\tau-t)} z(\tau)\,\mathrm{d}\tau p_\omega \Psi_{mn} \tag{60}$$

and

$$\begin{aligned}(Q_\omega^* Q_\omega)z(t) \; = \; & \sum_{m,n\geq 0}\sum_{k,l\geq 0} \langle h, \Psi_{mn}\rangle_{L^2(D)} \langle h, \Psi_{kl}\rangle_{L^2(D)} p_\omega \Psi_{mn} \\ & \times \int_t^T e^{\lambda_{mn}(r-t)} \int_0^r e^{\lambda_{kl}(r-\tau)} \langle g(\tau), \Psi_{kl}\rangle_{L^2(\omega)} \,\mathrm{d}\tau \,\mathrm{d}r \end{aligned} \tag{61}$$

with

$$\begin{aligned} p_\omega \Psi_{mn}(\xi_1,\xi_2) \; = \; & \sum_{i\in\{1,3\}} \mathbf{1}_{\omega_i}(\xi_1,\xi_2) R_i(n) \varphi_m^\alpha(\xi_1) \\ & + \sum_{i\in\{2,4\}} \mathbf{1}_{\omega_i}(\xi_1,\xi_2) R_i(m) \varphi_n^\beta(\xi_2), \end{aligned}$$

where

$$\begin{aligned} & R_1(m) = \sqrt{\frac{2}{\beta}}, && R_2(m) = \sqrt{\frac{2}{\alpha}}(-1)^m, \\ & R_3(m) = (-1)^m R_1(m), && R_4(m) = (-1)^m R_2(m) \quad \text{for } m \geq 1, \\ & R_1(0) = R_3(0) = \sqrt{\frac{1}{\beta}}, && R_2(0) = R_4(0) = \sqrt{\frac{1}{\alpha}}. \end{aligned}$$

4.3.1. Observations without Errors

In the case of observations without errors, the source s is given by

$$\Lambda_\omega s = Q_\omega^* y, \tag{62}$$

where $\Lambda_\omega = Q_\omega^* Q_\omega$ and y is the observation performed by the sensor (D,h). Therefore, for $i \in \{1,\dots,4\}$, the restriction $\Lambda_{\omega,i}$ of Λ_ω to Γ_i, is given by

$$\Lambda_{\omega,i} s = (Q_\omega^*)_i \, y$$

with

$$\begin{cases} (\Lambda_{\omega,i} s)(t) \; = \; \displaystyle\sum_{m,n\geq 0}\sum_{k,l\geq 0} \langle h, \Psi_{mn}\rangle_{L^2(D)} \langle h, \Psi_{k,l}\rangle_{L^2(D)} S_i(m,n) \\ \qquad\qquad\quad \times \displaystyle\int_t^T e^{\lambda_{mn}(r-t)} \int_0^r e^{\lambda_{kl}(r-\tau)} \langle g(\tau), \Psi_{kl}\rangle_\omega \,\mathrm{d}\tau \,\mathrm{d}r \\ (Q_\omega^*)_i \, y(t) \; = \; \displaystyle\sum_{m,n\geq 0} \langle h, \Psi_{mn}\rangle_{L^2(D)} S_i(m,n) \int_t^T e^{\lambda_{mn}(\tau-t)} y(\tau)\,\mathrm{d}\tau \end{cases}$$

and

$$S_i(m,n) = \begin{cases} R_i(n)\mathbf{1}_{\omega_i}\varphi_m^\beta & \text{if } i \in \{1,3\}, \\ R_i(m)\mathbf{1}_{\omega_i}\varphi_n^\beta & \text{if } i \in \{2,4\}. \end{cases}$$

Then for $i \in \{1,3\}$ (resp. for $i \in \{2,4\}$), by multiplying $\Lambda_{\omega,i}s = (Q_\omega^*)_i y$ by φ_k^α (resp. φ_k^β), $k \geq 0$, and for M large enough, $0 \leq k \leq M$, from (58) it follows that

$$\begin{aligned} \sum_{l,l'=0}^{M} \Big(&\langle g(\cdot,0), \varphi_{l'}^\alpha\rangle_{L^2(\omega_1)} \langle \varphi_l^\alpha, \varphi_{l'}^\alpha\rangle_{L^2(\omega_1)} \langle \Lambda_{\omega,i}\mathbf{1}_{\Gamma_1}\varphi_l^\alpha, \varphi_k^\alpha\rangle_{L^2(\omega_i)} \\ &+ \langle g(\alpha,\cdot), \varphi_{l'}^\beta\rangle_{L^2(\omega_2)} \langle \varphi_l^\beta, \varphi_{l'}^\beta\rangle_{L^2(\omega_2)} \langle \Lambda_{\omega,i}\mathbf{1}_{\Gamma_2}\varphi_l^\beta, \varphi_k^\alpha\rangle_{L^2(\omega_i)} \\ &+ \langle g(\cdot,\beta), \varphi_{l'}^\alpha\rangle_{L^2(\omega_3)} \langle \varphi_l^\alpha, \varphi_{l'}^\alpha\rangle_{L^2(\omega_3)} \langle \Lambda_{\omega,i}\mathbf{1}_{\Gamma_3}\varphi_l^\alpha, \varphi_k^\alpha\rangle_{L^2(\omega_i)} \\ &+ \langle g(0,\cdot), \varphi_{l'}^\beta\rangle_{L^2(\omega_4)} \langle \varphi_l^\beta, \varphi_{l'}^\beta\rangle_{L^2(\omega_4)} \langle \Lambda_{\omega,i}\mathbf{1}_{\Gamma_4}\varphi_l^\beta, \varphi_k^\alpha\rangle_{L^2(\omega_i)} \Big) = H_i^\omega(k) \end{aligned} \tag{63}$$

with

$$H_i^\omega(k) = \begin{cases} \langle (Q_\omega^*)_i\, y, \varphi_k^\alpha\rangle_{L^2(\omega_i)} & \text{if } i \in \{1,3\}, \\ \langle (Q_\omega^*)_i\, y, \varphi_k^\beta\rangle_{L^2(\omega_i)} & \text{if } i \in \{2,4\}, \end{cases}$$

To have an approximation of s, we have to calculate the coefficients

$$\langle g(\cdot,0), \varphi_{l'}^\alpha\rangle_{L^2(\omega_1)}, \quad \langle g(\alpha,\cdot), \varphi_{l'}^\beta\rangle_{L^2(\omega_2)}, \quad \langle g(\cdot,\beta), \varphi_{l'}^\alpha\rangle_{L^2(\omega_3)}, \quad \langle g(0,\cdot), \varphi_{l'}^\beta\rangle_{L^2(\omega_4)}$$

for $l' \in \{0,\ldots,M\}$. Then s is determined by solving (63).

Example 6. For $\alpha = 1$, $\beta = 2^{1/4}$, $\omega = \omega_2 \cup \omega_3 \cup \omega_4$ with $\omega_2 = \{\alpha\}\times]1/4,\beta[\subset \Gamma_2$, $\omega_3 =]0,1[\times\{\beta\} \subset \Gamma_3$, $\omega_4 \equiv \Gamma_4$, we consider the case when the output is given by a zone sensor (D,h) such that $D =]1/3,1/2[\times]1/8,1/6[$ and $h(\xi_1,\xi_2) = 1$. Let s be a source with the intensity g given by

$$g(\xi_1,\xi_2) = \begin{cases} \mathbf{1}_{]1/2,1[}(\xi_1,\xi_2)(-16\xi_2^3 + 36\xi_2^2 - 24\xi_2 + 5) + \mathbf{1}_{]1,\beta[}(\xi_1,\xi_2) & \text{on } \Gamma_2, \\ -2\xi_1^3 + 3\xi_1^2 & \text{on } \Gamma_3, \\ 0 & \text{otherwise.} \end{cases} \tag{64}$$

Its representation, for $M = 20$, is given in Fig. 3.

The calculated approximation $\bar{g}_\omega$ of g is given in Fig. 4. In order to compare $\bar{g}_\omega$ and g, we show their restrictions to Γ_i, $i \in \{1,2,3\}$ in Figs. 5, 6 and 7, respectively (g is represented by a dotted line and $\bar{g}_\omega$ is drawn using a solid line). ♦

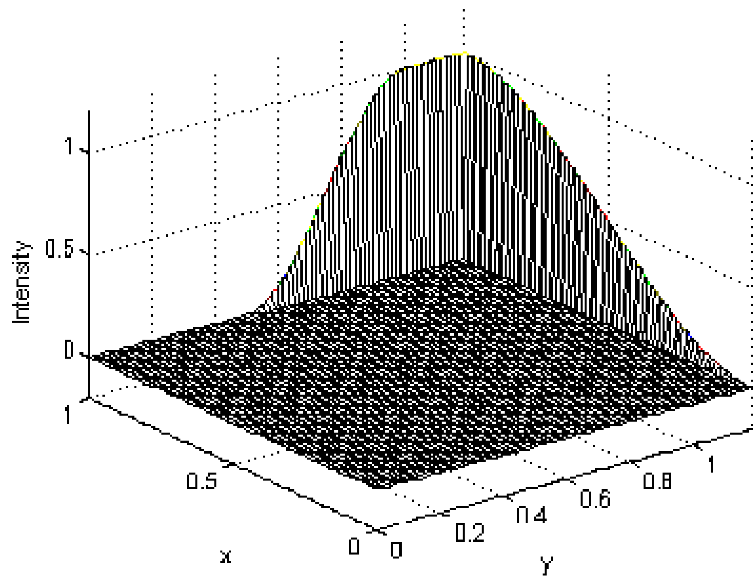

Fig. 3. Exact source intensity of Example 6.

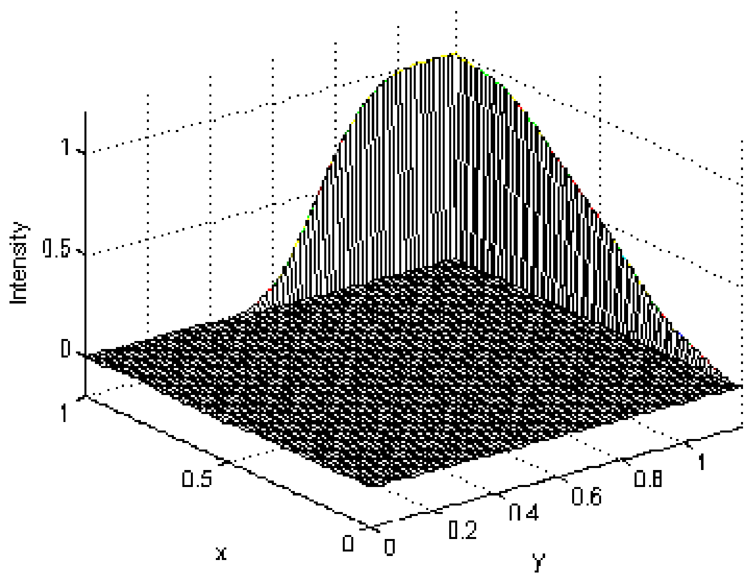

Fig. 4. Estimated source intensity.

4.3.2. Observations with Errors

In this part, we consider the system (47) disturbed by a zone constant source and augmented with the output

$$z(t) = \langle h, x(t) \rangle_{L^2(D)} + e_\omega(t), \quad t \in]0, T[, \tag{65}$$

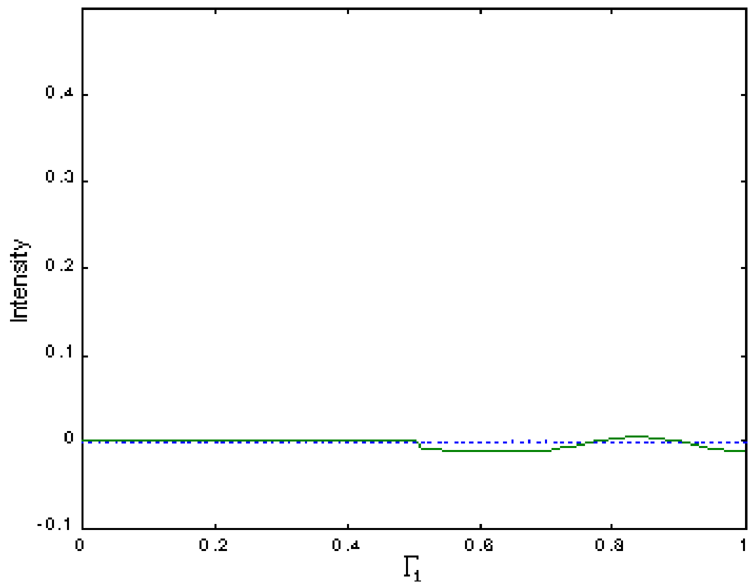

Fig. 5. Restriction of the solution from Example 6 to Γ_1.

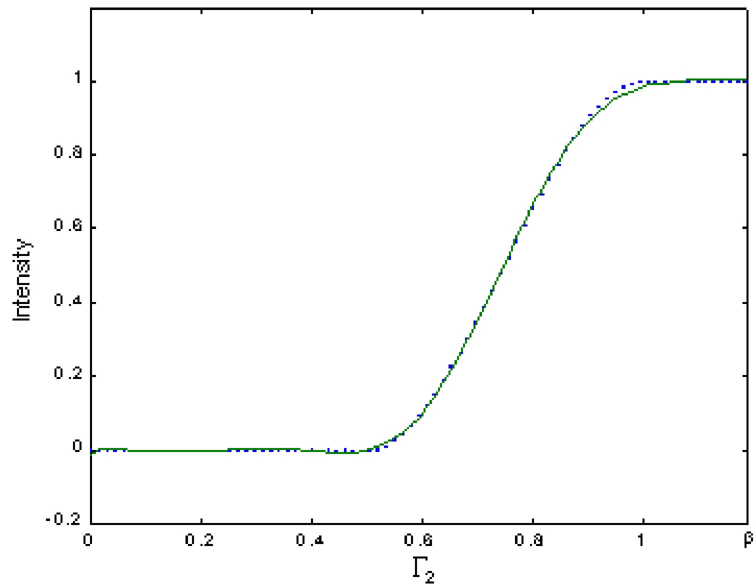

Fig. 6. Restriction of the solution from Example 6 to Γ_2.

where e_ω is an observation error. In this case, to have an approximation of s, we have to solve, for $0 \leq k \leq M$, the system (63) with

$$H_i^\omega(k) = \begin{cases} \langle (Q_\omega^*)_i\, y, \varphi_k^\alpha \rangle + \langle (Q_\omega^*)_i\, e_\omega, \varphi_k^\alpha \rangle & \text{if } \ i \in \{1,3\}, \\ \langle (Q_\omega^*)_i y, \varphi_k^\beta \rangle + \langle (Q_\omega^*)_i e_\omega, \varphi_k^\beta \rangle & \text{if } \ i \in \{2,4\}. \end{cases}$$

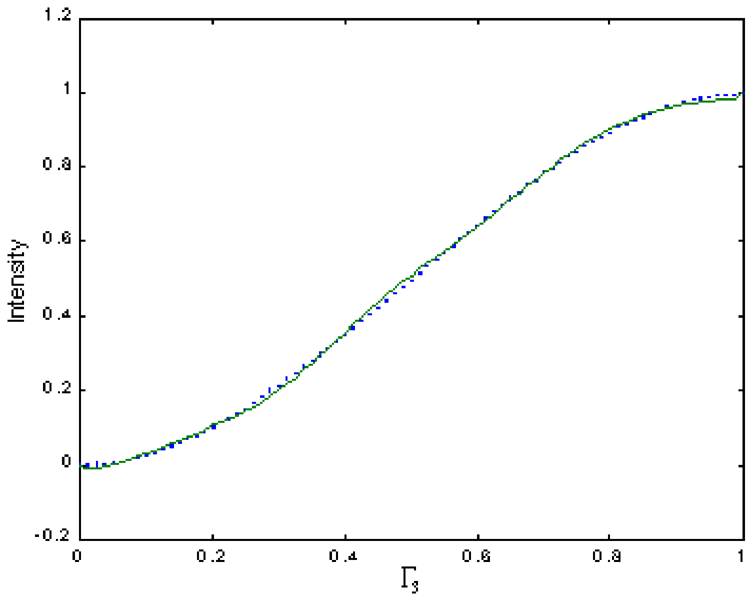

Fig. 7. Restriction of the solution from Example 6 to Γ_3.

Example 7. We consider the case of the region $\omega = \{\alpha\}\times]1/4, 10/13[\subset \Gamma_2$ and the zone sensor (D, h), with $D =]1/5, 2/5[\times]1/5, 2/5[$ and $h(\xi_1, \xi_2) = \xi_1 \sin(\pi\xi_2)$. Let g be an exact intensity of s and g_i be the estimated one corresponding to the error e_i, $i = 1, 3$. Then for $\alpha = 1$, $\beta = 2^{1/4}$, $e_1 = 0$, $e_2 = 10^{-6}$, $e_3 = 10^{-5}$, $M = 20$ and g such that

$$g(x, y) = \begin{cases} (-189y^2 + 189y - 42)\big(\mathbf{1}_{]1/3,2/3[}(y)\big) & \text{on } \Gamma_2, \\ 0 & \text{otherwise,} \end{cases}$$

the representations of g and g^i_ω, $i = 1, 3$ on Γ_2 are given in Fig. 8.

5. Conclusion

In this paper, we introduced the notion of regional detection and characterized internal or boundary regionally detectable sources, as well as the sensors ensuring their regional detection (regional spy sensors). Then we demonstrated that a source (internal or boundary) can be regionally detectable without being detectable in the whole domain, and hence that sensors may be ω-spies without being Ω-spies. We also showed how to reconstruct such sources from observations, with extensions to the case when the output is affected by an observation error. Numerical simulations were given in one- and two-dimensional spaces and various situations were examined. Finally, let us note that other aspects of the detection problem can be considered, and also, that the approach developed can be extented to other classes of systems.

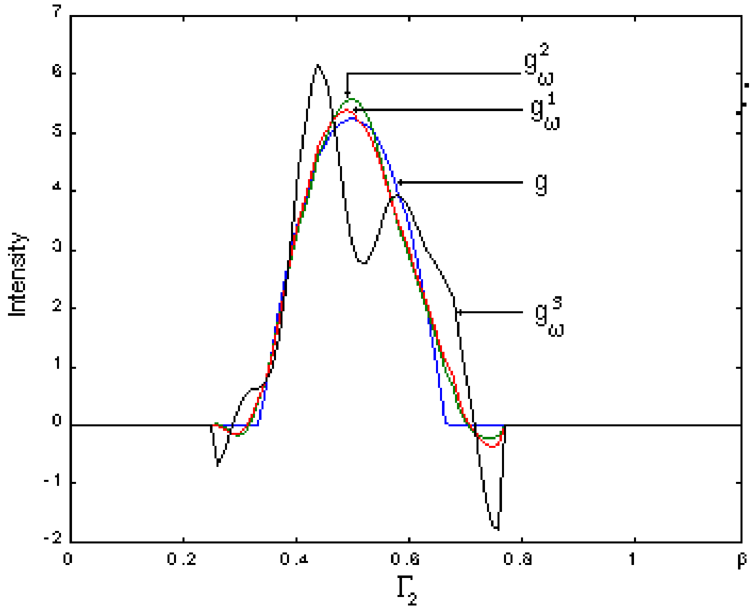

Fig. 8. Solutions to the reconstruction problem of Example 7.

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