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J-ENERGY PRESERVING WELL-POSED LINEAR SYSTEMS

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The following is a short survey of the notion of a well-posed linear system. We start by describing the most basic concepts, proceed to discuss dissipative and conservative systems, and finally introduce J-energy-preserving systems, i.e., systems that preserve energy with respect to some generalized inner products (possibly semi-definite or indefinite) in the input, state and output spaces. The class of well-posed linear systems contains most linear time-independent distributed parameter systems: internal or boundary control of PDE's, integral equations, delay equations, etc. These systems have existed in an implicit form in the mathematics literature for a long time, and they are closely connected to the scattering theory by Lax and Phillips and to the model theory by Sz.-Nagy and Foiaş. The theory has been developed independently by many different schools, and it is only recently that these different approaches have begun to converge. One of the most interesting objects of the present study is the Riccati equation theory for this class of infinite-dimensional systems (H^2- and H^∞-theories).

Keywords: well-posed linear system, system node, transfer function, Lax-Phillips semigroup, dissipative system, conservative system, model theory, conservative realization, J-energy-preserving system, Lyapunov equation, Riccati equation

1. Well-Posed Linear Systems

Many infinite-dimensional linear time-independent continuous-time systems can be described by the equations

$$\begin{aligned} x'(t) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) + Du(t), \quad t \geq 0, \\ x(0) &= x_0 \end{aligned} \tag{1}$$

on a triple of Hilbert spaces, namely, the input space U, the state space X, and the output space Y. We have $u(t) \in U$, $x(t) \in X$ and $y(t) \in Y$. The operator A is supposed to be the generator of a strongly continuous semigroup $t \mapsto \mathfrak{A}^t$. The

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generating operators A, B and C are usually unbounded, whereas D is always bounded.

By modifying this set of equations slightly we get the notion of a *well-posed linear system*. In the sequel, we think about the block matrix $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ as *one single (unbounded) operator* from $X \times U$ to $X \times Y$, and write (1) in the form

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0. \tag{2}$$

The operator S completely determines the system. Thus we may identify the system with such an operator, which we call the *node* of the system.

There are some *necessary conditions* which a node S must satisfy in order to generate a well-posed linear system. First of all, S must be closed and densely defined as an operator from $X \times U$ into $X \times Y$. Let us denote the domain of X by $\mathcal{D}(S)$. Then S can be split into $S = \left[\begin{smallmatrix} S_1 \\ S_2 \end{smallmatrix}\right]$, where S_1 maps $\mathcal{D}(S)$ into X and S_1 maps $\mathcal{D}(S)$ into Y. By analogy to the finite-dimensional case, let us set $S_1 = A\&B$ and $S_2 = C\&D$, so that $S = \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right]$ (the reader who finds this notation confusing may throughout replace $A\&B$ by S_1 and $C\&D$ by S_2). It is not true, in general, that $A\&B$ and $C\&D$ (defined on $\mathcal{D}(S)$) can be decomposed into $A\&B = [A \ B]$ and $C\&D = [C \ D]$; this is possible only in the case where $\mathcal{D}(S)$ can be written as the product of one subset of X times another subset of U. However, as we shall see below, certain extended versions of $A\&B$ and $C\&D$ can be decomposed as indicated above, so that *$A\&B$ and $C\&D$ are respectively the restrictions to $\mathcal{D}(S)$ of $[A \ B]$ and $[C \ D]$* for suitably defined operators A, B, C and D. The first of these two decompositions is more fundamental than the second, so we have incorporated it in the following definition of a *system node* (the second decomposition is commented on after Definition 3).

Definition 1. We call S a *system node* on the three Hilbert spaces (U, X, Y) if it satisfies condition (S) below:

(S) $S := \left[\begin{smallmatrix} A\&B \\ C\&D \end{smallmatrix}\right] : X \times U \supset \mathcal{D}(S) \to X \times Y$ is a closed linear operator. Here $A\&B$ is the restriction of $[A \ B]$ to $\mathcal{D}(S)$, where A is the *generator of a C_0 semigroup*, which has been extended to an operator in $\mathcal{L}(X; X_{-1})$, where X_{-1} is the completion of X under the norm $\|x\|_{X_{-1}} := \|(\alpha I - A)^{-1} x\|_X$ (α is an arbitrary number in the resolvent set of A). The operator B is an arbitrary operator in $\mathcal{L}(U; X_{-1})$, and $C\&D$ is an arbitrary linear operator from $\mathcal{D}(S)$ to Y. In addition, we require that

$$\mathcal{D}(S) = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in X \times U \mid Ax + Bu \in X \right\}.$$

In the sequel, we shall simply write $S = \left[\begin{smallmatrix} A & B \\ C\&D \end{smallmatrix}\right]$ and ignore the fact that $[A \ B]$ is defined on all of $X \times U$ (with values in X_{-1}) and not just on $\mathcal{D}(S)$ (with values

in X). Often X_{-1} is defined in a different but equivalent way as the dual of $\mathcal{D}(A^*)$ (we identify the dual of X with X itself). Let us also remark that we can replace the assumption that S is closed by the equivalent assumption that $C\&D$ *is continuous from* $\mathcal{D}(S)$ *(with the graph norm) to* Y.

As the following lemma shows, every system node induces a "dynamical system" of a certain type:

Lemma 1. *Let* S *be a system node on* (U, X, Y). *Then, for each* $x_0 \in X$ *and* $u \in W^{2,1}(\mathbb{R}^+; U)$ *with* $\left[\begin{smallmatrix} x_0 \\ u(0) \end{smallmatrix}\right] \in \mathcal{D}(S)$, *the equation*

$$\begin{bmatrix} \dot{x}(t) \\ y(t) \end{bmatrix} = S \begin{bmatrix} x(t) \\ u(t) \end{bmatrix}, \quad t \geq 0, \quad x(0) = x_0 \tag{3}$$

has a unique solution (x, y) *satisfying* $\left[\begin{smallmatrix} x(t) \\ u(t) \end{smallmatrix}\right] \in \mathcal{D}(S)$ *for all* $t \geq 0$, $x \in C^1(\mathbb{R}^+; X)$, *and* $y \in C(\mathbb{R}^+; Y)$.

Definition 2. By the *linear system* Σ generated by a system node S we understand the family Σ of maps defined by

$$\Sigma_0^t \begin{bmatrix} x_0 \\ \pi_{[0,t]} u \end{bmatrix} := \begin{bmatrix} x(t) \\ \pi_{[0,t]} y \end{bmatrix},$$

parametrized by $t \geq 0$, where x_0, $x(t)$, u, and y are as in Lemma 1, and $\pi_{[0,t]}u$ and $\pi_{[0,t]}y$ are respectively the restrictions of u and y to $[0, t]$. The *transfer function* of Σ is defined by

$$\widehat{\mathfrak{D}}(z) := C\&D \begin{bmatrix} (zI - A)^{-1}B \\ I \end{bmatrix}, \quad z \in \rho(A).$$

By taking *Laplace transforms* in (3) we find that if u is Laplace transformable with transform $\hat{u}$, then the output y is also Laplace transformable with transform

$$\hat{y}(z) = C(zI - A)^{-1}x_0 + \widehat{\mathfrak{D}}(z)\hat{u}(z)$$

for $\Re z$ large enough. Here

$$Cx := C\&D \begin{bmatrix} x \\ 0 \end{bmatrix} \text{ for } x \in \mathcal{D}(A).$$

Thus, our definition of the transfer function is equivalent to the standard definition in the classical case.

So far we have *defined* Σ_0^t *only for the class of smooth data* given in Lemma 1. In order to extend Σ_0^t to a larger class of data we need an extra *well-posedness assumption.*

Definition 3. A system Σ generated by a system node S is *well-posed* if the following additional condition holds:

(WP) For some $t > 0$ there is a finite constant $K(t)$ such that the solution (x, y) in Lemma 1 satisfies

$$\|x(t)\|^2 + \|y\|^2_{L^2(0,t)} \le K(t)\big(\|x_0\|^2 + \|u\|^2_{L^2(0,t)}\big).$$

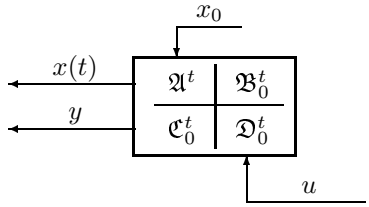

Fig. 1. Input/state/output diagram of Σ.

It is not difficult to show that if this condition holds for *one* $t > 0$, then it holds of *all* $t > 0$. If a system is well-posed, then Σ can be *extended by continuity* to a family of operators

$$\Sigma^t_0 := \left[\begin{array}{c|c} \mathfrak{A}^t & \mathfrak{B}^t_0 \\ \hline \mathfrak{C}^t_0 & \mathfrak{D}^t_0 \end{array}\right]$$

from $X \times L^2([0,t];U)$ to $X \times L^2([0,t];Y)$. (We still denote the extended family by Σ.)

As is shown in (Staffans and Weiss, 2002a), the system node S *of a well-posed system* can be always split into

$$S = \begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where $A\colon X \to X_{-1}$, $B\colon U \to X_{-1}$, $C\colon W \to Y$, and $D\colon U \to Y$. Here

$$\mathcal{D}(A) =: X_1 \subset W \subset X \subset X_{-1}$$

and $W = (\alpha I - A)^{-1}(X + BU)$ where $\alpha \in \rho(A)$. The operators A and B are unique, but C and D are *not always unique* (only the restriction of C to $X_1 = \mathcal{D}(A)$ is unique). The operators A, B and C may be *unbounded*, but D is always bounded. If Σ is well-posed, then $\widehat{\mathfrak{D}}$ *is bounded on some right half-plane*, and $\widehat{\mathfrak{D}}$ can be written in the familiar form

$$\widehat{\mathfrak{D}}(z) = C(zI - A)^{-1}B + D, \quad z \in \rho(A).$$

For more details, explanations and examples we refer the reader to (Adamajan and Arov, 1970; Arov, 1979; 1999, Arov and Nudelman, 1996; Curtain and Weiss, 1989; Salamon, 1987; 1989; Staffans, 1997; 1998a; 1998b; 1998c; 1999a; 2002; Staffans and Weiss, 2002a; 2002b; Weiss, 1989a; 1989b; 1989c; 1991; 1994a; 1994b; Weiss and

Weiss, 1997) and the references therein. Note that different groups of authors use different notations. For example, in an alternative set of notations (introduced by George Weiss) the system itself and the transfer function are denoted by

$$\Sigma_t := \left[\begin{array}{c|c} \mathbb{T}_t & \Phi_t \\ \hline \Psi_t & \mathbb{F}_t \end{array}\right] := \left[\begin{array}{c|c} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \hline \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{array}\right], \quad \mathbf{G}(s) := \widehat{\mathfrak{D}}(s),$$

respectively. In the notation of Grabowski and Callier (2001) the system node and the transfer function are respectively denoted by

$$\left[\begin{array}{cc} A & Ad \\ c^\sharp & 0 \end{array}\right] := S, \quad \hat{g}(s) := \widehat{\mathfrak{D}}(s).$$

Translating the notation used by Arov and Nudelman (1996) to our notation we obtain

$$\alpha := \Sigma, \quad \mathfrak{N}^- := U, \quad \mathfrak{N}^+ := Y,$$

$$\left[\begin{array}{cc} B & L \\ M & K \end{array}\right] := \left[\begin{array}{cc} A & B \\ C & D \end{array}\right], \quad N := C\&D, \quad \theta_\alpha(s) := \widehat{\mathfrak{D}}(-is).$$

In addition to the references mentioned above, there is a large range of relevant literature that *we do not cite explicitly*. This applies, in particular, to most of the *Russian literature*, represented by Adamjan, Arov, Brodskiĭ, Kreĭn, Livšic, Nudelman, Potapov, and Šmulijan, among others. Another large block of relevant results is found in the western *operator theory* and *scattering theory* group, represented by Ball, de Branges, Douglas, Lax, Fuhrman, Helson, Helton, Phillips, Rosenblum, and Rovnyak. A third group of missing results are those that have been developed in *stochastic identification theory*, represented by Byrnes, Georgiou, Gilliam, Lindquist, and Picci. (The above lists of researchers are far from being complete.)

2. Lax-Phillips Scattering

A generalized Lax–Phillips scattering model is a semigroup $\mathfrak{T}$ defined on

$$\mathcal{Y} \times X \times \mathcal{U} = L^2(\mathbb{R}^-; Y) \times X \times L^2(\mathbb{R}^+; U)$$

with certain additional properties. We call $\mathcal{U}$ the incoming subspace, X the central state space, and $\mathcal{Y}$ the outgoing subspace. In the classical cases treated in (Lax and Phillips, 1967; 1973), $\mathfrak{T}$ is required to be unitary (the conservative case) or to be a

contraction semigroup (the dissipative case). Below we use the following notation:

$$(\pi_J u)(s) := \begin{cases} u(s), & s \in J, \\ 0, & \text{otherwise.} \end{cases}$$

$$(\pi_+ u)(s) := \begin{cases} u(s), & s \in \mathbb{R}^+, \\ 0, & s \in \mathbb{R}^-, \end{cases}$$

$$(\tau^t u)(s) := u(t+s), \quad s,\ t \in \mathbb{R}.$$

$$\tau_+^t := \pi_+ \tau^t, \quad t \geq 0,$$

Theorem 1. *Let $\mathcal{Y} = L^2(\mathbb{R}^-; Y)$ and $\mathcal{U} = L^2(\mathbb{R}^+; U)$. For all $t \geq 0$ we define on $\mathcal{Y} \times X \times \mathcal{U}$ the operator $\mathfrak{T}^t$ by*

$$\mathfrak{T}^t = \begin{bmatrix} \tau^t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tau_+^t \end{bmatrix} \begin{bmatrix} I & \mathfrak{C}_0^t & \mathfrak{D}_0^t \\ 0 & \mathfrak{A}^t & \mathfrak{B}_0^t \\ 0 & 0 & I \end{bmatrix}. \tag{4}$$

Then $\mathfrak{T}$ is a strongly continuous semigroup. If x respectively y are the state trajectory and the output function of Σ with initial state $x_0 \in X$ and input function $u_0 \in \mathcal{U}$, and if we define $y(t) = y_0(t)$ for $t < 0$, then for all $t \geq 0$

$$\begin{bmatrix} \tau^t & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & \tau^t \end{bmatrix} \begin{bmatrix} \pi_{(-\infty,t]} y \\ x(t) \\ \pi_{[t,\infty)} u_0 \end{bmatrix} = \mathfrak{T}^t \begin{bmatrix} y_0 \\ x_0 \\ u_0 \end{bmatrix}. \tag{5}$$

Formula (5) shows that at any time $t \geq 0$ the first component of $\mathfrak{T}_t \left[\begin{smallmatrix} y_0 \\ x_0 \\ u_0 \end{smallmatrix}\right]$ represents the past output, the second component represents the present state, and the third component represents the future input.

The preceding theorem is taken from (Staffans and Weiss, 2002a), and it is also found in (Staffans, 2002). Special cases of this result (where either the input or the output is missing) appear in (Engel, 1998; Grabowski and Callier, 1996). The roots of Theorem 1 are very old, and the preceding references represent only *a fraction of all the relevant ones*. The setting which we describe above corresponds to the one found in *orthogonal scattering theory. Non-orthogonal* scattering has also been studied intensively. This case is important, for example, in the state space construction in stochastic identification theory.

3. Dissipative and Conservative Systems

Below we shall interpret the words "dissipative" and "conservative" in a rather restricted sense. Many authors use these words to represent (some subclass of) the more general classes of (R, P, J)-dissipative and (R, P, J)-conservative systems that

we define in Section 5. In particular, the following definition is a special case of the definitions in the two classical papers (Willems, 1972; 1972b).

Definition 4. A system Σ generated by a system node S is *dissipative* if the following *energy inequality* holds:

(D) For all $t > 0$, the solution (x, y) in Lemma 1 satisfies

$$\|x(t)\|^2 + \|y\|^2_{L^2(0,t)} \leq \|x_0\|^2 + \|u\|^2_{L^2(0,t)}.$$

Thus *every dissipative system is well-posed*: The dissipativity inequality (D) implies the well-posedness inequality (WP). Physically, dissipativity means that *there are no internal energy sources.*

Theorem 2. *The following conditions are equivalent:*

1. Σ *is dissipative.*
2. *The corresponding Lax–Phillips model is a contraction semigroup.*
3. *For all* $t > 0$*, the operator* $\Sigma_0^t = \left[\begin{smallmatrix} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{smallmatrix}\right]$ *is a contraction from* $X \times L^2([0,t);U)$ *to* $X \times L^2([0,t);Y)$.

Dissipativity can also be characterized by some algebraic operator inequalities involving the system node $S = \left[\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right]$ (which are **L**inear **M**atrix **I**nequalities in the finite-dimensional case). See (Staffans and Weiss, 2002a) for the general case and (Willems, 1972b) for the matrix case.

Definition 5. A system Σ generated by a system node S is *energy-preserving* if the following *energy balance equation* holds:

(E) For all $t > 0$, the solution (x, y) in Lemma 1 satisfies

$$\|x(t)\|^2 + \|y\|^2_{L^2(0,t)} = \|x_0\|^2 + \|u\|^2_{L^2(0,t)}.$$

Note that every energy-conserving system is dissipative, hence well-posed. Physically, a system is energy-preserving if *there are no internal energy sources or sinks.*

Definition 6. A system Σ generated by a system node S is *conservative* if *both the original system and the dual system are energy-preserving.*

The *dual system* is the one generated by S^*. (If S is a system node, then so is S^*.) A finite-dimensional system is *conservative if and only if it is energy-preserving and* $U = Y$. Some related (but more complicated) results are also true in infinite-dimensions.

Theorem 3. *The following conditions are equivalent:*

1. Σ *is conservative.*
2. *The corresponding Lax–Phillips model is a unitary semigroup.*
3. *For all* $t > 0$*, the operator* $\Sigma_0^t = \left[\begin{smallmatrix} \mathfrak{A}^t & \mathfrak{B}_0^t \\ \mathfrak{C}_0^t & \mathfrak{D}_0^t \end{smallmatrix}\right]$ *is a unitary operator from* $X \times L^2([0,t);U)$ *to* $X \times L^2([0,t);Y)$.

The preservation of energy can also be characterized by some algebraic operator identities involving the system node $S = [\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}]$: differentiating the energy balance equation with respect to t (by Lemma 1, this is possible) we find that, for all $[\begin{smallmatrix} x \\ u \end{smallmatrix}] \in \mathcal{D}(S) = \left\{ [\begin{smallmatrix} x \\ u \end{smallmatrix}] \in X \times U \;\middle|\; Ax + Bu \in X \right\}$,

$$\langle Ax + Bu, x\rangle_X + \langle x, Ax + Bu\rangle_X + \langle Cx + Du, Cx + Du\rangle_Y = \langle u, u\rangle_U. \quad (6)$$

(Here we have used the fact that we can write $C\&D\,[\begin{smallmatrix} x \\ u \end{smallmatrix}] = Cx + Du$ for $[\begin{smallmatrix} x \\ u \end{smallmatrix}] \in \mathcal{D}(S)$ since Σ is well-posed.) In the finite-dimensional case this set of equations decouples into the three independent equations

$$\begin{aligned} A + A^* + C^*C &= 0, \\ B + C^*D &= 0, \\ (B^* + D^*C &= 0,) \\ D^*D &= I. \end{aligned} \quad (7)$$

(The third equation above is the adjoint of the second.) In the infinite-dimensional case such a decoupling is much more difficult. This problem is discussed further in Section 6. The results presented in this section are taken from (Arov and Nudelman, 1996; Malinen *et al.*, 2002; Staffans and Weiss, 2002a; Weiss *et al.*, 2001), and they are also found in (Staffans, 2002).

4. The Universal Model of a Contraction Semigroup

There is a classical problem in mathematics:

Let $\mathfrak{A}$ *be an arbitrary contraction semigroup on some Hilbert space* X*. Is it always possible to find a* unitary dilation $\widetilde{\mathfrak{A}}$ *of* $\mathfrak{A}$ *defined on some larger space* $\widetilde{X}$*?*

By this we meant the following: X is a subset of $\widetilde{X}$, $\widetilde{\mathfrak{A}}$ is a unitary semigroup on $\widetilde{X}$, and for all $t \geq 0$ and $x \in X$,

$$\mathfrak{A}^t x = \pi_X \widetilde{\mathfrak{A}}^t x,$$

where π_X is the orthogonal projection of $\widetilde{X}$ onto X. We also say that $\mathfrak{A}$ is a *compression* of $\widetilde{X}$. The answer to this question is:

Theorem 4. *Given an arbitrary contraction semigroup* $\mathfrak{A}$ *on a Hilbert space* X*, it is always possible to find a conservative system* Σ *whose semigroup is the given semigroup* $\mathfrak{A}$*. The corresponding Lax-Phillips semigroup is a unitary dilation of* $\mathfrak{A}$*. The system* Σ *is* unique *(modulo unitary similarity transformations in the input and output spaces) if we require* $\widehat{\mathfrak{D}}$ *to be strictly contractive.*

Strict contractivity of $\widehat{\mathfrak{D}}$ means that there is no nontrivial subspace of U on which $\widehat{\mathfrak{D}}$ reduces to an isometric constant. This additional restriction on the transfer function is a *minimality requirement*: by factoring out $\mathcal{N}(B)$ from U and $\mathcal{R}(C)^{\perp}$ from Y we can always reduce a conservative system to a system whose transfer function is strictly contractive. The corresponding Lax-Phillips semigroup is then a *minimal unitary dilation* of $\mathfrak{A}$.

The preceding theorem also has a "converse":

Theorem 5. *Every contractive causal shift invariant operator $\mathfrak{D}$ from $L^2(\mathbb{R}^+;U)$ to $L^2(\mathbb{R}^+;Y)$ has a conservative realization, i.e., there exists a conservative system Σ with this input/output map. The system Σ is* unique *(modulo unitary similarity transformations in its state space) if we require $\mathfrak{A}$ to be completely non-unitary.*

Complete non-unitarity of the semigroup $\mathfrak{A}$ means that there is no nontrivial reducing subspace on which $\mathfrak{A}$ is unitary. Again the requirement that $\mathfrak{A}$ is completely non-unitary is a *minimality requirement*: We can always reduce a conservative system to a system with a completely non-unitary semigroup by factoring out the intersection of the unreachable and unobservable subspaces.

By combining the preceding results with a further representation result for the Lax-Phillips semigroup corresponding to a conservative system we arrive at the following universal model of a completely non-unitary contraction semigroup:

Theorem 6. *Every completely non-unitary contraction semigroup $\mathfrak{A}$ on some Hilbert space X is unitarily equivalent to a compression of the (bilateral) shift operator on some subspace $\widetilde{X}$ of $L^2(\mathbb{R};Z)$ (and there are formulas for how to find the space Z and the subspace $\widetilde{X}$).*

The results presented in this section are taken from the book (Sz.-Nagy and Foiaş, 1970), and the same book also contains a wealth of additional material on conservative systems, including results on *strong stability, controllability, observability*, and *transfer functions being inner* (or co-inner or bi-inner). This book is written primarily in *discrete time*, but the results apply equally well in continuous time. Recall that well-posedness does not cause any problems when the systems are *dissipative*. The *Cayley transform* maps a well-defined subclass of all discrete time dissipative (or conservative) systems *one-to-one onto* the class of continuous time dissipative (or conservative) linear systems, and it preserves controllability, observability, strong stability, inner transfer functions, etc. In other words, it preserves almost all the important properties of a system. The only exceptions are exponential stability and all properties related to the behavior of the system over a finite time interval (such as exact controllability/observability in finite time).

Of course, there is also a fair amount of additional literature on this subject, in particular, a huge *Russian literature*. See (Weiss and Tucsnak, 2001) for a detailed description of a particular case of the general construction presented in Theorem 4. Theorem 5 is also found in (Arov and Nudelman, 1996) (in continuous time). Theorem 6 is given in (Sz.-Nagy and Foiaş, 1970) (both in discrete and continuous time). A

modern presentation of the results presented up to now (in continuous time) appears in the book manuscript (Staffans, 2002).

5. (*R*,*P*,*J*)-Energy Preserving Systems

The standard energy inequality of a dissipative system can also be written with the help of inner products:

$$\langle x(t), x(t)\rangle_X + \int_0^t \langle y(s), y(s)\rangle_Y \,\mathrm{d}s \le \langle x_0, x_0\rangle_X + \int_0^t \langle u(s), u(s)\rangle_U \,\mathrm{d}s.$$

We can make this equation more general by introducing self-adjoint cost operators: the input cost operator $R\colon U \to U$, the state cost operator $P\colon X \to X$, and the output cost operator $J\colon Y \to Y$.

Definition 7. A system Σ generated by a system node S is (R,P,J)-*dissipative* if the following (R,P,J)-*energy inequality* holds:

(JD) For all $t > 0$, the solution (x,y) in Lemma 1 satisfies

$$\langle x(t), Px(t)\rangle_X + \int_0^t \langle y(s), Jy(s)\rangle_Y \,\mathrm{d}s \le \langle x_0, Px_0\rangle_X + \int_0^t \langle u(s), Ru(s)\rangle_U \,\mathrm{d}s.$$

The system Σ is (R,P,J)-*energy preserving* if the above inequality holds as an equality:

(JE) For all $t > 0$, the solution (x,y) in Lemma 1 satisfies

$$\langle x(t), Px(t)\rangle_X + \int_0^t \langle y(s), Jy(s)\rangle_Y \,\mathrm{d}s = \langle x_0, Px_0\rangle_X + \int_0^t \langle u(s), Ru(s)\rangle_U \,\mathrm{d}s.$$

Finally, Σ is (R,P,J)-*conservative* if both Σ and the dual system Σ^d are (R,P,J)-energy preserving.

It is important to observe that *in the above cases well-posedness is neither guaranteed, nor necessarily relevant.* This possible lack of well-posedness causes some additional difficulties, but it is still possible to extend some of the results mentioned in Section 3 to the class of systems described in Definition 7. For simplicity, let us restrict the discussion to the case where the system is (R,P,J)-energy-preserving. Differentiating the (R,P,J)-energy balance equation with respect to t (and using Lemma 1), we obtain the following *Lyapunov equation*:

$$\begin{aligned}\langle Ax+Bu, Px\rangle_X + \big\langle x, P(Ax+Bu)\big\rangle_X \\ + \left\langle C\&D \begin{bmatrix} x \\ u \end{bmatrix}, J[C\&D] \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle_Y = \langle u, Ru\rangle_U,\end{aligned} \tag{8}$$

valid for all $\left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] \in \mathcal{D}(S)$. In the finite-dimensional case this set of equations decouples into the three independent equations

$$\begin{aligned} A^*P + PA + C^*JC &= 0, \\ PB + C^*JD &= 0, \\ (B^*P + D^*JC &= 0,) \\ D^*JD &= R. \end{aligned} \tag{9}$$

Again, in the infinite-dimensional case such a decoupling is much more difficult (see the discussion in Section 6).

(R,P,J)-energy-preserving systems appear in optimal control, for example, in minimization problems (H^2-optimal control) and in minimax problems (H^∞-optimal control). The operator J is known, but the original system is not energy-preserving in any sense. Instead, we want to find a feedback operator K, an input cost operator R, and a state cost operator (Riccati operator) P such that the corresponding *closed loop system is* (R,P,J)*-energy-preserving*. By this we mean the following: We add another combined observation/feedthrough operator $K\&L$ to the bottom of the system node S (similar to the original $C\&D$), and we then feed this output back into the system, i.e., we take u to be of the form

$$u = v + K\&L \begin{bmatrix} x \\ u \end{bmatrix},$$

where v is the closed loop input. To simplify the discussion, let us assume that we can split $K\&L$ into $K\&L = [K \;\; L]$, and that we can take $L = 0$ (and let us also ignore the fact that the splitting of $K\&L$ into $[K \;\; L]$ need not be unique). Then $K\&L \left[\begin{smallmatrix} x \\ u \end{smallmatrix}\right] = Kx$, so that

$$u = v + Kx.$$

As we have mentioned above, the operator K should be chosen in such a way that the closed loop system is (R,P,J)-energy-preserving for some (unknown) cost operators R and P. By replacing u in the Lyapunov equation (8) by $v = u - Kx$ we get the *Riccati equation*

$$\begin{aligned} &\langle Ax + Bu, Px\rangle_X + \big\langle x, P(Ax + Bu)\big\rangle_X \\ &\qquad + \left\langle C\&D \begin{bmatrix} x \\ u \end{bmatrix}, J[C\&D] \begin{bmatrix} x \\ u \end{bmatrix} \right\rangle_Y = \big\langle u - Kx, R(u - Kx)\big\rangle_U, \end{aligned} \tag{10}$$

which is valid for all $\left[\begin{smallmatrix} x \\ u-Kx \end{smallmatrix}\right] \in \mathcal{D}(S)$. In the finite-dimensional case this set of equations decouples into the three equations

$$\begin{aligned} A^*P + PA + C^*JC &= K^*RK, \\ PB + C^*JD &= -K^*R, \\ (B^*P + D^*JC &= -RK,) \\ D^*JD &= R. \end{aligned} \tag{11}$$

It is especially in the case of the positive real lemma (which will be discussed below) that the system (11) is often referred to as the *Popov-Kalman-Szegö-Yakubovic* system. In the infinite-dimensional case the interpretation of these equations raises some serious questions.

By appropriately choosing R, P and J we get many of the standard one-block and two-block "optimal control" results. This time we work with the extended system

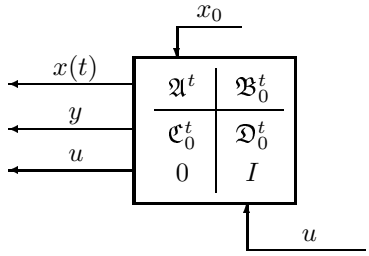

Fig. 2. Input added to output of Σ.

in Fig. 2, where we have added a copy of the input to the output (in addition, the system used in the full information H^∞ problem has two inputs instead of one):

1. $y := \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, $C := \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, $D := \left[\begin{smallmatrix} 0 \\ I \end{smallmatrix}\right]$, $J := \left[\begin{smallmatrix} I & 0 \\ 0 & R \end{smallmatrix}\right]$. This is the *standard LQR Riccati equation*. Here $R \geq 0$ is the same operator which appears in the definition of J, and both $R \geq 0$ and $J \geq 0$. We further require $P \geq 0$, and (11) becomes

$$A^*P + PA + C^*C = K^*RK,$$

$$RK = -B^*P.$$

Usually R is assumed to be invertible, in which case we can eliminate K to get the standard regulator Riccati equation

$$A^*P + PA + C^*C = PBR^{-1}B^*P.$$

2. $y := \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, $C := \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, $D := \left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right]$, $J := \left[\begin{smallmatrix} I & 0 \\ 0 & I \end{smallmatrix}\right]$. This is the *normalized coprime factorization problem*. We still have $J \geq 0$, $P \geq 0$, $R \geq 0$, and (11) becomes

$$A^*P + PA + C^*C = K^*RK,$$

$$RK = -(B^*P + D^*C),$$

$$R = I + D^*D.$$

Clearly, R is invertible, and we may eliminate R and K to get the normalized coprime factorization Riccati equation

$$A^*P + PA + C^*C = (PB + C^*D)(I + D^*D)^{-1}(B^*P + D^*C).$$

3. $y := \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, $C := \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, $D := \left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right]$, $J := \left[\begin{smallmatrix} I & 0 \\ 0 & -\gamma^2 I \end{smallmatrix}\right]$. This is the *bounded real lemma*. Here J is indefinite, and (11) becomes

$$A^*P + PA + C^*C = K^*RK,$$

$$RK = -(B^*P + D^*C),$$

$$R = D^*D - \gamma^2 I.$$

Usually γ is chosen so that $R \leq 0$ is invertible and $P \geq 0$. Eliminating R and K from the above system we get the bounded real lemma Riccati equation

$$A^*P + PA + C^*C = -(PB + C^*D)(\gamma^2 I - D^*D)^{-1}(B^*P + D^*C).$$

4. $y := \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, $C := \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, $D := \left[\begin{smallmatrix} D \\ I \end{smallmatrix}\right]$, $J := -\left[\begin{smallmatrix} 0 & I \\ I & 0 \end{smallmatrix}\right]$: This is the *positive real lemma*. The operator J is again indefinite, and (11) becomes

$$\begin{aligned} A^*P + PA &= K^*RK, \\ RK &= -B^*P + C, \\ R &= -(D + D^*). \end{aligned}$$

Clearly, $R \leq 0$, and we require that $P \geq 0$. If $D + D^*$ is invertible, then we can eliminate R and K to get the positive real lemma Riccati equation

$$A^*P + PA + C^*C = -(PB - C)(D + D^*)^{-1}(B^*P - C).$$

5. $u := \left[\begin{smallmatrix} w \\ u \end{smallmatrix}\right]$, $y := \left[\begin{smallmatrix} y \\ u \end{smallmatrix}\right]$, $B := \begin{bmatrix} B_1 & B_2 \end{bmatrix}$, $C := \left[\begin{smallmatrix} C \\ 0 \end{smallmatrix}\right]$, $D := \left[\begin{smallmatrix} D_{11} & D_{12} \\ 0 & I \end{smallmatrix}\right]$, $J := \begin{bmatrix} I & 0 \\ 0 & -\gamma^2 I \end{bmatrix}$. This is the H^∞ *full information problem*. In this problem $P \geq 0$, but J and R are indefinite. In the special case where $D_{12} = 0$, the system (11) becomes (the feedback operator K has two components $K = \left[\begin{smallmatrix} K_1 \\ K_2 \end{smallmatrix}\right]$ since we now have two inputs)

$$A^*P + PA + C^*C = \begin{bmatrix} K_1^* & K_2^* \end{bmatrix} R \begin{bmatrix} K_1 \\ K_1 \end{bmatrix},$$

$$R \begin{bmatrix} K_1 \\ K_1 \end{bmatrix} = -\left(\begin{bmatrix} B_1^* \\ B_2^* \end{bmatrix} P + \begin{bmatrix} D_{11}^* \\ D_{12}^* \end{bmatrix} C \right),$$

$$R = \begin{bmatrix} D_{11}^* D_{11} & 0 \\ 0 & -\gamma^2 I \end{bmatrix}.$$

(If $D_{12} \neq 0$, then the solution contains an additional feed-forward term from the disturbance to the control input.) For further comments on this example we refer the reader to standard textbooks on H^∞ control.

One important problem which is missing from the above list is the *Nehari problem*. It is related to the full information H^∞-problem discussed above, but it does not quite fit into the general framework of this section (a minimax approach to this problem is presented in (Staffans and Mikkola, 1998)). Another related problem is the

computation of the *n-th singular value* of the Hankel operator. In the last problem even the appropriate Riccati operator P is indefinite.

6. Open Questions and Some Solutions

Presently research is going on to extend the finite-dimensional Riccati equation theory to the setting of an infinite-dimensional well-posed linear system. Much has been done, but even more remains to be done. There are several problems which make the infinite-dimensional theory significantly more difficult than the finite-dimensional one. The first question to ask is: *What do we really want out of the Riccati equation*? Depending on the answer (optimal control, spectral factorization, computation of invariant subspaces, optimal identification, etc.), the "correct formulation" of the problem might be different. Once we have decided on the purpose of the Riccati equation, we are faced with the problem that *the closed loop system may not be a system*, or it may *not be well-posed*. If it is a system, then *what is the "correct form" of the Riccati equation*? For example, to compute the feedback operator K from the equation $K = -R^{-1}(B^*P + D^*JC)$ we need to know for which $x \in \mathcal{D}(C)$ it is true that $Px \in \mathcal{D}(B^*)$. In addition, the *non-uniqueness of the splitting of the observation/feedthrough operators* $C\&D$ into $[C\ D]$ and $K\&L$ into $[K\ L]$ comes into play. All the operators A, B, C, D, and K appear in the Riccati equation (L is supposed to be zero), but which are the "correct" versions of these operators? Can we always take $L = 0$? Is there always a "correct splitting" of $C\&D$ and $K\&L$ which makes the Riccati equation valid? Such a splitting does not depend only on the system itself, but also on the given cost operator J.

The difficulties described above have been approached in different ways. Some results on the "*correct splitting*" of $C\&D$ into $[C\ D]$ (i.e., a splitting which makes the Riccati equation valid with $L = 0$) are found in (Flandoli *et al.*, 1988) (this reference does not require the optimal closed loop system to be well-posed). If $\widehat{\mathfrak{D}}(\infty) = \lim_{\lambda\to+\infty} \widehat{\mathfrak{D}}(\lambda)$ exists, then the system is called *regular*, and it is possible to take $D = \widehat{\mathfrak{D}}(\infty)$. In general, this splitting is not compatible with the Riccati equation, and we have to add some correction terms to the Riccati equation, similar to those seen in the discrete time theory. In particular, the formula for R becomes more complicated. See (Mikkola, 2002; Staffans, 1995; 1996; 1997; 1998b; 1998c; 1998d; 1999b; Weiss and Weiss, 1997). These results are largely based on *spectral factorization*, and so are those in (Callier and Winkin, 1990; 1992; 1999; Grabowski, 1991; 1993; 1994). When the closed loop system is not well-posed, then we can use the (non-well-posed) *compensators with internal loop* introduced in (Curtain *et al.*, 1997) (see (Mikkola, 2002)). Another approach is to use the *Cayley transform* to map the continuous time system into a discrete time system and use the discrete time Riccati equation theory (joint work with Malinen is in progress). In discrete time all the operators become bounded, but some extra correction terms enter the Riccati equations. In many cases *additional smoothing properties* have been used (Pritchard-Salamon and parabolic cases); see, e.g., (Curtain and Ichikawa, 1996; Curtain and Oostveen, 1998; Lasiecka and Triggiani, 2000; van Keulen, 1993; Oostveen, 2000; Sasane and Curtain, 2001; 2001; Weiss, 1997).

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