Int. J. Appl. Math. Comput. Sci., 2003, Vol. 13, No. 4, 453–468

THE STABILITY OF AN IRRIGATION CANAL SYSTEM

HAMID BOUNIT*

* Department of Mathematics, Ibnou Zohr University
Faculty of Sciences, BP. 8106, 80000 Agadir, Morocco
e-mail: bounith@yahoo.fr

In this paper we examine the stability of an irrigation canal system. The system considered is a single reach of an irrigation canal which is derived from Saint-Venant's equations. It is modelled as a system of nonlinear partial differential equations which is then linearized. The linearized system consists of hyperbolic partial differential equations. Both the control and observation operators are unbounded but admissible. From the theory of symmetric hyperbolic systems, we derive the exponential (or internal) stability of the semigroup underlying the system. Next, we compute explicitly the transfer functions of the system and we show that the input-output (or external) stability holds. Finally, we prove that the system is regular in the sense of (Weiss, 1994) and give various properties related to its transfer functions.

Keywords: Saint-Venant equation, dimensionless, symmetric hyperbolic equation, internal stability, transfer function, input-output stability, regular systems

1. Introduction

Hydraulic systems of irrigation canals are inherently characterized by distributed parameter dynamics, giving rise to delayed responses. The essential nature of distributed parameter dynamics cannot be ignored if we wish to control these processes. The objective of our work is to develop a framework which would be useful for robust control of these distributed parameter processes. The approach adopted here is essentially infinite dimensional: the analysis will be carried out based on a PDE model in contrast to the approximation ideas, resulting in infinite dimensional models.

In this paper we discuss the internal and external stability of an irrigation canal system. The system considered is an irrigation canal which is partitioned as a single reach consisting of a single pool with two gates, G_u and G_d, located at its upstream and downstream ends, respectively (Fig. 1). This is a basic element of a complete irrigation canal. In what follows, the reach is assumed to be uniform with a trapezoidal cross-section of the slope β (see Appendix 2). The derivation of the Saint-Venant equations of the unsteady flow in open canals for shallow water conditions can be found in the literature (Chow, 1985; Mahmood and Yevjevich, 1975; Miller and Yevjevich, 1975). Then in these conditions, the flow dynamics in open canals are governed by the following nonlinear coupled hyperbolic partial differential equations (PDEs):

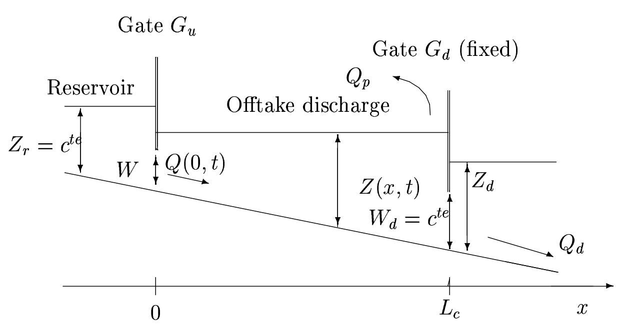

Fig. 1. The investigated reach.

$$\begin{cases} \dfrac{\partial S(x,t)}{\partial t} + \dfrac{\partial Q(x,t)}{\partial x} = q, & \text{(MASC)} \\[2mm] \dfrac{\partial Q(x,t)}{\partial t} + \dfrac{\partial (V \cdot Q(x,t))}{\partial x} & \\ \qquad + g\dfrac{\partial Z(x,t)}{\partial x} = -gSJ + q \cdot V & \text{(MOMC)} \end{cases} \tag{1}$$

for all $(x,t) \in]0, L[\times \mathbb{R}^+$ where x is the spatial location (m), t is time (s), S is the flow cross-section (m^2), Q is the flow discharge (m^3/s), q is the infiltration rate (m^2/s), V is the mean velocity (m/s), g is the gravity acceleration (m/s^2), Z is the water elevation (m), and J is the friction slope. The equation (MASC) is the conservation of mass, and (MOMC) is the conservation of momentum. They are complemented with initial conditions $Z(x,0)$ and $Q(x,0)$, and upstream, downstream and internal boundary conditions.

This nonlinear model is a simplified version of the model implemented in SIC (Simulation of Irrigation Canals), a commercially available package developed by

CEMAGREF (1992). It is applied to a single reach, consisting of a single pool with a single gate located at its downstream end. Its objective is to describe the dynamic behaviour of the flow discharges Q and the water elevations Z in the canal as boundary conditions change. For the studied reach, the appropriate boundary conditions need to be specified:

1. The upstream discharge Q_u^u.
2. The discharge equation of the upstream gate is
$$Q_u(t) = Q_u^u(t) = c_u L_u W_u \sqrt{2g(Z_u^u(t) - Z_d^u(t))}.$$
3. The discharge equation of the downstream gate is
$$Q_d(t) = Q_d^d(t) = c_d L_d W_d \sqrt{2g(Z_u^d(t) - Z_d^d(t))},$$
where Q_u^u and Q_d^d are the discharges through the upstream and downstream gates, c_u and c_d are the upstream and downstream gate discharge coefficients, L_u and L_d are the widths of the upstream and downstream gates (m), W_u and W_d are the upstream and downstream gate openings (m), Z_u^d and $Z_d = Z_d^d$ are the upstream and the downstream water elevations at the downstream gate, $Z_u = Z_u^u$ and Z_d^u are the upstream and downstream water elevations at the downstream gate, respectively.
4. The offtake outflow is $Q_p(t) = Q_u^d(t) - Q_d(t)$, where Q_u^d is the upstream discharge at the downstream gate.
5. The downstream water elevation Z_d at the downstream gate is given by the rating curve equation
$$Z_d(t) = qQ_d(t), \quad q > 0.$$

To transform the Saint-Venant equations to a dimensionless form, each variable is divided by a constant reference value with the same dimension. A possible system of reference variables consists of the reference variables corresponding to the steady flow $Q_0(x) = Q_0$ (see Appendix 2). Other choices are possible as indicated in (Baume and Sau, 1997; Clemments *et al.*, 1995). From the Saint-Venant equations, if dimensionless reference variables are used for the steady flow, the system linearized around a reference steady state $(Z_0(x), Q_0(x), S_0(x))$ given in Appendix 2 is governed by the following hyperbolic partial differential equations (Baume, 1990; Baume and Sau, 1997; Bounit, 2003a; Bounit *et al.*, 1997)):

$$\begin{aligned}&\frac{\partial Z^*(x^*,t^*)}{\partial t} + a_1 \frac{\partial Q^*(x^*,t^*)}{\partial x^*} = 0,\\ &\frac{\partial Q^*(x^*,t^*)}{\partial t^*} + a_2 \frac{\partial Z^*(x^*,t^*)}{\partial x^*} + a_3 \frac{\partial Q^*(x^*,t^*)}{\partial x^*}\\ &\quad + a_4 Z^*(x^*,t^*) + a_5 Q^*(x^*,t^*) = 0\end{aligned} \tag{2}$$

for all $(x^*, t^*) \in]0, L_c^*[\times \mathbb{R}^+$, where L_c^* is the dimensionless length of the canal.

Gate openings would be specified relative to the water depth at a steady flow for the reference section.

Then the above PDEs are coupled by the following boundary conditions:

1. The dimensionless linearized ($G_i's$, $i = u, d$) discharge equations
$$\begin{aligned}Q_u^*(t^*) &= Q^*(0,t^*) = c_1 W_u^*(t^*) - c_2 Z^*(0,t^*)\\ &= u^*(t^*),\end{aligned} \tag{3}$$
$$\begin{aligned}Q_d^*(t^*) &= c_3(Z^*(L_c^*,t^*) - Z_d^*(t^*))\\ &= Q^*(L_c^*,t^*) - Q_p^*(t^*).\end{aligned} \tag{4}$$
2. The dimensionless rating curve equation
$$Z_d^*(t^*) = q^* Q_d^*(t^*). \tag{5}$$

The dimensionless output considered is
$$y(t^*) = Z^*(L_c^*, t^*). \tag{6}$$

In (4)–(5), Z_d^* and Q_d^* are, respectively, the dimensionless downstream water elevation and discharge associated with the downstream gate G_d. W_u^* is the dimensionless deviation opening at the upstream gate. $Q_p^*(t^*)$ is the dimensionless offtake discharge. The downstream gate G_d is fixed *a priori* (i.e., $W_d = c^{te}$, cf. Fig. 1). The upstream water elevation for G_u is also constant (i.e., $Z_r = c^{te}$, Fig. 1). The constant parameters a_i are uniquely determined by the steady state and the dimensionless reference system (only a_2 and a_4 change the sign). Similarly, the constants c_i depend on the steady state, the dimensionless reference system used and the discharge at the upstream downstream gate and width coefficients (see Appendix 2). The state variables $Z^*(x^*, t^*)$ and $Q^*(x^*, t^*)$ are the dimensionless deviations in the water level and discharge, respectively, from the steady state in the canal. The input variable $u^*(t^*)$ which represents the variation in the inlet dimensionless discharge $Q^*(0, t^*)$ is boundary and affects the PDEs in (4). The output variable $y(t^*)$ in (1), representing the deviation in the dimensionless water level in the downstream, is also boundary and enters the boundary condition (5). The disturbance variable $d^*(t^*) = Q_p^*(t^*)$ represents the deviation in the unknown dimensionless outflow in the upstream of gate G_d and enters the boundary condition (5). So (2)–(6) describe an infinite dimensional linear system with boundary input and boundary output.

This irrigation canal system was studied in (Bounit, 2003b) in order to construct an H^∞-controller. The H^∞-control theory developed in (Francis and Zames, 1984;

Zames and Francis, 1983) combined with an approximation approach given in (Yoon and Lee, 1991) was applied to this system for minimizing the worst effect of the disturbance $d(t)$ on the output $y(t)$. By considering some other geometric configuration of the canal irrigation system, the canal is partitioned as a single reach with a simple gate positioned in its downstream which is the basic element of the complete irrigation canal. Then a robust low-gain P.I.-controller was proposed in (Bounit, 2003a).

The present paper shows how the system (2)–(6) is transformed into a dissipative symmetric hyperbolic system. Then we prove that the associated semigroup is exponentially stable using the theorem of (Rauch and Taylor, 1974), whence it is stabilizable and detectable. Input-output stability is also demonstrated. The concepts of the approximate and exact controllability and observability are not recovered here. The first concept is more convenient from a control engineering point of view because it is less restrictive than the exact controllability. Recall that there are many results on controllability and observability for a large class of symmetric hyperbolic systems in (Russel, 1978). Moreover, there exist computational tests for checking controllability (resp. observability) for large classes of linear distributed systems (Curtain and Zwart, 1995) and it would be interesting to study these concepts for our system.

We also prove that the system (2)–(6) is regular in the sense of (Weiss, 1994). The regularity of the controlled and observed systems with an exponentially stable semigroup guarantees that the plant transfer function $P(s)$ and the disturbance transfer function $W(s)$ are in H_0^∞ (that is analytic and bounded in the right-hand half plane). The fact that the system under consideration is regular has useful consequences on the design of the feedback controller for the system.

The class of regular linear systems is closed under feedback. The most important consequence is that internal and external stabilities are equivalent for a regular system which is both stabilizable and detectable as proved in (Rebarber, 1993). Using the theory of symmetric hyperbolic systems outlined in (Rauch and Taylor, 1974; Russell, 1978), we prove exponential stability for a much class of irrigation canals. Using the recent representation theory developed in (Weiss, 1994), we are able to characterize the transfer functions in terms of the semigroup operator, the input operator and the output operator. This characterization is useful for controller design purposes.

The paper is organized as follows: In Section 2, the system is transformed to an abstract boundary control system. In Section 3, the stability result of (Rauch and Taylor, 1974) is presented for a class of symmetric hyperbolic systems. Section 4 is divided into two subsections: In the first subsection, the system equations (2)–(6) are transformed into the form of a symmetric hyperbolic system. The exponential stability is obtained by applying the result presented in Section 3. In the second subsection, the transfer function representation of the system (2)–(6) is given. Next, we characterize an open set of numerical values of physical constants of the canal for which we can actually prove that the system is input-output stable. This fact is useful in H^∞-control and robust control, as will be discussed in another article (Bounit, 2000a; 2003b). In Section 5, we show that the system is regular and various properties are derived for its transfer functions. Section 6 contains conclusions.

2. Symmetric Hyperbolic Systems

Consider a symmetric hyperbolic system of the form

$$\begin{cases} \dfrac{\partial h(x,t)}{\partial t} = A(x)\dfrac{\partial h(x,t)}{\partial x} + B(x)h(x,t), \\ \qquad\qquad (x,t) \in]0,1[\times\mathbb{R}^+, \\ h^-(0,t) = D_0 h^+(0,t), \\ h^+(1,t) = D_1 h^-(1,t), \\ h(x,0) = h_0(x), \end{cases} \tag{7}$$

where $h^-(x,t) \in \mathbb{R}^p, h^+(x,t) \in \mathbb{R}^q$ and $h(x,t) = (h^-(x,t), h^+(x,t))$ is a $(p+q)\times 1$ vector function for $(x,t) \in [0,1]\times\mathbb{R}^+$, $A(x)$ and $B(x)$ are real $(p+q)\times(p+q)$ matrix functions and $A(x)$ is diagonal for $x \in [0,1]$. D_0 and D_1 are real constant matrices.

The diagonal matrix has the form

$$\begin{pmatrix} A^-(x) & 0 \\ 0 & A^+(x) \end{pmatrix},$$

with

$$A^-(x) = \mathrm{diag}(\lambda_i(x);\ i = 1,p),$$
$$A^+(x) = \mathrm{diag}(\lambda_i(x);\ i = p+1, p+q).$$

We denote by Λ the transposed matrix of Λ or the adjoint operator of Λ, as will be clear from the context, and by $A_x(x)$ the Jacobian of $A(x)$. For clarity, we assume that the following hypotheses are satisfied for the system (7):

$\mathbf{H_1}$: $B(\cdot) \in C^0([0,1];\mathbb{R}^{n\times n})$ and $A(x) \in C^1([0,1];\mathbb{R}^{n\times n})$,

$\mathbf{H_2}$: $\lambda_i(x) < 0,\ i = 1,2,\dots,p$ and $\lambda_i(x) > 0$, $i = p+1,\dots,p+q$ for any $x \in [0,1]$,

$\mathbf{H_3}$: For each $e^- \in \mathbb{R}^p$, $e^+ \in \mathbb{R}^q$ and $x \in [0,1]$, we have

(i) $$\begin{pmatrix} e^- \\ e^+ \end{pmatrix}^* \big(B(x) + B^*(x) - A_x(x)\big) \begin{pmatrix} e^- \\ e^+ \end{pmatrix} \le 0,$$

(ii) $$(e^-)^* \big(A^-(1) + D_1^* A^+(1) D_1\big)(e^-) \le -r^- \|e^-\|_p^2,$$

(iii) $$(e^+)^* \big(A^+(0) + D_0^* A^+(0) D_0\big)(e^+) \ge r^+ \|e^+\|_q^2,$$

and

$$r^+ \ge 0, \quad r^- \ge 0 : r^+ + r^- > 0.$$

Theorem 1. (Rauch and Taylor, 1974). *Assume that the system (7) satisfies the hypotheses (H_1)–(H_3). Then for each $h_0 \in (L^2[0,1])^n$, (7) has a unique solution:*

$$h(\cdot,t) \in C\big([0,+\infty); \big(L^2[0,1]\big)^n\big).$$

The semigroup of bounded linear operators $S(t)$ from $H = (L^2[0,1])^n$ into itself defined by $h(\cdot,t) = S(t)h_0$ is exponentially stable:

$$\|S(t)\|_{\mathcal{L}(H)} \le M e^{-\omega t},$$

for some constants $M, \omega > 0$.

3. Stability of the Irrigation Canal

In this section, we first show that the investigated irrigation canal system can be transformed into the form (7) and prove the exponential stability of the system by applying Theorem 1. Next, we compute the system transfer functions $P(s)$ and $W(s)$ and prove input-output stability.

3.1. Exponential Stability of the Semigroup

We have to convert the system (2)–(6) into a standard state space form. It is important to be very precise about the state space formulation, because the proof of exponential stability is system theoretic in nature. In the analysis below, this model is seen as a boundary distributed control system. Now, observe that the basic dynamical model (2)–(6) can be rewritten as a boundary control system (BCS):

$$\begin{aligned} \dot{\Phi}(t^*) &= A_\partial \Phi(t^*) + B\Phi(t^*), \\ \Gamma_\partial \Phi(t^*) &= \begin{pmatrix} 1 \\ 0 \end{pmatrix} u(t^*) + \begin{pmatrix} 0 \\ 1 \end{pmatrix} d(t^*), \\ y(t^*) &= C\Phi(t^*), \end{aligned}$$

where $\dot{\Phi}(t^*)$ denotes the derivative of $\Phi(\cdot,t^*)$ with respect to t^* and

$$A_\partial = \begin{pmatrix} 0 & -a_1 \\ -a_2 & -a_3 \end{pmatrix} \frac{\partial}{\partial x^*}, \quad B = \begin{pmatrix} 0 & 0 \\ -a_4 & -a_5 \end{pmatrix}.$$

The state $\Phi = (Z^*, Q^*)$ of the system (BCS) belongs to H, the control input $u \in U = \mathbb{R}$ and the output $y \in Y = \mathbb{R}$. The boundary operator Γ_∂ and the output operator C are given as follows:

$$\Gamma_\partial \Phi = \big(\Phi_2(0), \Phi_2(L_c^*) - k\Phi_1(L_c^*)\big)^T, \quad k = \frac{c_3}{1 + q\, c_3},$$

$$C\Phi = \Phi_1(L_c^*).$$

This description is analysed in detail in the following.

Since our objective is the dynamical analysis of an irrigation canal model using linear distributed parameter systems theory (see, e.g., Curtain and Zwart, 1995; Pazy, 1983; Weiss, 1989a; 1989b), an important preliminary step is to obtain a description of the model as an infinite dimensional Hilbert state space (BCS), where $\Phi(t)$ belongs to a real separable Hilbert space H and the spatial differential operator A_∂ is the infinitesimal generator of a strongly continuous C_0 semigroup $(e^{A_\partial t})_{t \ge 0}$ on H. Here, we use the (Hilbert) state space $H = L^2[0, L_c^*] \times L^2[0, L_c^*]$ obtained as the Cartesian product of the Hilbert space

$$L^2[0, L_c^*] = \Big\{ f \mid [0, L_c^*] \longrightarrow \mathbb{R}; \int_0^{L_c^*} |f(x)|^2 \,\mathrm{d}x < +\infty \Big\}.$$

$L^2[0, L_c^*]$ is a Hilbert space with the inner product and norm defined respectively by

$$\forall f, g \in L^2[0, L_c^*] : \quad \langle f, g \rangle_{L^2} = \int_0^{L_c^*} f(x) g(x) \,\mathrm{d}x,$$

$$\|f\|_2 = \Big(\int_0^{L_c^*} |f(x)|^2 \,\mathrm{d}x \Big)^{1/2}.$$

The Hilbert space H is endowed with the inner product $\langle \cdot, \cdot \rangle_H$ defined as follows: For any $f = (f_1, f_2)^T$ and $g = (g_1, g_2)^T \in H$,

$$\langle f, g \rangle_H = \langle f_1, g_1 \rangle_{L^2} + \langle f_2, g_2 \rangle_{L^2}.$$

The domain $D(A_\partial)$ of the unbounded operator $A_\partial : D(A_\partial) \longrightarrow H$ is given by $D(A_\partial) = H^1[0, L_c^*] \times H^1[0, L_c^*]$, where $H^1[0, L_c^*]$ is the Sobolev space:

$$H^1[0, L_c^*] = \Big\{ f \in L^2[0, L_c^*] \mid f \text{ is absolutely continuous and } \mathrm{d}f/\mathrm{d}x^* \in L^2[0, L_c^*] \Big\}.$$

Now, let us show how to transform the irrigation canal system into the form (7). We perform a normalization so that $L_c^* = 1$; the distance along the canal is referred to as its length. This last assumption is not necessary for our work, but it makes the calculation slightly simpler.

For simplicity, we shall write $\Phi(x,t)$ instead of $\Phi(x^*,t^*)$ whenever no confusion arises. Then the (BCS) can be written as

$$\begin{cases} \dfrac{\partial\Phi(x,t)}{\partial t} = A_c\dfrac{\partial\Phi}{\partial x} + B_c\Phi(x,t), \\ \qquad (x,t) \in]0,1[\times\mathbb{R}^+, \\ \Phi_2(0,t) = u(t), \\ \Phi_2(1,t) = k\Phi_1(1,t) + d(t), \\ y(t) = \Phi_1(1,t), \end{cases} \tag{8}$$

where

$$A_c = \begin{pmatrix} 0 & -a_1 \\ -a_2 & -a_3 \end{pmatrix}, \quad B_c = \begin{pmatrix} 0 & 0 \\ -a_4 & -a_5 \end{pmatrix}.$$

The matrix A_c is not diagonal as required in Theorem 1. So, let us diagonalize it. For this, we need to compute two characteristic roots of the following characteristic equation:

$$\det(\lambda I - A_c) = \lambda(\lambda + a_3) - a_1a_2.$$

It is easy to verify that $4a_1a_2 + a_3^2 > 0$ (see Appendix 2). Then, these roots are given by

$$\begin{cases} \lambda^- = \dfrac{1}{2}\big(-a_3 - \sqrt{a_3^2 + 4a_1a_2}\big) < 0, \\ \lambda^+ = \dfrac{1}{2}\big(-a_3 + \sqrt{a_3^2 + 4a_1a_2}\big) > 0, \end{cases} \tag{9}$$

with

$$\begin{cases} \lambda^+ - \lambda^- = \sqrt{a_3^2 + 4a_1a_2}, \\ \lambda^+ + \lambda^- = -a_3, \\ \lambda^+\lambda^- = -4a_1a_2. \end{cases} \tag{10}$$

So, the passage matrix and its inverse are respectively given by

$$P = \begin{pmatrix} -\dfrac{\lambda^+\lambda^-}{4} & -\dfrac{\lambda^+\lambda^-}{4} \\ \lambda^- & \lambda^+ \end{pmatrix},$$

$$P^{-1} = \frac{1}{(\lambda^+ - \lambda^-)}\begin{pmatrix} -\dfrac{4}{\lambda^-} & -1 \\ \dfrac{4}{\lambda^+} & 1 \end{pmatrix}.$$

Thus the matrix A_c can be written as $A_c = PAP^{-1}$, where

$$A = \begin{pmatrix} \lambda^- & 0 \\ 0 & \lambda^+ \end{pmatrix}.$$

Consider the following linear transformation, which makes A_c diagonal:

$$\Phi \xrightarrow{\mathcal{T}} \Psi = P^{-1}\Phi = (\Psi_1, \Psi_2).$$

Applying the transformation ($\mathcal{T}$) to the first equation in (8), we obtain the following PDEs:

$$\frac{\partial\Psi(x,t)}{\partial t} = A\frac{\partial\Psi}{\partial x} + B\Psi(x,t), \quad (x,t) \in]0,1[\times\mathbb{R}^+, \tag{11}$$

where

$$B = P^{-1}B_cP = \frac{1}{4(\lambda^+ - \lambda^-)} \times \begin{pmatrix} \lambda^-(4a_5 - a_4\lambda^+) & \lambda^+(4a_5 - a_4\lambda^-) \\ -\lambda^-(4a_5 - a_4\lambda^+) & -\lambda^+(4a_5 - a_4\lambda^-) \end{pmatrix}.$$

Applying once again the transformation ($\mathcal{T}$) to the boundaries of the system (8), we obtain

$$\begin{cases} \Psi_1(0,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* P^{-1}\begin{pmatrix} \Phi_1(0,t) \\ u(t) \end{pmatrix}, \\ \Psi_2(0,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* P^{-1}\begin{pmatrix} \Phi_1(0,t) \\ u(t) \end{pmatrix}, \end{cases} \tag{12}$$

$$\begin{cases} \Psi_1(1,t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}^* Q\begin{pmatrix} \Phi_1(1,t) \\ d(t) \end{pmatrix}, \\ \Psi_2(1,t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}^* Q\begin{pmatrix} \Phi_1(1,t) \\ d(t) \end{pmatrix}, \end{cases} \tag{13}$$

with

$$Q = P^{-1}\begin{pmatrix} 1 & 0 \\ k & 1 \end{pmatrix}.$$

Eliminating $\Phi_1(0,t)$ in (12) and $\Phi_1(1,t)$ in (13), we get

$$\Psi_1(0,t) = D_0\Psi_2(0,t) + E_0u(t),$$

$$\Psi_2(1,t) = D_1\Psi_1(1,t) + E_1d(t),$$

where

$$D_0 = -\frac{\lambda^+}{\lambda^-}, \quad E_0 = -\frac{1}{\lambda^-},$$

$$D_1 = -\frac{\lambda^-(4 + k\lambda^+)}{\lambda^+(4 + k\lambda^-)}, \quad E_1 = \frac{4}{\lambda^+(4 + k\lambda^-)}.$$

A straightforward computation gives

$$y(t) = D\big(\Psi_1(1,t) + \Psi_2(1,t)\big),$$

with

$$D = -\frac{\lambda^+\lambda^-}{4}.$$

Thus, under the transformation $(\mathcal{T})$, the system (8) can be written as follows:

$$\begin{cases} \dfrac{\partial\Psi(x,t)}{\partial t} = A\dfrac{\partial\Psi(x,t)}{\partial x} + B\Psi(x,t), \\ \qquad (x,t) \in]0,1[\times\mathbb{R}^+, \\ \Psi_1(0,t) = D_0\Psi_2(0,t) + E_0u(t), \\ \Psi_2(1,t) = D_1\Psi_1(1,t) + E_1d(t), \\ y(t) = D(\Psi_1(1,t) + \Psi_2(1,t)). \end{cases} \tag{14}$$

Let $\mathcal{A}$ be the unbounded operator defined by

$$D(\mathcal{A}) = \Big\{(f_1,f_2)^T \in H^1[0,1]) \times H^1[0,1] \mid f_1(0) = D_0f_2(0),\ f_2(1) = D_1f_1(1)\Big\},$$

and, for each $f \in D(\mathcal{A})$,

$$\mathcal{A}f(x) = A\frac{\partial f(x)}{\partial x} + Bf(x).$$

Now, we state our stability result:

Theorem 2. *The operator $\mathcal{A}$ generates a C_0-group of contractions denoted by $(e^{t\mathcal{A}})_{t\geq 0}$, which is exponentially stable.*

Proof. In our case $p = q = 1 = n/2$, and D_0 and D_1 are both invertible. Thus, from (Rauch and Taylor, 1974; Russel, 1978) it is well known that $\mathcal{A}$ is the generator of a C_0-group of contractions on H. Clearly, the assumptions (H_1)–(H_2) are satisfied for the system (14).

Let us show that the assumption (H_3) is also satisfied by (14). According to (9) and (10), we have

$$A^+ + D_0^*A^-D_0 = \lambda^+ + \frac{(\lambda^+)^2}{(\lambda^-)^2}\lambda^- = \frac{\lambda^+(\lambda^+ + \lambda^-)}{\lambda^-} > 0.$$

Then there exists $r^+ > 0$ such that

$$(e^+)^*(A^+ + D_0^*A^-D_0)e^+ \geq r^+(e^+)^2.$$

Hence, Condition (iii) is satisfied:

$$A^- + D_1^*A^+D_1 = \lambda^- + D_1^*\lambda^+D_1 = \lambda^- + D_1^2\lambda^+ = \frac{(a_1a_2k - \lambda^+)^2\lambda^- + (a_1a_2k - \lambda^-)^2\lambda^+}{(a_1a_2k - \lambda^+)^2}.$$

Set $\theta = a_1a_2$, and consider

$$\begin{aligned}(\theta k - \lambda^+)^2\lambda^- &+ (\theta k - \lambda^+)^2\lambda^+ \\ &= \lambda^-({\lambda^+}^2 + \theta^2k^2 - 2\theta k\lambda^+) + \lambda^+({\lambda^-}^2 + \theta^2k^2 - 2\theta k\lambda^-), \\ &= (\lambda^+ + \lambda^-)\theta^2k^2 - 4\lambda^-\lambda^+\theta k + \lambda^-\lambda^+(\lambda^+ + \lambda^-).\end{aligned}$$

Since $\lambda^-\lambda^+ = -4a_1a_2 = -4\theta$, $\lambda^- + \lambda^+ = -a_3$, we obtain

$$\begin{aligned}P(k) &= (\theta k - \lambda^+)^2\lambda^- + (\theta k - \lambda^+)^2\lambda^+ \\ &= \theta(-a_3\theta k^2 + 16\theta k + 4a_3).\end{aligned}$$

Two roots of the equation $P(k) = 0$ are

$$k_1 = \frac{-4\theta + \sqrt{4\theta(4\theta + a_3^2)}}{a_3\theta} > 0,$$

$$k_2 = \frac{-4\theta - \sqrt{4\theta(4\theta + a_3^2)}}{a_3\theta} < 0.$$

The admissible root is the positive one, i.e., k_1. Then with an appropriate choice of the reference for the dimensionless gate system we can obtain $k > k_1$, which implies

$$\exists r^- > 0,\ (e^-)^*(\lambda^- + D_1^*\lambda^+D_1)(e^-) \leq -r^-(e^-)^2.$$

So, Condition (ii) is also satisfied.

A direct calculation gives

$$B + B^* = \frac{1}{2(\lambda^+ - \lambda^-)} \times \begin{pmatrix} \lambda^-(4a_5 - a_4\lambda^+) & 2a_5(\lambda^+ - \lambda^-) \\ 2a_5(\lambda^+ - \lambda^-) & -\lambda^+(4a_5 - a_4\lambda^-) \end{pmatrix}.$$

Set

$$a = \lambda^-(4a_5 - a_4\lambda^+), \quad b = 2a_5(\lambda^+ - \lambda^-),$$

$$c = -\lambda^+(4a_5 - a_4\lambda^-).$$

Since $a < 0$, Condition (i) is satisfied iff

$$b^2 - ac < 0. \tag{15}$$

Then it is indeed necessary to make a good choice of the dimensionless system to satisfy the condition (15). Finally, Theorem 1 gives the desired result. ■

3.2. Transfer Function and Input-Output Stability

In this subsection, the linearized system governed by (2)–(6) can be used to get the transfer matrix of the reach. This transfer matrix can be useful to design the H^∞-optimal control (Bounit, 2003b). To this end, we show that our irrigation canal system is input-output stable.

Now, determine explicitly the dimensionless plant and disturbance transfer functions of our irrigation canal system, i.e., $P(s)$ and $W(s)$. Taking the Laplace transform of both the sides in (2), we get

$$s\hat{Z}^*(x,s) + a_1 \frac{\partial \hat{Q}^*(x,s)}{\partial x} = 0,$$
$$s\hat{Q}^*(x,s) + a_2 \frac{\partial \hat{Z}^*(x,s)}{\partial x} + a_3 \frac{\partial \hat{Q}^*(x,s)}{\partial x} + a_4 \hat{Z}^*(x,s) + a_5 \hat{Q}^*(x,s) = 0 \tag{16}$$

for all $x \in]0,1[$.

This system has the following form:

$$\begin{pmatrix} a_1 & 0 \\ a_3 & a_2 \end{pmatrix} \begin{pmatrix} \dfrac{\partial \hat{Q}^*(x,s)}{\partial x} \\ \dfrac{\partial \hat{Z}^*(x,s)}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & -s \\ -(s+a_5) & -a_4 \end{pmatrix} \begin{pmatrix} \hat{Q}^*(x,s) \\ \hat{Z}^*(x,s) \end{pmatrix}. \tag{17}$$

For clarity, we set

$$A = \begin{pmatrix} a_1 & 0 \\ a_3 & a_2 \end{pmatrix}, \quad B(s) = \begin{pmatrix} 0 & -s \\ -(s+a_5) & -a_4 \end{pmatrix}.$$

Since $a_1 \neq 0$ (see Appendix 2), the constant matrix A is invertible iff $a_2 \neq 0$.

In the remainder of the paper, we suppose that $a_2 \neq 0$. Now, set $X(s) = A^{-1}B(s)$. Then (17) becomes

$$\begin{pmatrix} \dfrac{\partial \hat{Q}^*(x,s)}{\partial x} \\ \dfrac{\partial \hat{Z}^*(x,s)}{\partial x} \end{pmatrix} = X(s) \begin{pmatrix} \hat{Q}^*(x,s) \\ \hat{Z}^*(x,s) \end{pmatrix}. \tag{18}$$

In order to study stability, we have to compute the explicit expressions of the transfer functions $P(s)$ and $W(s)$.

The solution of the linear differential equation (18) leads to

$$\begin{pmatrix} \hat{Q}^*(x,s) \\ \hat{Z}^*(x,s) \end{pmatrix} = \exp(X(s)x) \begin{pmatrix} \hat{Q}^*(0,s) \\ \hat{Z}^*(0,s) \end{pmatrix}.$$

To compute $\exp(X(s)x)$, let us determine the eigenvalues of $X(s)$. It is easy to verify that

$$\det(X(s) - \lambda I) = 0 \iff \lambda^2 - X_{22}\lambda + k_s L_0^* s X_{21} = 0. \tag{19}$$

Let $\lambda_i(s),\ i = 1,2$ be two roots of (19), which are explicitly given bellow. So, the transition matrix and its inverse are

$$T(s) = \begin{pmatrix} -k_s L_0^* s & -k_s L_0^* s \\ \lambda_1(s) & \lambda_2(s) \end{pmatrix},$$

$$T^{-1}(s) = \frac{1}{k_s L_0^* s(\lambda_1(s) - \lambda_2(s))} \begin{pmatrix} \lambda_2(s) & k_s L_0^* s \\ -\lambda_1(s) & -k_s L_0^* s \end{pmatrix},$$

respectively.

It follows that the above matrix $X(s)$ can be written as

$$X(s) = T(s)D(s)T^{-1}(s), \quad D(s) = \operatorname{diag}(\lambda_i(s)).$$

Setting $D'(s) = \operatorname{diag}(e^{\lambda_i(s)})$, we deduce that

$$\begin{pmatrix} \hat{Q}^*(x,s) \\ \hat{Z}^*(x,s) \end{pmatrix} = T(s)D'(s)T^{-1}(s) \begin{pmatrix} \hat{Q}^*(0,s) \\ \hat{Z}^*(0,s) \end{pmatrix} = L(s) \begin{pmatrix} \hat{Q}^*(0,s) \\ \hat{Z}^*(0,s) \end{pmatrix},$$

with

$$L_{11}(s) = \frac{\lambda_1(s)e^{\lambda_2(s)x} - \lambda_2(s)e^{\lambda_1(s)x}}{\lambda_1(s) - \lambda_2(s)},$$

$$L_{12}(s) = \frac{k_s L_0^* s(e^{\lambda_2(s)x} - e^{\lambda_1(s)x})}{\lambda_1(s) - \lambda_2(s)},$$

$$L_{21}(s) = \frac{\lambda_1(s)\lambda_2(s)(e^{\lambda_1(s)x} - e^{\lambda_2(s)x})}{k_s L_0^* s(\lambda_1(s) - \lambda_2(s))},$$

$$L_{22}(s) = \frac{\lambda_1(s)e^{\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}}{\lambda_1(s) - \lambda_2(s)}.$$

Then we define the transfer matrix $M(s)$ of the reach as

$$\begin{pmatrix} \hat{Q}^*(x,s) \\ \hat{Z}^*(0,s) \end{pmatrix} = M(s) \begin{pmatrix} \hat{Q}^*(0,s) \\ \hat{Z}^*(x,s) \end{pmatrix}, \tag{20}$$

where

$$M_{11}(s) = \frac{(\lambda_1(s) - \lambda_2(s))e^{(\lambda_1(s)+\lambda_2(s))x}}{\lambda_1(s)e^{\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}},$$

$$M_{12}(s) = \frac{k_s L_0^* s(e^{\lambda_2(s)x} - e^{\lambda_1(s)x})}{\lambda_1(s)e^{(\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}},$$

$$M_{21}(s) = \frac{\lambda_1(s)\lambda_2(s)(e^{\lambda_2(s)x} - e^{\lambda_1(s)x})}{k_s L_0^* s(\lambda_1(s)e^{(\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}},$$

$$M_{22}(s) = \frac{\lambda_1(s) - \lambda_2(s)}{\lambda_1(s)e^{\lambda_1(s)x} - \lambda_2(s)e^{\lambda_2(s)x}}.$$

This modelling has the advantage of keeping distributed parameter system characteristics and therefore the infinite state space dimension.

Now, an easy computation gives

$$\lambda_1(s) = \frac{1}{2}\left(a(s) - \left(a^2(s) + b(s)s\right)^{1/2}\right),$$

$$\lambda_2(s) = \frac{1}{2}\left(a(s) + \left(a^2(s) + b(s)s\right)^{1/2}\right),$$

where

$$a(s) = \varepsilon_1 + \varepsilon_2 s, \qquad \varepsilon_1 = \frac{a_4}{a_2}, \qquad \varepsilon_2 = -\frac{2a_3}{a_2 a_1},$$

$$b(s) = \varepsilon_3 + \varepsilon_4 s, \qquad \varepsilon_3 = -\frac{4a_5}{a_2 a_1}, \qquad \varepsilon_4 = \frac{4}{a_2 a_1}.$$

So, the functions λ_i have the form

$$\lambda_1(s) = \frac{1}{2}\left[\varepsilon_1 + \varepsilon_2 s - \left((\varepsilon_1 + \varepsilon_2 s)^2 + (\varepsilon_3 + \varepsilon_4 s)s\right)^{1/2}\right],$$

$$\lambda_2(s) = \frac{1}{2}\left[\varepsilon_1 + \varepsilon_2 s + \left((\varepsilon_1 + \varepsilon_2 s)^2 + (\varepsilon_3 + \varepsilon_4 s)s\right)^{1/2}\right].$$

Set

$$\begin{aligned} F(s) &= \left((\varepsilon_1 + \varepsilon_2 s)^2 + (\varepsilon_3 + \varepsilon_4 s)s\right)^{1/2} \\ &= \left(\varepsilon_1^2 + (\varepsilon_3 + 2\varepsilon_1\varepsilon_2)s + (\varepsilon_2^2 + \varepsilon_4)s^2\right)^{1/2}. \end{aligned}$$

In the above expressions the complex square root is defined as follows:

$$\begin{aligned} (A + iB)^{1/2} &= \left(\frac{A + \sqrt{A^2 + B^2}}{2}\right)^{1/2} \\ &\quad + i\,\mathrm{sign}(B)\left(\frac{-A + \sqrt{A^2 + B^2}}{2}\right)^{1/2} \end{aligned} \tag{21}$$

for any real A and B.

Now, applying the Laplace transform to both sides in the boundary conditions (3)–(5), we obtain

$$\hat{Q}^*(0, s) = c_1 \hat{W}_u^*(s) - c_2 \hat{Z}^*(0, s), \tag{22}$$

$$\hat{Q}_d^*(s) = c_3\left(\hat{Z}^*(1, s) - \hat{Z}_d^*(s)\right), \tag{23}$$

$$\hat{Q}_d^*(s) = \hat{Q}^*(1, s) - \hat{d}^*(s), \tag{24}$$

$$\hat{Z}_d^*(s) = \alpha^* \hat{Q}_d^*(s). \tag{25}$$

Combining (23)–(25), we get

$$\hat{Q}^*(1, s) = k\hat{Z}^*(1, s) + \hat{d}^*(s). \tag{26}$$

Substituting (26) into (20) at $x = L_c^* = 1$ yields

$$\hat{Z}^*(1, s) = \frac{M_{11}(s)}{k - M_{12}(s)}\hat{Q}^*(0, s) - \frac{1}{k - M_{12}(s)}\hat{d}^*(s).$$

Since $\hat{Z}^*(1, s)$, $\hat{d}^*(s)$ and $\hat{Q}^*(0, s)$ are respectively the output, disturbance and control for the linearized system, the plant and the disturbance transfer functions are given as follows:

$$P(s) = \frac{M_{11}(s)}{k - M_{12}(s)}, \quad W(s) = \frac{-1}{k - M_{12}(s)}, \tag{27}$$

and the input-output map is

$$\hat{y}^*(s) = P(s)\hat{u}^*(s) + W(s)\hat{d}^*(s).$$

We write

$$\mathbb{C}_\alpha = \left\{z \in \mathbb{C} \mid \Re(z) > \alpha\right\},$$

where $\alpha \in \mathbb{R}$, and let H_α^∞ be the set of all $\mathbb{C}$-valued functions which are bounded and analytic on $\mathbb{C}_\alpha$.

Physical intuition clearly requires that any irrigation canal system be stable in the sense that its transfer function is in H_0^∞. In fact, it has been proved above that this system in the open-loop scheme is exponentially stable (Theorem 2). So, it is reasonable to think that our irrigation canal system is also stable with respect to both the control and the disturbance.

We make the following assumptions:

(A_1): $\quad a_2 < 0$ and $a_4 < 0$,

(A_2): $\quad a_1 a_4^2 + 4a_2 \neq 0$ (i.e. $\varepsilon_1^2 + \varepsilon_4 \neq 0$).

Let us define the constants

$$\alpha_1 = \frac{1}{2}\left(\varepsilon_1 + \frac{\varepsilon_3 + 2\varepsilon_1\varepsilon_2}{2(\varepsilon_1^2 + \varepsilon_4)}\right), \qquad \alpha_2 = \frac{\varepsilon_2^2 + \varepsilon_4 + \varepsilon_2}{2},$$

$$\beta_1 = \frac{1}{2}\left(-\varepsilon_1 + \frac{\varepsilon_3 + 2\varepsilon_1\varepsilon_2}{2(\varepsilon_1^2 + \varepsilon_4)}\right), \quad \beta_2 = \frac{\varepsilon_2^2 + \varepsilon_4 - \varepsilon_2}{2}.$$

From the assumption (A_1) and the definition of ε_i, we have $\varepsilon_i > 0$, $i = 1, 3$ and $\varepsilon_4 < 0$. Hence the coefficients $\alpha_j, \beta_j, j = 1, 2$ can change the signs. So, we will add the following assumption:

(A_3): $\quad \alpha_i > 0$ and $\beta_i > 0$, $i = 1, 2$.

Lemma 1. *Assume that (A_1)–(A_3) hold. Then for* $s = x + iy$ *with* $x \geq 0$, *we have*

$$\begin{aligned} &\alpha_1 - \beta_1 + (\alpha_2 + \beta_2)x \\ &\qquad \leq \Re\big(F(s)\big) \leq \alpha_1 + \beta_1 + (\alpha_2 + \beta_2)x, \end{aligned} \tag{28}$$

$$\left|\Im\left(F(s)\right)\right| \geq (\alpha_2 + \beta_2)|y|, \tag{29}$$

$$\mathrm{sign}\left(\Im\big(F(s)\big)\right) = \mathrm{sign}(y). \tag{30}$$

Outline of the proof. It is easy to see that $F(s)$ and λ_i' s can be written as

$$\begin{aligned} F(s) &= \big[(\alpha_1-\beta_1)^2 + 2(\alpha_1+\beta_1)(\alpha_2+\beta_2)s \\ &\quad +(\alpha_2+\beta_2)s^2\big]^{1/2}, \\ \lambda_1(s) &= \frac{1}{2}\big[\alpha_1-\beta_1+(\alpha_2-\beta_2)s - F(s)\big], \\ \lambda_2(s) &= \frac{1}{2}\big[\alpha_1-\beta_1+(\alpha_2-\beta_2)s + F(s)\big], \end{aligned} \tag{31}$$

with

$$\alpha_j > \beta_j > 0, \quad j = 1, 2.$$

So, the proof of Lemma 1 is a straightforward procedure using (21) (Gauthier and Xu, 1989). Thus it is omitted. ■

From Lemma 1 and the definition of $\lambda_i(s)$ in (31), we get

$$-\beta_1 - \beta_2 x \le \Re\big(\lambda_1(s)\big) \le -\beta_2 x, \tag{32}$$

$$\alpha_1 - \beta_1 + \alpha_2 x \le \Re\big(\lambda_2(s)\big) \le \alpha_1 + \alpha_2 x, \tag{33}$$

which implies

$$-\beta_1 \le \Re\big(\lambda_1(s) + \beta_2 s\big) \le 0, \tag{34}$$

$$-\beta_1 \le \Re\big(\lambda_2(s) - \alpha_1 - \alpha_2 s\big) \le 0. \tag{35}$$

Thus $e^{(\lambda_2(s)-\alpha_1-\alpha_2 s)}$ and $e^{(\lambda_1(s)+\beta_2 s)}$ are H^∞-units (i.e., elements of H^∞ with inverses also belonging to H^∞).

Set $b = a_1^{-1}$. Using (22) and the expression for $M(s)$, $P(s)$ and $W(s)$ can be written as

$$\begin{aligned} P(s) &= \frac{(\lambda_1(s)-\lambda_2(s))e^{(\lambda_1(s)+\lambda_2(s))}}{(k\lambda_1(s)+bs)e^{\lambda_1(s)} - (k\lambda_2(s)+bs)e^{\lambda_2(s)}} \\ &= \frac{F(s)e^{\lambda_1(s)}}{(k\lambda_2(s)+bs) - (k\lambda_1(s)+bs)e^{-F(s)}}, \end{aligned} \tag{36}$$

$$\begin{aligned} W(s) &= \frac{\lambda_1(s)e^{\lambda_1(s)} - \lambda_2(s)e^{\lambda_2(s)}}{(k\lambda_1(s)+bs)e^{\lambda_1(s)} - (k\lambda_2(s)+bs)e^{\lambda_2(s)}} \\ &= \frac{\lambda_1(s)e^{-F(s)} - \lambda_2(s)}{(k\lambda_1(s)+bs)e^{-F(s)} - (k\lambda_2(s)+bs)}. \end{aligned} \tag{37}$$

We are now able to show the following result.

Theorem 3. *On the assumptions (A_1)–(A_3), the transfer functions $P(s)$ and $W(s)$ are in H_0^∞ and satisfy*

$$\lim_{\mathbb{R}\ni s\to+\infty} P(s) = 0, \quad \lim_{\mathbb{R}\ni s\to+\infty} W(s) = D_d \in \mathbb{R}.$$

Proof. From Lemma 1 and (35), the real parts of $-F(s)$ and $\lambda_1(s)$ are negative for $\Re(s) \ge 0$. Then the exponential terms of the numerators and the denominators in the transfer functions $P(s)$ and $W(s)$ given by (36) and (37) have strictly negative real parts. On the other hand, the polynomial parts have the same order for both the numerator of $P(s)$ and the denominator of $W(s)$, respectively. It follows that $P(s)$ and $W(s)$ are bounded for all $\Re(s) \ge 0$.

Now let us show that the transfer functions $P(s)$ and $W(s)$ are analytic for $\Re(s) \ge 0$. By Lemma 1, it is clear that $F(s)$ and its inverse are analytic in $\Re(s) \ge 0$ and $F^{-1}(s)$ lies in H^∞. Then $\lambda_i(s)$ and the exponential functions $e^{\lambda_i(s)}$ are also analytic. Regarding expressions (36) and (37), we only have to prove that

$$K(s) = \big(k\lambda_2(s) + bs\big) - \big(k\lambda_1(s) + bs\big)e^{-F(s)} \ne 0$$

for all $\Re(s) \ge 0$.

Set $K_i(s) = k\lambda_i(s) + bs$, $i = 1, 2$. Since $\Re(K_2(s)) > 0$ for all $\Re(s) \ge 0$ (see (33)), it suffices to prove that

$$E(s) = \frac{K_1(s)e^{-F(s)}}{K_2(s)} \ne 1 \ \text{ for all } \Re(s) \ge 0. \tag{38}$$

We have

$$|E(s)| = \Big|\frac{K_1(s)}{K_2(s)}e^{-F(s)}\Big| = \Big|\frac{K_1(s)}{K_2(s)}\Big|\,|e^{-F(s)}|.$$

Since $\Re(-F(s)) < 0$ for all $\Re(s) \ge 0$, it follows that

$$|E(s)| < \Big|\frac{K_1(s)}{K_2(s)}\Big|. \tag{39}$$

Now, set $s = x + iy$ with $x \ge 0$.

$$\begin{aligned} \Big|\frac{K_1(s)}{K_2(s)}\Big| &= \Big|\frac{k\Re(\lambda_1(s)) + bx + i(k\Im(\lambda_1(s)) + by)}{k\Re(\lambda_2(s)) + bx + i(k\Im(\lambda_2(s)) + by)}\Big| \\ &= \Big|\frac{m_1(s)}{m_2(s)}\Big|, \end{aligned}$$

where

$$\begin{aligned} m_1(s) &= \frac{k}{2}\Re(\varepsilon_1 + \varepsilon_2 s - F(s)) + bx \\ &\quad + i\Big(\frac{k}{2}\Im\big(\varepsilon_1 + \varepsilon_2 s - F(s)\big) + by\Big), \\ m_2(s) &= \frac{k}{2}\Re(\varepsilon_1 + \varepsilon_2 s + F(s)) + bx \\ &\quad + i\Big(\frac{k}{2}\Im\big(\varepsilon_1 + \varepsilon_2 s + F(s)\big) + by\Big). \end{aligned}$$

Finally, we obtain

$$\Big|\frac{K_1(s)}{K_2(s)}\Big| = \Big|\frac{n_1(s)}{n_2(s)}\Big|,$$

where

$$n_1(s) = \frac{k}{2}\varepsilon_1 + \Big(\frac{k}{2}\varepsilon_2 + b\Big)x - \Re\big(F(s)\big) + i\left(\Big(\frac{k}{2}\varepsilon_2 + b\Big)y - \Im\big(F(s)\big)\right),$$

$$n_2(s) = \frac{k}{2}\varepsilon_1 + \Big(\frac{k}{2}\varepsilon_2 + b\Big)x + \Re\big(F(s)\big) + i\left(\Big(\frac{k}{2}\varepsilon_2 + b\Big)y + \Im\big(F(s)\big)\right).$$

Since k, b and ε_i, $i = 1, 2$ are positive, using Lemma 1 it is easy to verify that for $s = x + iy$ with $x \geq 0$, we get

$$\Big(\frac{k}{2}\varepsilon_1 + \Big(\frac{k}{2}\varepsilon_2 + b\Big)x - \Re\big(F(s)\big)\Big)^2 < \Big(\frac{k}{2}\varepsilon_1 + \Big(\frac{k}{2}\varepsilon_2 + b\Big)x + \Re\big(F(s)\big)\Big)^2,$$

$$\Big(\frac{k}{2}\varepsilon_2 + b\Big)y - \Im\big(F(s)\big)^2 \leq \Big(\frac{k}{2}\varepsilon_2 + b\Big)y + \Im\big(F(s)\big)^2.$$

This implies that for all $\Re(s) \geq 0$, we have

$$\Big|\frac{K_1(s)}{K_2(s)}\Big| < 1. \tag{40}$$

Combining (39) and (40), we obtain (38), which proves the analyticity of $1/K(s)$. Finally, the functions $P(s)$ and $W(s)$ are in H_0^∞. Hence the system (2)–(6) is input-output stable.

The plant transfer function $P(s)$ can be written as the product of an H^∞-unit $P_0(s)$ and an inner function (i.e., H^∞-factorization)

$$P(s) = P_0(s)e^{-\beta_2 s},$$

where

$$P_0(s) = \frac{F(s)e^{(\lambda_1(s)+\beta_2 s)}}{(k\lambda_2(s) + bs) - (k\lambda_1(s) + bs)e^{-F(s)}}$$

is in H_0^∞ together with its inverse, which is an H^∞-unit by the same argument as above and the properties of $\lambda_1(s)$ in (34). The inner function $e^{-\beta_2 s}$ is a pure time delay of the plant transfer function. It follows that

$$\lim_{\mathbb{R}\ni s\in\to+\infty} P(s) = \lim_{\mathbb{R}\ni s\to+\infty} P_0(s)e^{-\beta_2 s} = 0.$$

Recall that the disturbance transfer function $W(s)$ is given as follows:

$$W(s) = \frac{\lambda_1(s)e^{-F(s)} - \lambda_2(s)}{(k\lambda_1(s) + bs)e^{-F(s)} - (k\lambda_2(s) + bs)},$$

where

$$\lim_{\mathbb{R}\ni s\to+\infty} F(s) = +\infty.$$

due to (28).

Finally, taking into account the form of $\lambda_i's$ and $F(\cdot)$, it is easy to show that

$$\lim_{\mathbb{R}\ni s\to+\infty} W(s) = D_d,$$

where

$$D_d = \frac{a_1(\alpha_2 - \beta_2 + \sqrt{\alpha_2 + \beta_2})}{ka_1(\alpha_2 - \beta_2 + \sqrt{\alpha_2 + \beta_2}) + 2}.$$

This completes the proof. ■

4. System Regularity

From Theorem 2, the operator $\mathcal{A}^{-1}$ exists and the growth bound $\omega_0(\mathcal{A}) = \lim_{t\to+\infty} t^{-1}\log\|e^{t\mathcal{A}}\|$ of the analysed semigroup is negative (Engel and Nagel, 2000). Now, we have to define Hilbert spaces H_1 and H_{-1} as follows: H_1 is $\mathcal{D}(\mathcal{A})$ with the norm $\|h\|_1 = \|\mathcal{A}h\|_H$ and H_{-1} is its completion with respect to the norm $\|h\|_{-1} = \|\mathcal{A}^{-1}h\|_H$, also known as the extrapolated space associated with $e^{t\mathcal{A}}$. So we have $H_1 \subset H \subset H_{-1}$, densely and with continuous embedding. The operator $\mathcal{A}$ has a unique extension to the whole space H because it is defined on a dense set $\mathcal{D}(\mathcal{A})$ in H and is continuous from H to H_{-1}. The semigroup $e^{t\mathcal{A}}$ can be extended to a C_o-semigroup on H_{-1}, whose generator is nothing else than the extended operator $\mathcal{A} \in \mathcal{L}(H, H_{-1})$ (Engel and Nagel, 2000)).

We define the duality product on $H_{-1} \times \mathcal{D}(\mathcal{A}^*)$ by a continuous extension of the inner product on H: For all $h \in H$ and all $g \in \mathcal{D}(\mathcal{A}^*)$, $\langle h, g\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = \langle h, g\rangle_H$. For each $h \in H_{-1}$, by taking $h_n \in H$ such that $\|h - h_n\|_{-1} \xrightarrow[n\to+\infty]{} 0$, we set

$$\langle h, g\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = \lim_{n\to+\infty} \langle h_n, g\rangle_H, \quad \forall g \in \mathcal{D}(\mathcal{A}^*).$$

For each $h \in H_{-1}$, the mapping defined by $g \longrightarrow \langle h, g\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)}$ is a continuous linear form on $\mathcal{D}(\mathcal{A}^*)$. Conversely, given a continuous linear form ψ on $\mathcal{D}(\mathcal{A}^*)$, there exists a unique $h_\psi \in H_{-1}$ such that

$$\psi(h) = \langle h_\psi, f\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)}, \quad \forall f \in \mathcal{D}(\mathcal{A}^*).$$

In other words, the mapping $J : H_{-1} \longrightarrow \mathcal{D}'(\mathcal{A}^*)$ (the prime means the topological dual of $\mathcal{D}(\mathcal{A}^*)$) such that $Jh(f) = \langle h, f\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)}$ is an isomorphism.

The adjoint operator $\mathcal{A}^*$ is defined by

$$\mathcal{D}(\mathcal{A}^*) = \big\{(f_1, f_2)^T \in H^1[0,1]) \times H^1[0,1] \mid f_1(1) = D_1^* f_2(1), \ \ f_2(0) = D_0^* f_1(0)\big\},$$

where

$$D_0^* = -\frac{\lambda^-}{\lambda^+} D_0, \quad D_1^* = -\frac{\lambda^+}{\lambda^-} D_1,$$

and for each $f \in \mathcal{D}(\mathcal{A}^*)$ we have

$$\mathcal{A}^* f(x) = -A \frac{\partial f(x)}{\partial x} + B^* f(x).$$

The boundary output operator C is a continuous linear form on $(H^1[0,1])^2$ defined by $Cf = D(f_1(1) + f_2(1))$ and the restriction to H_1, which is denoted with the same letter, is also a continuous form on H_1. Moreover,

$$Cf = D(1 + D_1) f_1(1) = F f_1(1), \quad \forall f \in H_1. \tag{41}$$

Theorem 4. *Given $u(\cdot), d(\cdot) \in C_0^\infty(]0, +\infty[)$, the system (14) emanating from $\Psi_0 = 0$ at $t = 0$ has a unique solution $\Psi(\cdot,\cdot) \in C([0,+\infty[, H)$. Moreover, there exist $B_u, B_d \in H_{-1}$ such that*

$$\begin{cases} \Psi(\cdot,t) = \int_0^t \big(\beta^- e^{(t-\tau)\mathcal{A}} B_u u(\tau) \\ \qquad\qquad + \beta^+ e^{(t-\tau)\mathcal{A}} B_d d(\tau)\big)\,\mathrm{d}\tau, \\ \beta^- = \lambda^- E_0 = -1, \beta^+ = \lambda^+ E_1. \end{cases} \tag{42}$$

Proof. Since the system (14) is of the form (7), the existence and uniqueness of the solution $\Psi(\cdot,\cdot) \in C([0,+\infty[, H)$ is guaranteed by (Russell, 1978, Thm. 3.1). It only remains to prove that this solution $\Psi(\cdot,\cdot)$ given by (42) is continuous from $[0,+\infty[$ into H and satisfies (14). Introduce the adjoint system associated with (14):

$$\begin{cases} \frac{\partial p(x,t)}{\partial t} = A \frac{\partial p(x,t)}{\partial x} - B^* p(x,t), \\ \qquad\qquad (x,t) \in]0,1[\times [0,T[, \\ p_1(1,t) = D_1^* p_2(1,t), \\ p_2(0,t) = D_0^* p_1(0,t), \\ p(\cdot,T) = p_0 \in \mathcal{D}(\mathcal{A}^*). \end{cases} \tag{43}$$

An easy computation gives

$$\begin{aligned} \Big\langle \frac{\partial \Psi(\cdot,t)}{\partial t}, p(\cdot,t) \Big\rangle_H &= \Big\langle A \frac{\partial \Psi(\cdot,t)}{\partial x} + B^* \Psi(\cdot,t), p(\cdot,t) \Big\rangle_H \\ &= \beta^- u(t) p_1(0,t) + \beta^+ d(t) p_2(1,t) \\ &\quad - \langle \Psi(\cdot,t), \frac{\partial p(\cdot,t)}{\partial t} \rangle_H. \end{aligned}$$

This yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \big\langle \Psi(\cdot,t), p(\cdot,t) \big\rangle_H = \beta^- u(t) p_1(0,t) + \beta^+ d(t) p_2(1,t). \tag{44}$$

Integrating (44) and evaluating the result at $t = T$, we obtain

$$\langle \Psi(\cdot,T), p_0 \rangle_H = \int_0^T \big(\beta^- u(t) p_1(0,t) + \beta^+ d(t) p_2(1,t)\big)\,\mathrm{d}t.$$

Define the continuous (boundary operators) linear forms Γ_1 and Γ_2 on $(H^1[0,1])^2$ as follows:

$$\Gamma_1 : \big(H^1[0,1]\big)^2 \longrightarrow \mathbb{R}, \quad f \longmapsto f_1(0) = \Gamma_1 f,$$

$$\Gamma_2 : \big(H^1[0,1]\big)^2 \longrightarrow \mathbb{R}, \quad f \longmapsto f_2(1) = \Gamma_2 f.$$

Of course, $e^{t\mathcal{A}^*}$ is exponentially stable, which implies that $(\mathcal{A}^*)^{-1}$ is bounded on H. Therefore, the linear forms $\Gamma_1(\mathcal{A}^*)^{-1}$ and $\Gamma_2(\mathcal{A}^*)^{-1}$ are continuous on H. From Riesz's representation theorem, there are unique elements $\gamma_1, \gamma_2 \in H$ such that

$$\forall f \in H, \begin{cases} \Gamma_1(\mathcal{A}^*)^{-1} f = \langle \gamma_1, f \rangle_H, \\ \Gamma_2(\mathcal{A}^*)^{-1} f = \langle \gamma_2, f \rangle_H. \end{cases} \tag{45}$$

Since

$$p_1(0,t) = \Gamma_1 p(\cdot,t) = \Gamma_1 e^{(T-t)\mathcal{A}^*} p_0,$$

$$p_2(1,t) = \Gamma_2 p(\cdot,t) = \Gamma_2 e^{(T-t)\mathcal{A}^*} p_0,$$

we obtain

$$\begin{aligned} &\langle \Psi(\cdot,T), p_0 \rangle_H \\ &\quad = \int_0^T \big(\beta^- u(t) \Gamma_1 e^{(T-t)\mathcal{A}^*} p_0 \\ &\qquad + \beta^+ d(t) \Gamma_2 e^{(T-t)\mathcal{A}^*} p_0\big)\,\mathrm{d}t, \\ &\quad = \int_0^T \big(\beta^- u(t) \Gamma_1 (\mathcal{A}^*)^{-1} \mathcal{A}^* e^{(T-t)\mathcal{A}^*} p_0 \\ &\qquad + \beta^+ d(t) \Gamma_2 (\mathcal{A}^*)^{-1} \mathcal{A}^* e^{(T-t)\mathcal{A}^*} p_0\big)\,\mathrm{d}t. \end{aligned} \tag{46}$$

Substituting (45) into (46), we get

$$\begin{aligned} &\langle \Psi(\cdot,T), p_0 \rangle_H \\ &\quad = \int_0^T \big\{\beta^- \langle \gamma_1, \mathcal{A}^* e^{(T-t)\mathcal{A}^*} p_0 \rangle_H u(t) \\ &\qquad + \beta^+ \langle \gamma_2, \mathcal{A}^* e^{(T-t)\mathcal{A}^*} p_0 \rangle_H d(t)\big\}\,\mathrm{d}t. \end{aligned} \tag{47}$$

Integrating the first term in (47) by parts, we obtain

$$\begin{aligned} &\Psi(\cdot,t) \\ &\quad = -\beta^- \gamma_1 u(t) - \beta^+ \gamma_2 d(t) + \int_0^t \big(\beta^- e^{(t-\tau)\mathcal{A}} \gamma_1 u'(\tau) \\ &\qquad + \beta^+ e^{(t-\tau)\mathcal{A}} \gamma_2 d'(\tau)\big)\,\mathrm{d}\tau, \end{aligned} \tag{48}$$

where $u'(t)$ and $d'(t)$ are the derivatives of $u(t)$ and $d(t)$, respectively

It is easy to check that the function $\Psi(\cdot,t)$ given by (48) is a classical solution of the system (14).

For the previously defined Γ_1 and Γ_2 there exist unique $B_u, B_d \in H_{-1}$ such that

$$\forall f \in \mathcal{D}(\mathcal{A}^*), \begin{cases} \langle B_u, f\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = \Gamma_1 f = f_1(0), \\ \langle B_d, f\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = \Gamma_2 f = f_2(1). \end{cases} \tag{49}$$

Moreover, it is easy to see that for all $\Psi \in H$ and $p \in \mathcal{D}(\mathcal{A}^*)$, we have

$$\langle e^{t\mathcal{A}}\mathcal{A}\Psi, p\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} = \langle \Psi, e^{t\mathcal{A}^*}\mathcal{A}^* p\rangle_H, \quad \forall t \geq 0. \tag{50}$$

Using (47) and (50), a direct computation gives

$$\begin{aligned} &\langle \Psi(\cdot,t), p\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} \\ &\quad = \langle \Psi(\cdot,t), p\rangle_H, \\ &\quad = \int_0^t \big(\beta^- \langle e^{(T-\tau)\mathcal{A}}\mathcal{A}\gamma_1, p\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} u(\tau) \\ &\qquad + \beta^+ \langle e^{(T-\tau)\mathcal{A}}\mathcal{A}\gamma_2, p\rangle_{H_{-1},\mathcal{D}(\mathcal{A}^*)} d(\tau)\big)\,\mathrm{d}\tau \end{aligned}$$

for all $p \in D(A^*)$.

According to (45) and (49), we obtain

$$\begin{aligned} \Psi(\cdot,t) = \int_0^t &\big(\beta^- e^{(T-\tau)\mathcal{A}} B_u u(\tau) \\ &+ \beta^+ e^{(T-\tau)\mathcal{A}} B_d d(\tau)\big)\,\mathrm{d}\tau. \end{aligned} \tag{51}$$

This completes the proof. ■

Remark 2. Since γ_1 and γ_2 in (45) are in $(H^1[0,1])^2$ (see Appendix 1) and the integral terms are in $\mathcal{D}(\mathcal{A})$ (Pazy, 1983), the output function is well defined:

$$\begin{aligned} y(t) = &-\beta^- C\gamma_1 u(t) - \beta^+ C\gamma_2 d(t) \\ &+ C\int_0^t (\beta^- e^{(T-\tau)\mathcal{A}}\gamma_1 u'(\tau) \\ &+ \beta^+ e^{(T-\tau)\mathcal{A}}\gamma_2 d'(\tau))\,\mathrm{d}\tau. \end{aligned}$$

Performing the Laplace transform in (48) and (51), we obtain the following identity:

$$\begin{aligned} \hat{y}(s) &= \big(-\beta^- C\gamma_1 + \beta^- C(sI-\mathcal{A})^{-1}\gamma_1 s\big)\hat{u}(s) \\ &\quad + \big(-\beta^+ C\gamma_2 + \beta^+ C(sI-\mathcal{A})^{-1}\gamma_2 s\big)\hat{d}(s) \\ &= \beta^- C(sI-\mathcal{A})^{-1} B_u \hat{u}(s) \\ &\quad + \beta^+ C(sI-\mathcal{A})^{-1} B_d \hat{d}(s). \end{aligned}$$

Finally, from Theorem 3 we deduce that

$$\forall \Re(s) > 0, \begin{cases} P(s) = \big(-\beta^- C\gamma_1 \\ \qquad + \beta^- C(sI-\mathcal{A})^{-1}\gamma_1 s\big) \\ \quad = \beta^- C(sI-\mathcal{A})^{-1} B_u, \\ W(s) = \big(-\beta^+ C\gamma_2 \\ \qquad + \beta^+ C(sI-\mathcal{A})^{-1}\gamma_2 s\big) \\ \quad = \beta^+ C(sI-\mathcal{A})^{-1} B_d. \end{cases} \tag{52}$$

Now we formulate our main result.

Theorem 5. *The triple* $(C, \mathcal{A}, [B_u, B_d])$ *is a regular system satisfying*

$$\forall \Re(s) > \omega_0(\mathcal{A}), \begin{cases} P(s) = \beta^- C_L(sI_-\mathcal{A})^{-1} B_u, \\ W(s) = \beta^+ C_L(sI_-\mathcal{A})^{-1} B_d + D_d, \end{cases} \tag{53}$$

where C_L *is the Lebesgue extension of* C.

Proof. Let $\Psi(\cdot,t)$ be the solution of (14) with $u(t) = d(t) = 0$ emanating from $\Psi_0 \in \mathcal{D}(\mathcal{A})$ at $t = 0$ and $V(t) = \frac{1}{2}\|\Psi(\cdot,t)\|_H^2$. Since $\Psi(\cdot,t)$ lies in $\mathcal{D}(\mathcal{A})$ (Pazy, 1983), differentiating $V(t)$ along the trajectory of (14) and integrating by parts yields

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &= \langle \mathcal{A}\Psi(\cdot,t), \Psi(\cdot,t)\rangle_H, \\ &= \Psi(1,t)^* A\Psi(1,t) - \Psi(0,t)^* A\Psi(0,t) \\ &\quad + \int_0^t \Psi(x,t)^*(B+B^*)\Psi(x,t)\,\mathrm{d}x. \end{aligned}$$

Using (i) in the assumption (H_3) and the boundary conditions in (14), we have

$$\begin{aligned} \frac{\mathrm{d}V(t)}{\mathrm{d}t} &\leq (\lambda^- + \lambda^+ D_1^2)\Psi_1^2(1,t) \\ &\quad - (\lambda^+ + \lambda^- D_0^2)\Psi_2^2(0,t) \\ &\leq (\lambda^- + \lambda^+ D_1^2)\Psi_1^2(1,t) \end{aligned}$$

which results from using (ii) and (iii) of H_3.

Using the boundary conditions in (14), we obtain

$$\frac{\mathrm{d}V(t)}{\mathrm{d}t} \leq -r^-\Psi_1^2(1,t) \leq 0, \quad (r^- = (\lambda^- + D_1^2\lambda^+) > 0). \tag{54}$$

Integrating inequality (54) from 0 to T, we deduce that

$$r^- \int_0^T \Psi_1^2(1,t)\,\mathrm{d}t \leq \frac{1}{2}\big(\|\Psi_0\|_H^2 - \|\Psi(\cdot,T)\|_H^2\big) \leq \frac{1}{2}\|\Psi_0\|_H^2.$$

Since $\Psi(\cdot,t) \in \mathcal{D}(\mathcal{A})$, from (41) we get $C\Psi(\cdot,t) = Ce^{t\mathcal{A}}\Psi_0 = F\Psi_1(1,t)$. Moreover,

$$\int_0^T (Ce^{t\mathcal{A}}\Psi_0)^2\,\mathrm{d}t \leq \kappa\|\Psi_0\|_H^2, \quad (\kappa = F^2/2r^-).$$

This proves that the operator C is $\mathcal{A}$-admissible (Weiss, 1989).

By Theorem 4, it is clear that both the control operator B_u and the disturbance operator B_d are admissible (Weiss, 1989b). Now denote by C_L the Lebesgue extension of $C \in \mathcal{L}(H_1, \mathbb{R})$ in the sense of (Weiss, 1989a). From Lemma 1, the transfer functions $P(s)$ and $W(s)$ have strong limits at $+\infty$ along the real axis. Therefore, applying Theorems 1.3 and 4.7 of (Weiss, 1994), respectively, we assert that $(C_L, \mathcal{A}, B_u)$ and $(C_L, \mathcal{A}, B_d)$ are regular triples and that for all $\Re(s) > \omega_0$,

$$\begin{cases} P(s) = \beta^- C_L (sI_- \mathcal{A})^{-1} B_u \in H^\infty_{\omega_0}, \\ W(s) = \beta^+ C_L (sI_- \mathcal{A})^{-1} B_d + D_d \in H^\infty_{\omega_0}. \end{cases} \tag{55}$$

Finally, identifying (53) and (55), we deduce that for all $\Re(s) > 0$, we have

$$\begin{cases} P(s) &= \beta^- C_L (sI_- \mathcal{A})^{-1} B_u = \beta^- C (sI - \mathcal{A})^{-1} B_u \\ &= (-\beta^- C \gamma_1 + \beta^- C (sI - \mathcal{A})^{-1} \gamma_1 s), \\ W(s) &= \beta^+ C_L (sI_- \mathcal{A})^{-1} B_d + D_d \\ &= \beta^+ C (sI - \mathcal{A})^{-1} B_d \\ &= \big(-\beta^+ C \gamma_2 + \beta^+ C (sI - \mathcal{A})^{-1} \gamma_2 s \big). \end{cases} \tag{56}$$

According to Theorem 2.3 of (Weiss, 1994), the system (14) is written as

$$\begin{cases} \dot{\Psi}(\cdot, t) = \mathcal{A} \Psi(\cdot, t) + \beta^- B_u u(t) + \beta^+ B_d d(t), \\ y(t) = C_L \Psi(\cdot, t) + D_d d(t), \end{cases} \tag{57}$$

a.e. for $t \geq 0$. ■

5. Conclusion

We have studied the notions, of internal and external stability for an irrigation canal system. This process is representative for a class of hyperbolic systems. The paper shows how this kind of systems is transformed into the form of classical symmetric hyperbolic systems, which allows us to prove exponential stability using Rauch and Taylor's theorem. The transfer function is derived, and we show that the system is input-output stable. The irrigation canal system has unbounded input and output operators. It is shown that the system is regular and the transfer functions of the system are in H^∞_μ for some $\mu < 0$. The results presented here are essential for various controller design methods to be applied for this class of hydraulic systems, e.g., H^∞-control (Francis and Zames, 1984; Gauthier and Xu, 1991; Zames and Francis, 1983), P.I. controllers (Bounit, 2003a; Pohjolainen, 1985a; 1985b; Xu and Jerbi, 1995) and output feedback (Curtain, 1988; Rebarber, 1993).

Acknowledgement

The author started this project together with J.P. Baume from the Division of Irrigation at CEMAGREF of Montpellier and profited greatly from many joint discussions.

References

Baume J.P. and Sau H. (1997): *Steady of irrigation canal dynamics for control purpose*. — Proc. 1st Int. Workshop *Regulation of Irrigation Canals, RIC'97*, Marrakech, Morocco, pp. 3–12.

Baume J.P. (1990): *Régulation des cannaux d'irrgation: Étude du sous-systèmes bief avec vanne*. — M.Sc. thesis, Université des Sciences et Techniques du Languedoc (USTL), Montpellier, France.

Bounit H. (2003a): *Robust PI-controller for an irrigation canal system*. — (In preparation).

Bounit H. (2003b): *H^∞-controller for an irrigation canal system*. — (In preparation).

Bounit H., Hammouri H. and Sau J. (1997): *Regulation of an irrigation canal through the semigroup approach*. — Proc. 1st Int. Workshop *Regulation of Irrigation Canals, RIC'97*, Marrakech, Morocco, pp. 261–267.

Burt C.M., Clemmens A.J. and Streslkoff T.S. (1998): *Influence of canal geometry and dynamics on controllability*. — J. Irrig. Drain. Eng., ASCE, Vol. 124, No. 1, pp. 16–22.

Chow V.T. (1985): *Open Channels Hydraulics*. — New York: Mac Graw Hill .

Clemmens A.J., Streslkoff T.S. and Gooch R.S. (1995): *Influence of canal geometry and dynamics on controllability*. — Proc. 1st Int. Conf. *Water Res. Eng., ASCE*, Reston, VA, pp. 21–25.

Curtain R.F. (1988): *Equivalence of input-output stability and exponential stability for infinite dimensional systems*. — Math. Syst. Theory, Vol. 21, pp. 19–48.

Curtain R.F. and Zwart H. (1995): *An Introduction to Infinite Dimensional Linear System Theory*. — New York: Springer.

Engel K.J. and Nagel R. (2000): *One Parameter Semigroups for Linear Evolution Equations*. — New York: Springer.

Francis B.A. and Zames G. (1984),: *On H^∞-optimal sensitivity theory for SISO feedback systems*. — IEEE Trans. Automat. Contr., Vol. AC–29, No. 1, pp. 9–16.

Gauthier J.P. and Xu C.Z. (1991): *H^∞-control of a distributed parameter system with non-minimun phase*. — Int. J. Contr., Vol. 53, No. 1, pp. 45–79.

Mahmood M.A. and Yevjevich V. (1975): *Unsteady Flow in Open Channels, Vols. 1 and 2*. — Fort Collins USA: Water Resources Publications.

Miller M.A. and V. Yevjevich V. (1975): *Unsteady Flow in Open Channels, Vol. 3*. — Fort Collins USA: Water Resources Publications.

Pazy A. (1983): *Semigroups of Linear Operators and Applications to Partial Differential Equations*. — New York: Springer.

Pohjolainen S.A. (1985a): *Robust controller for infinite systems with exponential strongly continuous semigroups*. — J. Math. Anal. Appl., Vol. 111, pp. 622–636.

Pohjolainen S.A. (1985b): *Robust multivariable PI-controllers for infinite dimensional systems*. — IEEE Trans. Automat. Contr., Vol. 27, pp. 17–30.

Rauch J. and Taylor M. (1974): *Exponential decay of solution to hyperbolic equations in bounded domain*. — Indiana Univ. Math. J., Vol. 24, pp. 79–86.

Rebarber R. (1993): *Conditions for the equivalence of internal and external stability for distributed parameter systems*. — IEEE Trans. Automat. Contr., Vol. 38, No. 6, pp. 994–998.

Russel D.L. (1978): *Controllability and stabilizability theory for linear partial differential equation: Recent progress and open questions*. — SIAM Review, Vol. 20, No. 4, pp. 639–739.

Weiss G. (1989a): *Admissible observation operators for linear semigroups*. — Israel J. Math., Vol. 65, pp. 17–43.

Weiss G. (1989b): *Admissibility of unbounded control operators*. — SIAM. J. Contr. Optim., No. 27, pp. 527–545.

Weiss G. (1994): *Transfert function of regular linear systems. Part I: Characterization of regularity*. — Trans. Amer. Math. Soci., Vol. 342, pp. 827–854.

Xu C.Z. and Jerbi H. (1995): *A robust PI-controller for infinite dimensional systems*. — Int. J. Contr., Vol. 61, No. 1, pp. 33–45.

Yoon M.G. and Lee B.H. (1991): *An approximation approach to H^∞ control problems for distributed parameter systems*. — Automatica, Vol. 33, No. 11, pp. 2049–2052.

Zames G. and Francis B.A. (1983): *Feedback, minimax sensitivity and optimal robustness*. — IEEE Trans. Automat. Contr., Vol. AC-28, No. 5, pp. 585–600.

Appendices

1. Regularity of γ_1 and γ_2

Computing γ_1 and γ_2 is equivalent to solving the following differential equations:

$$\begin{cases} \dfrac{\partial p(x)}{\partial x} = A^{-1}Bp(x) - A^{-1}f(x), \\ p_1(1) = D_1^* p_2(1), \\ p_2(0) = D_0^* p_1(0) \\ \langle \gamma_1, f\rangle_H = p_1(0), \\ \langle \gamma_2, f\rangle_H = p_2(1). \end{cases} \tag{58}$$

This problem has a unique solution $p \in \mathcal{D}(\mathcal{A}^*)$ for all $f \in H$. It is given by the following integral equation:

$$p(x) = e^{A^{-1}Bx}p(0) - \int_0^x e^{A^{-1}B(x-\tau)}A^{-1}f(\tau)\,\mathrm{d}\tau. \tag{59}$$

Evaluating (59) at $x = 1$ and recalling that $p_2(0) = D_0^* p_1(0)$, we have

$$\begin{aligned} p_1(1) &= \begin{pmatrix}1\\0\end{pmatrix}^* e^{A^{-1}B}p(0) \\ &\quad - \begin{pmatrix}1\\0\end{pmatrix}^* \int_0^1 e^{A^{-1}B(1-x)}A^{-1}f(x)\,\mathrm{d}x \\ &= \begin{pmatrix}1\\0\end{pmatrix}^* e^{A^{-1}B}\begin{pmatrix}1\\D_0^*\end{pmatrix}p_1(0) \\ &\quad - \begin{pmatrix}1\\0\end{pmatrix}^* \int_0^1 e^{A^{-1}B(1-x)}A^{-1}f(x)\,\mathrm{d}x, \end{aligned}$$

$$\begin{aligned} p_2(1) &= \begin{pmatrix}0\\1\end{pmatrix}^* e^{A^{-1}B}p(0) \\ &\quad - \begin{pmatrix}0\\1\end{pmatrix}^* \int_0^1 e^{A^{-1}B(1-x)}A^{-1}f(x)\,\mathrm{d}x, \\ &= \begin{pmatrix}0\\1\end{pmatrix}^* e^{A^{-1}B}\begin{pmatrix}1\\D_0^*\end{pmatrix}p_1(0) \\ &\quad - \begin{pmatrix}0\\1\end{pmatrix}^* \int_0^1 e^{A^{-1}B(1-x)}A^{-1}f(x)\,\mathrm{d}x. \end{aligned}$$

Now, since $p_1(1) = D_1^* p_2(1)$, a trivial verification shows that, for all $f \in H$, w have

$$\begin{aligned} p_1(0) &= \left(\begin{pmatrix}1\\-D_1^*\end{pmatrix}^* e^{A^{-1}B}\begin{pmatrix}1\\D_0^*\end{pmatrix}\right)^{-1}\begin{pmatrix}1\\-D_0^*\end{pmatrix}^* \\ &\quad \times \int_0^1 e^{A^{-1}B(1-x)}A^{-1}f(x)\,\mathrm{d}x, \end{aligned}$$

$$\begin{aligned} p_2(1) &= \begin{pmatrix}0\\1\end{pmatrix}^* e^{A^{-1}B}\begin{pmatrix}1\\D_0^*\end{pmatrix} \\ &\quad \times \left(\begin{pmatrix}1\\-D_1^*\end{pmatrix}^* e^{A^{-1}B}\begin{pmatrix}1\\D_0^*\end{pmatrix}\right)^{-1}\begin{pmatrix}1\\-D_0^*\end{pmatrix}^* \\ &\quad \times \int_0^1 e^{A^{-1}B(1-x)}A^{-1}f(x)\,\mathrm{d}x \\ &\quad - \begin{pmatrix}0\\1\end{pmatrix}^* \int_0^1 e^{A^{-1}B(1-x)}A^{-1}f(x)\,\mathrm{d}x. \end{aligned}$$

Thus we obtain

$$\begin{aligned}\gamma_1^* &= \left(\begin{pmatrix}1\\-D_1^*\end{pmatrix}^* e^{A^{-1}B}\begin{pmatrix}1\\D_0^*\end{pmatrix}\right)^{-1}\\ &\quad\times\begin{pmatrix}1\\-D_0^*\end{pmatrix}^* e^{A^{-1}B(1-x)}A^{-1},\end{aligned}$$

$$\begin{aligned}\gamma_2^* &= \begin{pmatrix}0\\1\end{pmatrix}^* e^{A^{-1}B}\begin{pmatrix}1\\D_0^*\end{pmatrix}\\ &\quad\times\left(\begin{pmatrix}1\\-D_1^*\end{pmatrix}^* e^{A^{-1}B}\begin{pmatrix}1\\D_0^*\end{pmatrix}\right)^{-1}\\ &\quad\times\begin{pmatrix}1\\-D_0^*\end{pmatrix}^* e^{A^{-1}B(1-x)}A^{-1}\\ &\quad-\begin{pmatrix}0\\1\end{pmatrix}^* e^{A^{-1}B(1-x)}A^{-1}.\end{aligned}$$

Finally, the elements γ_1 and γ_2 are analytic functions of x. ■

2. Notation and Coefficients

The following symbols are used in this paper:

- Q discharge (L^3/T)
- Q_0 steady (reference) discharge (L^3/T)
- Y_0 water depth at reference discharge (L)
- S cross-sectional area (L^2)
- S_0 cross-sectional area at reference discharge (L^2)
- Z water-surface elevation (L)
- P_0 wetted perimeter at reference discharge (L)
- R hydraulic radius (cross-sectional area/wetted perimeter) (L)
- F_0 Froude number of reference discharge at reference depth in reference section
- J_0 friction slope at reference discharge
- g gravity acceleration (L/T^2)
- C_i G_i's discharge coefficient for $i = u, d$
- L_i G_i's width for $i = u, d$
- L_c canal length (L)
- L_0 top width at depth, Y_0 at reference discharge (L) (see Fig. 2)
- L_b bottom width in trapezoidal section (L) (see Fig. 2)
- s side slope in trapezoidal section (see Fig. 2)
- K_0 strickler coefficient in reference section $(L^{1/3}/T)$
- K_s section form coefficient
- K_p section form coefficient
- x distance along canal (L)
- t time (T)
- T_0 reference time (T)
- W Gate opening (L)
- '*' dimensionless counterpart

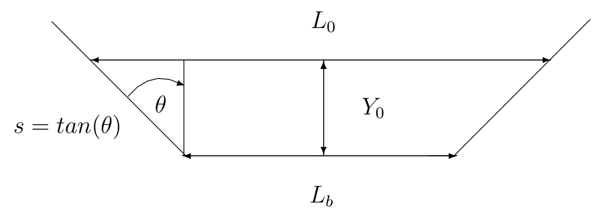

Fig. 2. Cross-sectional steady flow area.

Furthermore, we have

$$S_0 = Y_0(L_b + sY_0), \qquad P_0 = L_b + 2Y_0\sqrt{1+s^2},$$

$$Q_0 = \frac{J_0^{1/2} S_0^{5/3}}{K_0 P_0^{2/3}}, \qquad T_0 = \frac{S_0 L_c}{Q_0},$$

$$F_0^2 = \frac{Q_0^2}{g S_0^2 Y_0}, \qquad \chi = \frac{J_0 L_c}{Y_0},$$

$$K_s = \frac{L_0 Y_0}{S_0}, \qquad K_p = \frac{Y_0}{P_0}.$$

The dimensionless variables are

$$x^* = \frac{x}{L_c}, \quad Z^* = \frac{Z}{Y_0}, \quad S^* = \frac{S}{S_0}, \quad Q^* = \frac{Q}{Q_0},$$

$$R^* = \frac{R}{R_0}, \quad K^* = \frac{K}{K_0}, \quad t^* = \frac{t}{T_0}.$$

From the dimensionless nonlinear Saint-Venant equation and for a canal with a uniform geometry of the slope β (Bounit *et al.*, 1997), the steady state is described by the following conditions:

1. $K_0(x)$, $S_0(x)$ and $Q_0(x)$ are constant,
2. $\mathrm{d}Z_0(x)/\mathrm{d}x = -\beta$,
3. $-\beta + \chi J_0^* = 0$ (i.e., $-\beta K_0^{*2} R_0^{*4/3} S_0^{*2} + \chi Q_0^{*2} = 0$).

The coefficients a_i' and c_i are defined as follows:

$$a_1 = \frac{1}{K_s L_0^*}, \quad a_2 = \frac{S_0^*}{F_0^2} - \frac{K_s Q_0^{*2} L_0^*}{S_0^{*2}}, \quad a_3 = 2\frac{Q_0^*}{S_0^*},$$

$$a_4 = -\beta \frac{K_s L_0^*}{F_0^2} - \frac{7}{3}\frac{\chi K_s J_0^* L_0^*}{F_0^2} + \frac{4}{3}\frac{\chi S_0^* J_0^* K_p}{F_0^2 P_0^*} 2\sqrt{1+s^2},$$

$$a_5 = 2\frac{\chi S_0^* J_0^*}{F_0^2 Q_0^*},$$

$$c_1 = \frac{C_u L_u \sqrt{2gY_0^3 \delta h_u}}{Q_0}, \quad c_2 = \frac{C_d L_d \sqrt{2gY_0^3}}{Q_0} \frac{W_{us}^*}{2\sqrt{\delta h_u}},$$

$$c_3 = \frac{C_d L_d \sqrt{2gY_0^3}}{Q_0} \frac{W_{ds}^*}{2\sqrt{\delta h_d}}.$$

What is more, $\delta h_i = \delta Z_{us}^{*,i} - \delta Z_{ds}^{*,i}$ for $i = u, d$, where $\delta Z_{us}^{*,i}$ and $\delta Z_{ds}^{*,i}$ are respectively the dimensionless upstream and downstream water elevation variations corresponding to the reference steady state. W_{is}^* is the dimensionless opening of G_i corresponding to the reference steady state.

Received: 18 December 2002
Revised: 12 September 2003