DYNAMICS OF SOCIAL NETWORKS: A DETERMINISTIC APPROACH

DAVID W. PEARSON*, MARK MCCARTNEY**

* EURISE (Roanne Research Group), University Jean Monnet, Saint Etienne
I.U.T. de Roanne, 20, avenue de Paris, 42334 Roanne, France
e-mail: `david.pearson@univ-st-etienne.fr`

* School of Computing and Mathematical Sciences, University of Ulster
Newtownabbey, County Antrim, Northern Ireland
e-mail: `m.mccartney@ulst.ac.uk`

Our aim is to model the dynamics of social networks, which comprises the problem of how people get to know each other, like each other, detest each other, etc. Most existing models are stochastic in nature and, obviously, based on random events. Our approach is deterministic and based on ordinary differential equations. This should not be seen as a challenge to stochastic models, but rather as a complement.

Keywords: social networks, dynamic systems, quantitative sociodynamics

1. Introduction

Social networks can be many things. For example, they can be simply a set of people who get to know each other, like children in a new class at school. They can be groups of people who already know each other and we want to look at how their relationships develop in time, whether they like each other or hate each other. We can also look at a group of people in one room who need to make a decision, how people negotiate with others, try to convince others that their choice is the right one. In general, a social network is a set of variables concerning a set of people in a social environment. The variables can be psychological, social or other, and our main interest is modelling the dynamic behaviour of these variables and the numerical simulation of the network.

Sociodynamics is a relatively new and exciting area of research, combining theoretical notions from mathematics, statistics, informatics and physics applied to the modelling of dynamic social phenomena (Gilbert and Doran, 1994; Helbing, 1995; Weidlich, 2000). Researchers in the field have been successful in modelling phenomena such as traffic (or rather driver) behaviour, pedestrian behaviour, human population migration, etc. What is equally interesting is that these methods are not only used to model modern "observed" societies. They are also used to give us more insight into human social behaviour in ancient societies (Gilbert and Doran, 1994).

Most models of social networks are stochastic in nature. In fact, the inspiration for the current work was drawn from (Snijders and van Duijn, 1997), who proposed a stochastic model. The reader can also consult the references cited in this paper for other ideas and stochastic models. What we would like to do is, in a way, to simulate the behaviour of the model presented in (Snijders and van Duijn, 1997) with a deterministic model based on differential equations. Of course, the differences between deterministic differential equations and discrete-time stochastic models refrain us from obtaining equivalent behaviour. However, we are not striving to simulate the stochastic model exactly. That would be rather silly because we might as well just use the stochastic model. Our approach is rather to look at some of the principal points and the temporal behaviour of the stochastic model and to see if we can recreate them with our model. Having thus formed a basic model, we hope to enrich it and to introduce other possibilities.

It is important to note that we are not challenging the stochastic approach, we are simply looking for a different approach which could, we hope at least, complement the stochastic one. It must be said that we do not believe that human behaviour, individual or collective, is random. Stochastic models are used when we cannot model or even observe all the phenomena, and they are very successful. Our interest is directed towards a deterministic approach because we would like to get as close as possible to real psychological and social phenomena which, as has been stated previously, we do not perceive as random in nature.

As far as we are aware, very few researchers have sought a deterministic approach to the problem of social system modelling. Most of the work done in this area is based on more traditional methods taught to psychologists and social scientists, i.e. hypothesis testing, χ^2, etc., where the dynamics of networks are not really taken into account. The innovative work of the last two decades into sociodynamics has been carried out mainly by physicists mostly interested in applying stochastic models originating from physics.

Our own work was originally inspired by Kurt Lewin, a social psychologist who emigrated from Germany to the United Sates before World War II. Perhaps his most famous work published in English was (Lewin, 1936). A more recent example of work by social scientists in the area of dynamic behaviour is (Barber, 1992). The main objective of our research is to use social/psychological models such as those presented in (Barber, 1992) and to develop mathematical models from them. Having read the works of and worked with social scientists and psychologists, we hope that our mathematical models will be as close to reality as possible.

We have already sought an approach based on fuzzy logic (Pearson and Dray, 2001) and we are still continuing our work along these lines. We hope that the concepts presented in this paper, based on straightforward differential equations, will be linked together with a fuzzy logic model in order to give a greater depth, richness and reality to our method.

The paper is split into four sections. In Section 2 we present the mathematical model and the reasoning behind it. The model is analysed in Section 3. Some simulation examples are presented in Section 4 before summing up in Section 5.

2. Mathematical Model

A social network can be represented graphically as in Fig. 1. The individuals in the network are represented by the nodes, and the arcs represent the relations between the individuals.

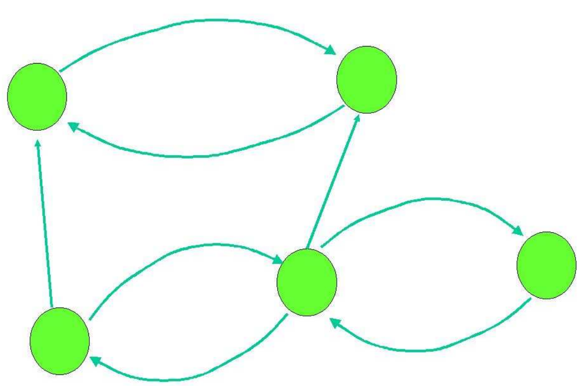

Fig. 1. Representation of a social network.

We remark that individuals can be assumed to mean individual people or indeed individual *groups* of people, although this does not concern us in our present work. The arcs are directed because, as is the case in Fig. 1, the relations are not necessarily reciprocal.

For a given set of individuals we want to model the dynamic behaviour of the relations and thus the configuration of the arcs in Fig. 1. Assume that there are n individuals in the network, the configuration matrix associated with the network being simply an $n \times n$ matrix $X = [x_{ij}]$ of zeros and ones. The value of 1 in row i and column j indicates that there is an arc from i to j, and that of 0 means that there is no arc. In this paper we do not consider self relations, such as self-confidence or self-esteem, therefore we set $x_{ii} = 0$ for $i = 1, \dots, n$. In fact, we study self relations in another related work (Pearson and Boudarel, 2001).

In (Snijders and van Duijn, 1997) the dynamic behaviour of a configuration matrix X is modelled as a discrete sequence X_t, $t = 0, 1, \dots$, where all the matrices are zero/one matrices with zero diagonal elements. The authors of this paper consider the sequence to be embedded in a continuous Markov process. Our idea is similar in that we develop a system of differential equations which defines a continuous process. A sequence of matrices is then provided by taking the initial matrix, which does not necessarily have to be a zero/one matrix, integrating the equations for a sufficiently long time so that the system reaches an equilibrium, and then taking the equilibrium point to be the next matrix in the sequence. The problem is therefore that of convergence and making sure that all the equilibrium points of the system correspond to valid zero/one matrices.

Rather than directly dealing with the entire matrix X, we simplify the matter by making use of the fact that the diagonal elements are all zeroes. For new coordinates we simply apply the off-diagonal elements in some order. The order is not important mathematically, but it may be important computationally when dealing with indexing. For example, we could choose the lexicographical order, working column by column for each row in turn. Thus, e.g., if $n = 3$, we would have

$$x^1 = x_{12}, \quad x^2 = x_{13}, \quad x^3 = x_{21},$$
$$x^4 = x_{23}, \quad x^5 = x_{31}, \quad x^6 = x_{32},$$

where the x^k's are the new coordinates and the x_{ij}'s the old ones.

For n individuals in the network there will be $m = n(n-1)$ variables x^k. Let I denote the unit interval $[0, 1]$. Our aim is that each of the variables should converge to a vertex of I, thus indicating whether an arc exists or not. We denote by I^m the m-dimensional unit hypercube and by 2^m the set of points corresponding to its vertices, simply because there are numerically 2^m of these points.

We begin the development of our model with a simple differential equation in one variable, $\dot{z} = f(z)$, where $\dot{z} = \mathrm{d}z/\mathrm{d}t$. First of all, we look for a function f such that $f(0) = 0$, $f(1) = 0$ and such that $f'(0)$ and $f'(1)$ are both negative, where $f' = \mathrm{d}f/\mathrm{d}z$. This merely guarantees that 0 and 1 are sinks for the differential equation. There are clearly numerous choices for such a function. We have chosen a polynomial because of ease of manipulation and numerical properties. Here and in the rest of this paper we fix the following form for f:

$$f(z) = -\alpha z + (2\alpha + \beta)z^2 - (\alpha + \beta)z^3. \tag{1}$$

The reader can easily verify that the above conditions are satisfied with $f'(0) = -\alpha$ and $f'(1) = -\beta$. Due to the fact that f in (1) is a cubic polynomial with three real roots with negative derivatives at the two roots 0 and 1, the third root will be found in the interval $(0, 1)$, where the derivative will be positive. Thus there will be an unstable fixed point somewhere between 0 and 1, dependent on the values chosen for the parameters α and β. In the remainder of the paper we assume that the third root is at the point $r \in (0, 1)$ and that $f'(r) = \gamma$, where $\gamma > 0$. Since this third fixed point is unstable, it does not provent us from proceeding with our modelling. In fact, it somewhat enriches the model. Two examples of the function (1) are shown in Fig. 2, in the top image the parameters are set to $\alpha = 1$, $\beta = 2$ and in the bottom image they are $\alpha = 3$, $\beta = 2$.

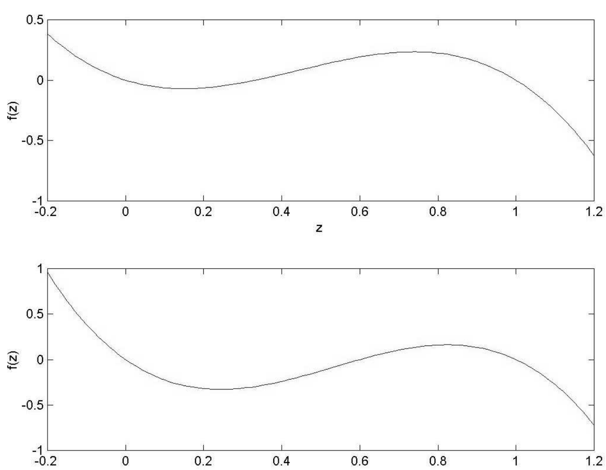

Fig. 2. Examples of the function f.

Now let the variable x be the vector containing the social network variables x^i. We extend the function f to a mapping ϕ via

$$\phi(x) = \begin{bmatrix} f(x^1) \\ f(x^2) \\ \vdots \\ f(x^m) \end{bmatrix}. \tag{2}$$

Now, to model interactions between the individuals, we introduce a matrix A with elements being constants or functions of x. The state vector x varies dynamically as a function of all the interactions and so, in the vector form, our model is

$$\dot{x} = A(x)\phi(x). \tag{3}$$

We now take a somewhat closer look at the matrix A in (3). First of all, the simplest model would be the one with a constant matrix A. Consider the i-th row of (3) in this case, i.e.

$$\dot{x}^i = a_{ij} f^j,$$

where we have simplified the notation by letting $f^j = f(x^j)$ and by using the summation convention on repeated indices, i.e.

$$a_{ij} f^j = \sum_{j=1}^{m} a_{ij} f^j.$$

We could interpret the constants a_{ij} as being a sort of a desire for the link to be created in the network, with the function values f^j indicating an attractive force either towards or against the creation of the link. In the simplest example of two individuals, we have $x^1 = x_{12}$ and $x^2 = x_{21}$, and so the system reads as

$$\dot{x}^1 = a_{11} f^1 + a_{12} f^2,$$

$$\dot{x}^2 = a_{21} f^1 + a_{22} f^2.$$

In the first row, the diagonal term $a_{11} f^1$ could represent some innate desire of the first individual to create a link with the second one, assuming that $a_{11} > 0$. However, there is a threshold value of r below which $a_{11} f^1$ becomes negative, indicating, e.g., a loss of interest in the creation of the link. The second term, $a_{12} f^2$, is reactive in nature. Assuming that $a_{12} > 0$, this term will become positive when $x^2 > r$, indicating that the second individual wishes to create a link with the first. In this case the first individual may react positively and vice versa. If $a_{12} < 0$, then this may indicate a sort of a reverse effect where the first individual will only be interested in creating the link if the second individual is *not* interested. The second row can be interpreted in the same way. In Section 3 we will see how important the diagonal terms are.

To develop the possibilities of our model a little further, we also consider the case where the elements of A

are functions of x. In this paper we present one idea. Consider the following for the i-th row of (3) (note that the summation convention is suppressed):

$$\dot{x}^i = a_{ii} f^i + \sum_{\substack{j=1 \\ j\neq i}}^{m} a_{ij} e^{-\mu_{ij}|x^i - x^j|} f^j. \qquad (4)$$

In (4), a_{ij} and $\mu_{ij} \geq 0$ are fixed parameters. The first term represents, as before, the innate desire to create a link. The change here is that the other terms are multiplied by $e^{-\mu_{ij}|x^i - x^j|}$, and they can be seen as attenuating factors coming into play when x^i and x^j are in opposition.

3. Model Analysis

Let $(A\phi)_\star$ denote the differential of the mapping $A\phi$ (the Jacobian matrix in a given coordinate system) and $\partial_k = \partial/\partial x^k$. Then the ij-th element of the differential is given by

$$(A\phi)_{\star ij} = a_{ij}\partial_j f^j + f^k \partial_j a_{ik}, \qquad (5)$$

which results from the particular structure of (2) (note the use of the summation convention).

Now, at each equilibrium point, the stability of this equilibrium point depends upon the eigenvalues of the matrix made up of the elements (5) evaluated at the equilibrium point. To satisfy the property that all the points of 2^m should be stable equilibrium points and that the other equilibrium points should be unstable, we require that all the eigenvalues at the points of 2^m have negative real parts and that the eigenvalues at the other equilibrium points have at least one with positive real part. Due to the design of the function f, we can easily evaluate (5) at the equilibrium points. Note that at all equilibrium points the second term vanishes because it is multiplied by f, which itself vanishes at an equilibrium point. We thus have

$$(A\phi)_{\star ij}(2^m) = -\alpha a_{ij} \ \text{or} \ -\beta a_{ij} \ \text{or} \ \gamma a_{ij}, \qquad (6)$$

where γ is a positive constant satisfying $f'(r) = \gamma$.

From (6) we can deduce the following property:

Proposition 1. *If the matrix A^T in (3) is diagonally dominant with positive diagonal elements at all equilibrium points, then 2^m equilibrium points are all stable and all other equilibrium points are unstable.*

The proof of this property is very straightforward. If an equilibrium point belongs to 2^m, then all the columns of matrix A are multiplied by either $-\alpha$ or $-\beta$. We can then apply Gerschgorin's theorem (Golub and Van Loan, 1986), working column by column, which states that all of the eigenvalues (λ) lie in the discs in the complex plain defined by

$$\left|\lambda - \delta a_{jj}\right| \leq \sum_{\substack{i=1 \\ i\neq j}}^{m} \left|\delta a_{ij}\right| \ \text{for} \ j = 1, \dots, m,$$

where $\delta = \alpha$ or β. If the matrix A satisfies the properties in the proposition, then the above shows that all the eigenvalues are in the left half complex plane. If, however, the equilibrium point does not belong to 2^m, then there is at least one column of the matrix multiplied by γ and satisfying

$$\left|\lambda + \gamma a_{jj}\right| \leq \sum_{\substack{i=1 \\ i\neq j}}^{m} \left|\gamma a_{ij}\right|.$$

From this it is easy to see that the eigenvalue lies in the right half complex plane.

4. Examples

In this section we present some simulation examples. There are clearly a lot of combinations of network dimensions, situations, parameter values, etc. For this reason we confine our study to a network of two individuals here. In this way we can look in depth at our model and visualise the associated vector fields. Thus we do not need to rely on simulation alone and we can see the attractors, sinks and nodes of the vector fields.

We would like to see the effects of the matrix A on the network, and so we fix certain parameters for all the simulations. We set $\alpha = 5$ and $\beta = 7$, and from that we can easily determine $r = 0.4167$ and $\gamma = 2.9167$.

In each of the following simulations we visualised the vector field defined by (3) and then calculated 100 trajectories of the vector field starting at random initial points. The trajectories were then superimposed on the vector field for graphical presentation with a star indicating the initial point. After each simulation the random seed was reinitialised to its original value, so the same initial points were used in each simulation.

The first matrix that we used was

$$\begin{bmatrix} 1 & 0.7 \\ 0.5 & 1 \end{bmatrix}.$$

We can determine the spectra of the differentials at the equilibrium points via (6). For example, denoting by

$\lambda(\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right])$ the spectrum at the point $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$, we have

$$\lambda\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \{-2.0420, -7.9580\},$$

$$\lambda\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \{-9.6401, -2.3599\},$$

$$\lambda\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \{-2.3599, -9.6401\},$$

$$\lambda\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \{-2.8587, -11.1413\},$$

whilst, e.g.,

$$\lambda\left(\begin{bmatrix}0\\r\end{bmatrix}\right) = \{-4.8619, 0.2785\}.$$

This shows that the system has the right qualitative properties. The vector field and trajectories for this example are shown in Fig. 3. We note that this example has all real eigenvalues at the four sinks. At the four corners of Fig. 3 we see typical behaviour of trajectories, where they line themselves with the corresponding eigenvector directions as they approach the sinks. The majority of the trajectories converge to the point $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, indicating that this is, in a way, a friendly network.

For the second example, we modified the matrix to the following:

$$\begin{bmatrix}1 & -0.7\\0.5 & 1\end{bmatrix}.$$

As could be expected, the negative value introduces a torsion effect into the force field and, for example, we have

$$\lambda\left(\begin{bmatrix}0\\0\end{bmatrix}\right) = \{-5.0000 \pm 2.9580i\},$$

$$\lambda\left(\begin{bmatrix}1\\1\end{bmatrix}\right) = \{-7.0000 \pm 4.1413i\}.$$

The vector field and trajectories for this example are shown in Fig. 4. This is an example of a system with complex eigenvalues and we can clearly see the spiralling effect that these have on the vector field close to the sinks at the four corners of Fig. 4. A trajectory will tend to a spiral round a sink before converging to it, showing perhaps a slightly indecisive nature of an individual. In this example the two points $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ attract most of the trajectories.

For the third example, we introduce the attenuating functions as in (4). First of all we set all the parameters $\mu_{ij} = 2$ and use the same constant matrix as in the first example above. As has been expected, the spectra at the two points $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ are identical to those in the first example. However, due to the attenuating functions, we have

$$\lambda\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \{-7.1065, -4.8935\},$$

$$\lambda\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \{-4.8935, -7.1065\},$$

which are clearly different from those in the first example in that they are less extreme. In Fig. 5 we can see the vector field and trajectories for this example. In spite of the

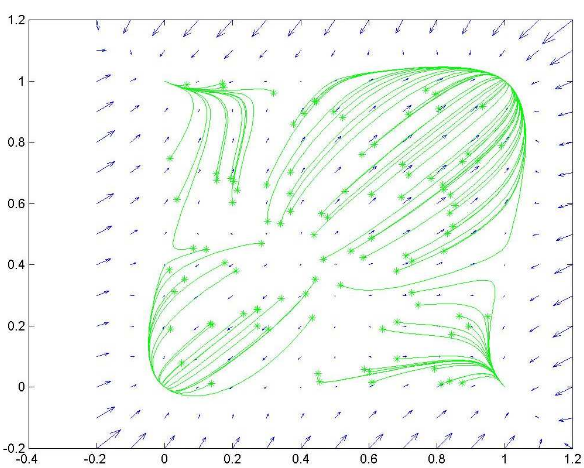

Fig. 3. First example.

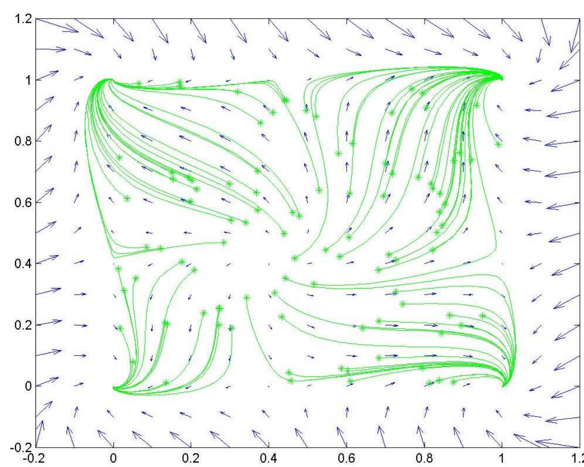

Fig. 4. Second example.

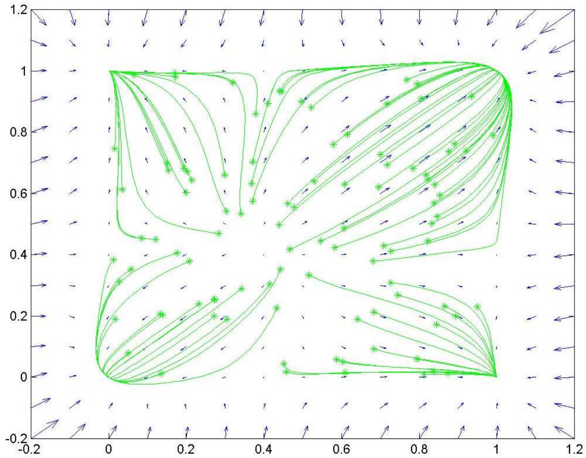

Fig. 5. Third example.

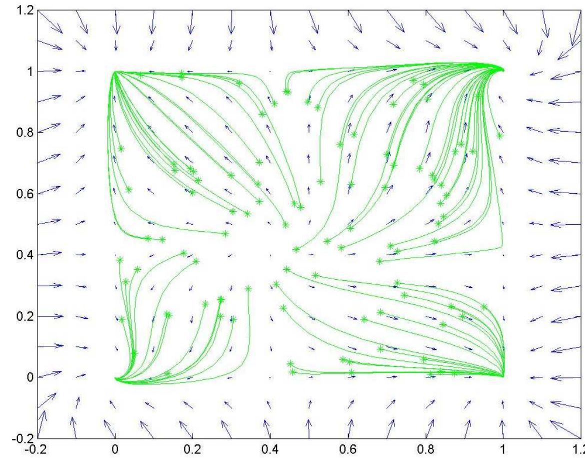

Fig. 6. Fourth example.

change in numerical values of the eigenvalues, the qualitative behaviour of this system is very much the same as in the first example.

In the fourth example, we use the same attenuating functions with the same parameters, but we apply the same constant matrix as in the second example. The spectra at the points $\left[\begin{smallmatrix}0\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$ are identical to those in the second example. However, the changes at the other two stable equilibrium points are somewhat more dramatic than the corresponding changes between the first and the third example. For example, without the attenuating functions we have

$$\lambda\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \{-6.0000 \pm 3.3541i\},$$

$$\lambda\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \{-6.0000 \pm 3.3541i\},$$

whilst with the attenuating functions we have

$$\lambda\left(\begin{bmatrix}1\\0\end{bmatrix}\right) = \{-6.8807, -5.1193\},$$

$$\lambda\left(\begin{bmatrix}0\\1\end{bmatrix}\right) = \{-6.8807, -5.1193\},$$

i.e., the eigenvalues have changed from complex to real. The simulation results for this example can be seen in Fig. 6, where we notice that the vector field has lost its spiralling effect at the two corners $\left[\begin{smallmatrix}1\\0\end{smallmatrix}\right]$ and $\left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ as predicted by the eigenvalues, and convergence to these points is qualitatively different from the trajectories in Fig. 4.

5. Conclusions

Although we have achieved some of our aims, this work is very much ongoing in nature. We have developed a model that does exhibit the characteristics that we require of it, but more still needs to be done.

One of our main lines of research concerns parameter identification. In (Snijders and van Duijn, 1997) and the references therein, a lot of work was done on the parameter estimation problem, or fitting the model to a set of data. We call it parameter identification in our model to differentiate between stochastic and deterministic cases. We are currently working on this and we hope to fit our model to data sets and, wherever possible, compare our results with those concerning stochastic models.

We are also looking for ways to give us better control over the eigenvalues. In other words, we are not only looking at the stability problem, but also at the levels of stability. This is important because we could have variables changing at widely different speeds, and this can sometimes reflect a real situation. For example, daily mood changes in individuals may have a local effect on a network, but no global effect in the long run.

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