Int. J. Appl. Math. Comput. Sci., 2014, Vol. 24, No. 2, 249–257
DOI: 10.2478/amcs-2014-0018

TIKHONOV REGULARIZATION AND CONSTRAINED QUADRATIC PROGRAMMING FOR MAGNETIC COIL DESIGN PROBLEMS

BARTŁOMIEJ GARDA, ZBIGNIEW GALIAS

Department of Electrical Engineering
AGH University of Science and Technology, al. Mickiewicza 30, 31-231 Cracow, Poland
e-mail: {bgarda,galias}@agh.edu.pl

In this work, the problem of coil design is studied. It is assumed that the structure of the coil is known (i.e., the positions of simple circular coils are fixed) and the problem is to find current distribution to obtain the required magnetic field in a given region. The unconstrained version of the problem (arbitrary currents are allowed) can be formulated as a Least-SQuares (LSQ) problem. However, the results obtained by solving the LSQ problem are usually useless from the application point of view. Moreover, for higher dimensions the problem is ill-conditioned. To overcome these difficulties, a regularization term is sometimes added to the cost function, in order to make the solution smoother. The regularization technique, however, produces suboptimal solutions. In this work, we propose to solve the problem under study using the constrained Quadratic Programming (QP) method. The methods are compared in terms of the quality of the magnetic field obtained, and the power of the designed coil. Several 1D and 2D examples are considered. It is shown that for the same value of the maximum current the QP method provides solutions with a higher quality magnetic field than the regularization method.

Keywords: coil design problem, constrained quadratic programming, Tikhonov regularization.

1. Introduction

The coil design problem is to find the shape and/or current distribution in the coil to excite the required magnetic field in a specific region, located in the interior of the coil. The problem belongs to the class of magnetic synthesis issues.

The problem has many applications, e.g., in medical imaging diagnostics. Magnetic resonance imaging devices are built of three types of coils. The first one is the coil exciting strong and homogeneous magnetic field. Magnetic homogeneity has a great impact on the imaging quality. The second coil, called the "gradient coil", is built to produce the gradient field (a linearly changing magnetic field), and the third coil is of radio frequency (Jin, 1999). Coils of the first type are usually made of superconducting materials due to the necessity of using very large currents. The other two coils are usually standard resistive ones. A homogeneous magnetic field is also required in devices like magnetic separators or filters.

In this work, we concentrate on the problem of designing coils exciting strong homogeneous magnetic fields. Although the main aspect of the design is the homogeneity of the magnetic field, energy issues should also be taken into account. This is especially important for standard (resistive) coils which dissipate energy due to the flow of the current. Therefore, one of the design goals should be the minimization of the energy dissipated in the coil. For standard coils the energy dissipated by the coil is proportional to the sum of squares of currents of simple coils. For superconducting coils this factor becomes important in the case of an abnormal work of the device.

Assuming that the shape of the coil is fixed (for example, the coil is composed of a fixed number of simple coils, and their positions are fixed), the problem under study can be described as a system of linear equations. Usually the number of simple coils is smaller than that of points in the target region. Therefore, the linear problem is overdetermined and the system of linear equations has no solution. A standard approach to approximate the solution of overdetermined systems is to use the Least-SQuares (LSQ) method. It appears that the solutions found using this method are often useless from the application point of view. The currents found are very large and have opposite signs in neighboring simple coils (Garda and Galias, 2010).

In order to obtain solutions useful from a practical point of view, one has to introduce some additional constrains. One of the frequently used methods to give preference to a particular solution with desirable properties is the Tikhonov regularization method (Sikora *et al.*, 1980; Zhu *et al.*, 2012). This method of solving LSQ problems is widely used when the linear system of equations is ill-conditioned. Another approach is to introduce additional constrains on the currents. The simplest one is to assume that all currents should be non-negative (Garda, 2012). This leads to the Non-Negative Least SQuares (NNLSQ) problem. Garda and Galias (2012) show that the NNLSQ approach outperforms the Tikhonov regularization based method both in terms of the quality of the solution and computation time. However, values of currents found using the NNLSQ method are very large, which makes the solution impractical because of the energy aspects. Heuristic methods (for example, genetic algorithms) are sometimes used to solve the constrained coil design problem (Fisher *et al.*, 1997). However, in the problem of linear coil design these methods are outperformed by standard linear algebra tools (Garda and Galias, 2010) and will not be studied here.

In this work, we consider a constrained coil design problem, where the additional assumption is that currents in simple coils belong to a certain interval, i.e. $i_k \in [i_{\min}, i_{\max}]$. In the simplest case, when one assumes that all currents have the same sign, i.e., $i_{\min} = 0$, $i_{\max} = \infty$, this problem reduces to the NNLSQ one. Using finite lower and upper bounds on currents limits the energy dissipated in the coil (the energy is proportional to the sum of squared currents), and therefore the energy aspect is automatically addressed by this method. It is shown that the problem considered can be solved using the Quadratic Programming (QP) method with box constrains.

The structure of the paper is the following. In Section 2, the coil design problem under study is defined. The Tikhonov regularization approach and the quadratic programming method are briefly described in Section 3. In Section 4, the four methods (LSQ, Tikhonov regularization, NNLSQ, and constrained QP) are compared in terms of the homogeneity of the magnetic field obtained, and the energy dissipated in the coil.

It is shown that the constrained quadratic programming approach provides solutions with a more homogeneous magnetic field and lower power than the regularization method.

2. Problem description

The coil design problem can be solved using the idea of the "target field" approach (Turner, 1986). Figure 1 shows the cross-section of the coil. It is assumed that the coil is composed of $n \times k$ coaxial coils, called in the following "simple coils". Here n and k are the numbers of simple coils in the z and r directions, respectively. The goal is to excite the desired magnetic field in the target points. The number of target points is denoted by m. In the example

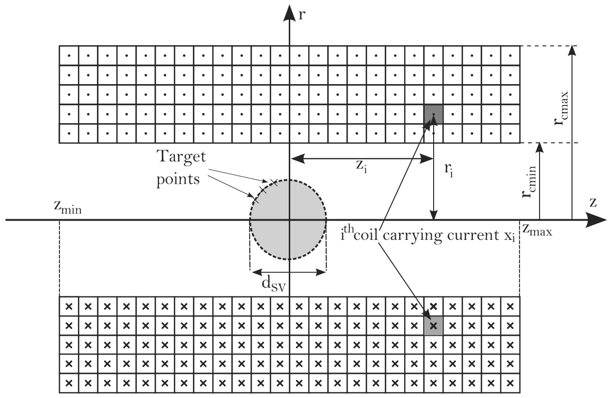

Fig. 1. Structure of the coil and target points.

shown in Fig. 1, target points lie on the circle centered at the origin of the coordinate system. We also consider the case when target points are located on the z axis. Let x_i denote the current flowing in the i-th coil.

The design problem is to find the vector of currents $x = (x_1, x_2, \ldots, x_w)^T$, where $w = n \cdot k$ is the total number of simple coils. Throughout the paper, the cases $k = 1$ and $k > 1$ will be referred to as the one-dimensional (1D) case and the two-dimensional (2D) case, respectively.

Each coil yields a contribution to the magnetic field at each target point. For the system of coaxial circular coils, only the B_z and B_r components of the magnetic field in the z and r directions need to be considered. Macovski *et al.* (2000) show that B_r is much smaller than B_z and has negligible contribution to the total magnetic field. Therefore, it is sufficient to consider the B_z component only. The contribution from the i-th simple coil with the current x_i located at the position (r_i, z_i) to the magnetic field at the target point (r_j, z_j) can be computed as

$$B_{j,i} = \frac{\mu_0}{2\pi\sqrt{(r_i + r_j)^2 + (z_j - z_i)^2}} \cdot \left(K(k) - \frac{r_i^2 + r_j^2 + (z_j - z_i)^2}{(r_i - r_j)^2 + (z_i - z_j)^2} E(k) \right) x_i, \tag{1}$$

where $k = \sqrt{4r_i r_j / ((r_i + r_j)^2 + (z_i - z_j)^2)}$ and $K(\cdot)$, $E(\cdot)$ denote the elliptic integrals of the first and second kind, respectively. When the target point is located on the z axis ($r_j = 0$), (1) reduces to

$$B_{j,i} = \frac{\mu_0 r_i^2}{2(r_i^2 + (z_i - z_j)^2)^{\frac{3}{2}}} \cdot x_i. \tag{2}$$

Let us denote by b_j the desired value of the magnetic field at the j-th target point. The goal is to find the values of currents x_i such that at each target point the difference

$$b_j - \sum_{i=1}^{w} B_{j,i}$$

between the desired field and the field exited at the target point is as small as possible.

Since the relation between the field at target points and the current at a given coil is linear, and m is larger than w, one can formulate the problem as an overdetermined set of linear equations,

$$A\,x = b, \tag{3}$$

where $A \in \mathbb{R}^{m\times w}$ is the coefficient matrix (the element $A_{j,i}$ is the coefficient at x_i in (1) or (2)), $x \in \mathbb{R}^w$ is the vector of the currents to be found, and $b \in \mathbb{R}^m$ is the vector defining the required field at target points.

3. Design methods

3.1. Least squares method. The least squares solution is the one that minimizes the sum of squared residual errors for all target points. This can be expressed by

$$\hat{x} = \arg\min_x \|A\,x - b\|_2^2, \tag{4}$$

where $\|\cdot\|_2$ denotes the Euclidean norm.

It is well known that the minimum (4) can be found by solving the set of normal equations

$$A^T A\,\hat{x} = A^T\,b. \tag{5}$$

The matrix A is usually ill-conditioned (especially for large w and m), which may lead to propagation of numerical errors and wrong solutions. Moreover, the LSQ method does not control the magnitude of the solution, and therefore solutions found in this way may be useless from a practical point of view. Examples will be given in Section 4.

3.2. Tikhonov regularization. Tikhonov regularization is a method frequently used for solving ill-posed problems (Tikhonov and Arsenin, 1977; Hansen, 1998; Prasath, 2011). In this method, the problem (4) is regularized by introducing an additional term to the minimized function,

$$\hat{x} = \arg\min_x \left(\|Ax - b\|_2^2 + \|\Lambda x\|_2^2\right), \tag{6}$$

where Λ is a suitably chosen Tikhonov matrix. Selecting $\Lambda = \lambda I$ gives preferences to solutions with a lower Euclidean norm.

Here, we use the Tikhonov method to solve the optimization problem (4) with an additional constraint that $x_i \geq 0$ for each i. This is achieved by finding a minimum value of λ (called in the following λ_{opt}) such that the solution of (6),

$$\hat{x}(\lambda) = \arg\min_x \left\{\|Ax - b\|_2^2 + \lambda^2\|x\|_2^2\right\},$$

satisfies the condition $\hat{x}_i \geq 0$ for all i, i.e.,

$$\lambda_{\text{opt}} = \min\{\lambda\colon \hat{x}_i(\lambda) \geq 0,\ \forall i\}.$$

The coefficient λ controls the smoothness of the solution. By increasing λ we can easily obtain a solution with all positive entries. However, we want to keep λ as small as possible so that the term $\|Ax - b\|^2$ remains important when finding $\hat{x}$.

λ_{opt} can be found using the bisection algorithm. The algorithm starts with two values of λ, denoted by $\lambda_{\min}$ and $\lambda_{\max}$. Initially, $\lambda_{\min}$ is set to zero and $\lambda_{\max}$ is selected so that $\hat{x}_i(\lambda_{\max}) \geq 0$ for each i. In each step of the algorithm, a new value $\lambda = 0.5\cdot(\lambda_{\min} + \lambda_{\max})$ is tested. If $\hat{x}_i(\lambda) \geq 0$ for each i, then we set $\lambda_{\max} = \lambda$. Otherwise, we set $\lambda_{\min} = \lambda$. When the difference $\lambda_{\max} - \lambda_{\min}$ is sufficiently small, the computations are stopped and $\lambda_{\text{opt}} = \lambda_{\max}$ is returned.

3.3. Quadratic programming method. Since

$$\|Ax - b\|_2^2 = 2\left(\frac{1}{2}x^T A^T A x - (A^T b)^T x + \frac{1}{2}b^T b\right),$$

defining $Q = A^T A \in \mathbb{R}^{w\times w}$ and $q = -A^T b \in \mathbb{R}^w$, the problem of finding the minimum of $\|Ax - b\|_2^2$ with the box constrains $x_i \in [\underline{x}_i, \overline{x}_i]$ can be formulated as a constrained quadratic programming problem,

$$\hat{x} = \arg\min_{x\colon\, \underline{x}\leq x\leq \overline{x}} \frac{1}{2}x^T Q x + q^T x, \tag{7}$$

where $\underline{x}, \overline{x} \in \mathbb{R}^w$ are the vectors of lower and upper constraints. The active-set algorithm can be used to solve the problem formulated above (Voglis and Lagaris, 2004). For the problem (7), the associated Lagrangian has the form

$$L(x,\gamma,\alpha) = \frac{1}{2}x^T Q x + q^T x - \gamma^T(x - \underline{x}) + \alpha^T(x - \overline{x}),$$

where γ and α are the Lagrange coefficients representing the upper and lower constraints, respectively.

The triple $(\hat{x}, \hat{\gamma}, \hat{\alpha})$ is the solution of the constrained quadratic programming problem when it fulfills the Karush–Kuhn–Tucker (KKT) conditions,

$$\begin{aligned} Q\hat{x} + q - \hat{\gamma} + \hat{\alpha} &= 0,\\ \hat{\gamma} \geq 0, \quad \hat{\alpha} &\geq 0,\\ \hat{\gamma}(\hat{x} - \underline{x}) = 0, \quad \hat{\alpha}(\overline{x} - \hat{x}) &= 0,\\ \underline{x} \leq \hat{x} &\leq \overline{x}. \end{aligned} \tag{8}$$

For these KKT conditions, the implementation of the exterior point active set algorithm is presented by Voglis and Lagaris (2004). In the algorithm, two active sets are used. These two sets consist of indices of unknowns x_i which the algorithm tries to set outside the lower and upper constraints, respectively. The remaining indices constitute the passive set. When a variable x_i is in one of the active sets, its value is $x_i = \underline{x}_i$ or $x_i = \overline{x}_i$, depending on which active set it belongs to. Note that in the algorithm, the solution is found only for the unknowns belonging to the passive set. This set is usually significantly reduced in the course of the algorithm. Therefore, the active set approach has the advantage of limiting the dimensionality of the problem to be solved and is recommended for the problem described in the paper (cf. Szynkiewicz and Błaszczyk, 2011).

As mentioned before, the non-negative least squares problem can be considered a special case of the quadratic programming method with box constraints (the empty set of upper constraints and bounds $\underline{x} = (0, \ldots, 0)^T$ are used). It follows that to solve the NNLSQ problem, a single active set should be used. The original NNLSQ algorithm was proposed by Lawson and Hanson (1987). Its improved version can be found in the work of Bro and Jong (1997).

4. Computational results and discussion

The problem of designing a coil of length z_{coil} is considered. Simple coils are coaxial circles of radius $r \in [r_{\min}, r_{\max}]$ centered at the z axis. In the 1D case, all simple coils have the same radius $r_{\min} = r_{\max}$. In the 2D case, radii of the simple coils change uniformly between $r_{\min}$ and $r_{\max}$. Two target areas are considered. In the first case, target points are located on the interval of length z_{req} enclosed in the z-axis and centered at the origin. In the second case, target points are located on the circle of diameter d_{SV} (diameter spherical volume) positioned at the center of the coordinate system. In all the tests, the desired magnetic field is homogeneous (it is equal at all target points) and has the value μ_0 being the magnetic permeability of the vacuum. As mentioned in the Introduction, producing a homogeneous magnetic field in a certain spherical volume is crucial from the application point of view, especially in the design of magnets for MRI devices. Parameters of the design problem have been chosen according to the relations proposed by Xu *et al.* (1999),

$$z_{\text{coil}} = 0.8d + 1.2d_{\text{SV}}, \tag{9}$$

where $d = 2r_{\min}$ is the interior diameter of the coil. The above formula binds the relations of the length and diameter of the coil with d_{SV}. It was developed under the constraints to keep the high homogeneity and minimize the cross-section area of the coil. Values of the construction parameters used in simulations are presented in Table 1. The proposed dimensions are typical for whole-body MRI systems used nowadays.

The quality of the solution is assessed by three coefficients. The first one, called the *field quality coefficient*, is the square of the Euclidean norm of the difference between the required field and the field excited by the coil,

$$f(\hat{x}) = \|b - A\hat{x}\|_2^2. \tag{10}$$

Table 1. Dimensions of the coil and target areas (in meters).

Target area	z_{coil}	$r_{\min}$	$r_{\max}$	d_{SV}	z_{req}
Case 1, 1D	1.02	0.3	–	–	0.9
Case 2, 1D	1.02	0.3	–	0.45	–
Case 1, 2D	1.02	0.3	0.4	–	0.9
Case 2, 2D	1.02	0.3	0.4	0.45	–

This coefficient is equal to the cost function minimized in the LSQ optimization process. The second coefficient, called the *energy coefficient*, is the square of the Euclidean norm of the solution,

$$\|\hat{x}\|_2^2 = \sum_{i=1}^{w} \hat{x}_i^2. \tag{11}$$

This coefficient is proportional to the power dissipated in the coil (in the case of standard resistive 1D coil). The third coefficient used is the maximum value of the current,

$$\|\hat{x}\|_\infty = \max_{x \in 1,\ldots,w} |\hat{x}_i|. \tag{12}$$

This coefficient is also very important form the application point of view (the maximum value of the current is limited by properties of the power supply).

4.1. Least squares method. First, let us study the case when the problem dimension w is small. Let us consider the case $n = 6$, $k = 1$, $w = 6$, $m = 1000$, and the linear target area. The current distribution obtained using the LSQ method and the corresponding magnetic field are shown in Fig. 2. One can see that all currents are positive.

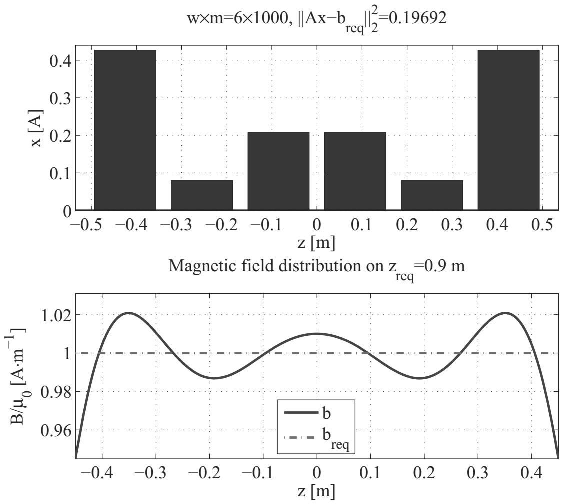

Fig. 2. LSQ solution for $n \times k = 6 \times 1$ and the corresponding field distribution for the linear target area.

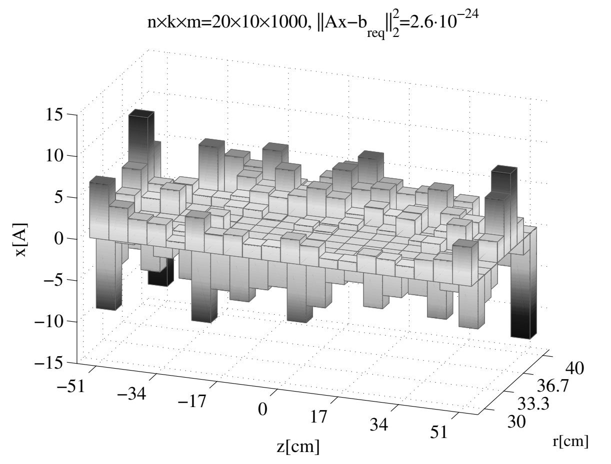

Fig. 3. LSQ solution for $n \times k = 20 \times 10$ and the linear target area.

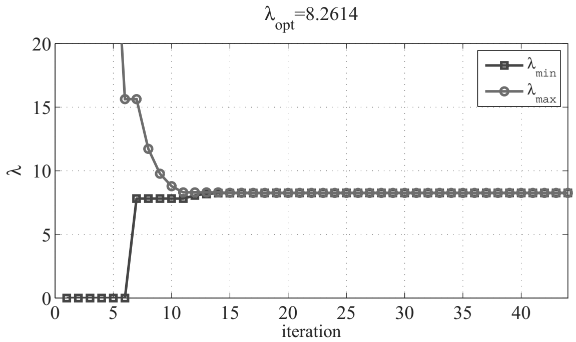

Fig. 4. Convergence of the bisection algorithm to find the optimal regularization parameter, $n \times k = 20 \times 10$, $m = 1000$.

More generally, for small w ($w \leq 6$ for the linear target area and $w \leq 10$ for the circular target area) the non-negativity constrains for the LSQ solution are fulfilled and, in consequence, the NNLSQ method and the constrained QP method with sufficiently large upper bounds produce the same results as the LSQ method.

Now, let us consider the case when the dimension w is larger. Assume $n = 20$, $k = 10$, $w = 200$, $m = 1000$, and the linear target area. The solution found using the LSQ method is presented in Fig. 3. Let us note that in many cases neighboring coils have currents of opposite directions and they are much larger than for small w.

Another important observation is the lack of symmetry of the solution. Since the problem is symmetric, we expect a symmetric solution. The lack of symmetry indicates instability of the computation procedure. These properties of the solution are consequences of the fact that for larger w the problem becomes ill-posed. Although the field quality coefficient for this solution is very small, the solution is useless from the practical point of view, due to its properties mentioned above. Its energy coefficient is very large $\left(\|\hat{x}\|_2^2 \approx 2.2 \cdot 10^3\right)$.

4.2. Tikhonov regularization. Let us now solve the same problem ($n = 20$, $k = 10$, $m = 1000$, the linear target area) using the Tikhonov regularization method. To apply this method, first we have to select the value of the parameter λ. The minimum value of the regularization parameter λ satisfying the constraints $\hat{x}_i \geq 0$ for all i is found using the bisection algorithm presented in Section 3.2. Figure 4 presents intermediate computation results as a function of the iteration number. After 37 iterations an optimal regularization parameter is found with a precision of 10^{-10}.

Figure 5 presents the solution obtained using the Tikhonov regularization method with the regularization parameter $\lambda_{\rm opt}$ and the corresponding magnetic field at the target area. As expected, the solution is smooth and symmetric. Note that the field quality coefficient $f(\hat{x}) \approx 0.2$ is much worse than for the LSQ solution $f(\hat{x}) \approx 3 \cdot 10^{-24}$. This can be regarded as a cost of smoothing the solution. However, the energy coefficient is approximately $1.4 \cdot 10^5$ times smaller and the maximum current is approximately 640 times smaller than for the LSQ solution. These properties make the solution found using the Tikhonov regularization method much better from the application point of view.

For the same parameters of the coil, when the target points are located on the circle, the optimal value of the regularization parameter is smaller ($\lambda_{\rm opt} \approx 0.4683$) and the field quality coefficient is significantly smaller ($f(\hat{x}) \approx 1.8899 \cdot 10^{-4}$). This means that, for the circular target area, it is easier to obtain good solutions. However, note that, due to a more complicated formula of the cost function (it involves calculation of elliptic integrals), the computation time for the circular case is larger (approximately four times) than for the linear case.

Figure 6 shows the solution obtained using the Tikhonov regularization method and the corresponding magnetic field in the circular target area for the 1D case ($n \times k = 200 \times 1$). This solution was obtained for the optimal regularization parameter $\lambda_{\rm opt} = 1.764 \cdot 10^{-2}$. The current distribution in the coil is smooth and symmetric. Observe that the field quality coefficient $f(\hat{x}) \approx 3.77 \cdot 10^{-7}$ is about 500 times smaller than for the 2D case ($n \times k = 20 \times 10$), although the total number of simple coils is the same. This can be explained by noting that increasing n makes it possible to obtain better control over coils located closer to the target area.

4.3. Non-negative least squares method. Let us now solve the same problem ($n \times k = 200 \times 1$, the circular target area) using the NNLSQ method. The solution and

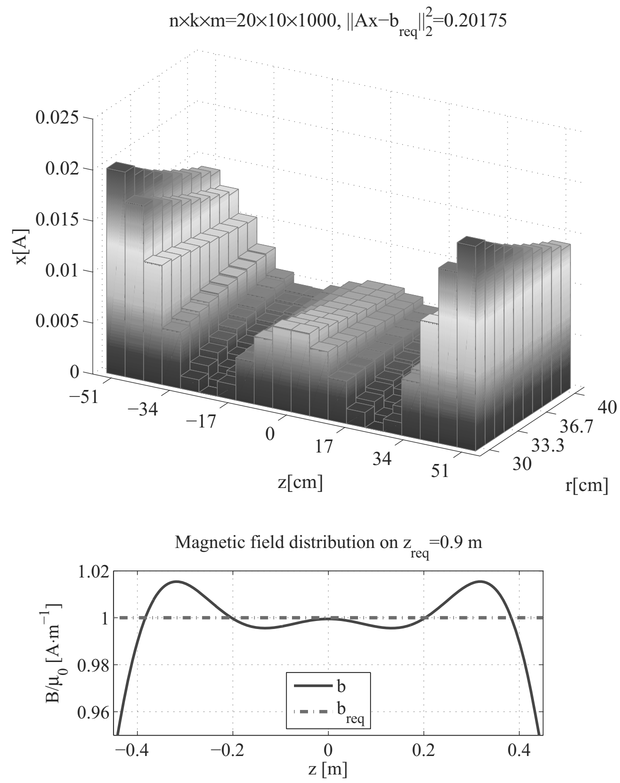

Fig. 5. Solution for $n \times k = 20 \times 10$ and the linear target area obtained with the Tikhonov regularization method for the optimal regularization parameter and the corresponding magnetic field.

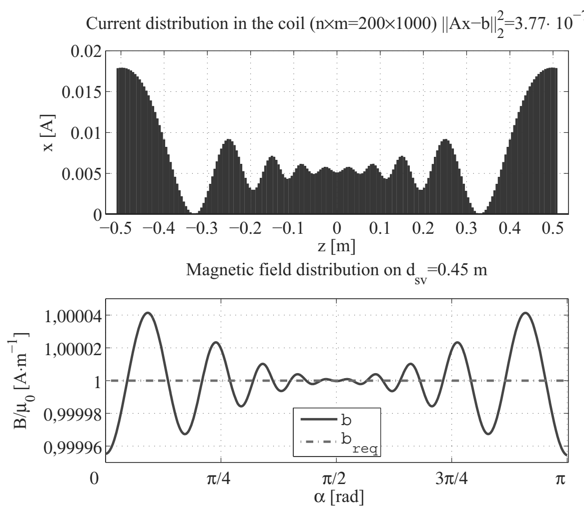

Fig. 6. Solution for $n \times k = 200 \times 1$ and the circular target area obtained with the Tikhonov regularization method, $\lambda_{\mathrm{opt}} = 1.76 \cdot 10^{-2}$.

the corresponding magnetic field are plotted in Fig. 7. One can see that only 32 out of 200 simple coils have non-zero currents. A general observation is that the number of coils with a non-zero current when using the NNLSQ method is usually small. This explains the efficiency of the active set algorithm and its capability to solve high dimensional problems. In this particular case, 168 variables belong to the active set. At the final step of the algorithm, their values are set to zero and the optimization is performed with the remaining 32 variables only. Note that the field quality coefficient is better than for the Tikhonov regularization technique. However, the maximum value of the current $\|x\|_\infty$ is approximately 20 times larger than for the regularization method. This results in an increase in the energy coefficient, which is more than 19 times larger than for the regularization method.

It is interesting to note that when the size of the passive set exceeds a certain limit, the standard NNLSQ algorithm does not work properly. The solution contains negative currents; the energy factor and the maximum current become extremely large. This is a consequence of the fact that for a large size of the passive set the subproblem of solving the unconstrained LSQ problem for the variables from the passive set becomes ill-conditioned.

4.4. Constrained quadratic programming method. The constrained quadratic programming method makes it possible to improve the field quality coefficient of the Tikhonov regularization method without degrading the energy coefficient. In order to compare the constrained QP method with the Tikhonov regularization method, the lower and upper constraints for the solution are chosen to match the bounds for the solution found with the regularization method. The lower constraint ensures that the solution is non-negative. The upper constraint ensures that currents are not larger than the maximum current obtained with the regularization technique. More precisely, we use $\underline{x}_i = 0$ and $\overline{x}_i = \|\hat{x}(\lambda_{\mathrm{opt}})\|_\infty$, where $\hat{x}(\lambda_{\mathrm{opt}})$ is the solution found by the regularization method with the optimum value of λ.

Figure 8 shows the solution for the case $n \times k = 200 \times 1$ and the circular target area obtained with the constrained QP method. The upper constraint is equal to the maximum current of the solution presented in Fig. 6, i.e., $\overline{x}_i = 1.791 \cdot 10^{-2}$ A. In the solution shown in Fig. 8(a), the active set representing the lower constraint (zero current) contains 110 elements and the upper constraint active set (maximum current) contains 78 elements. It follows that the passive set contains 22 elements only. The field quality coefficient is approximately 6.3 times smaller than for the solution obtained with the regularization algorithm, while the energy coefficient is approximately 0.6 times larger. Obviously, the maximum value of the current is the same for both the methods. When we compare this solution with the NNLSQ one, it can be seen that the quality of the magnetic field is much worse (the field quality coefficient is 1500 times larger). This is a consequence of additional constraints (maximum value of currents). However, both

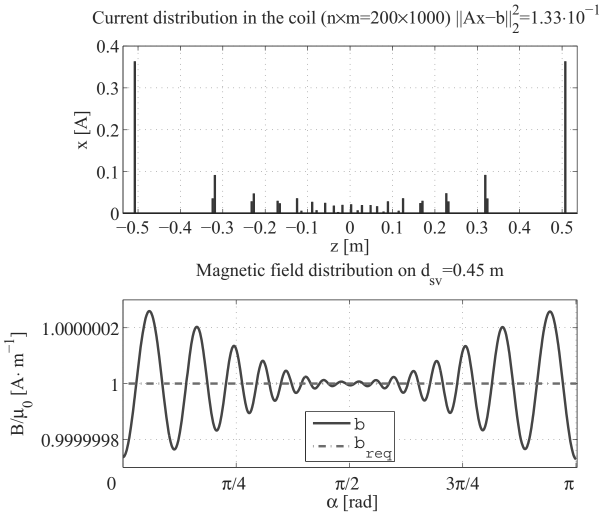

Fig. 7. NNLSQ solution for $n \times k = 200 \times 1$ and the circular target area.

the energy coefficient and the maximum current are approximately 20 times smaller than for the NNLSQ solution.

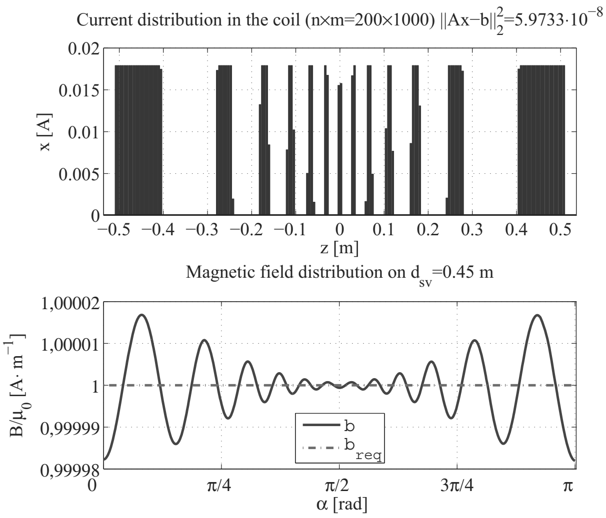

Fig. 8. Constrained QP solution for $n \times k = 200 \times 1$ and the circular target area

The results for the 2D case with $n \times k = 20 \times 10$ are shown in Fig. 9. The white, black and gray colors represent the variables from the lower active set, the upper active set and the passive set, respectively. The lower active set, the upper active set, and the passive set contain 120, 74, and 6 elements, respectively. The field quality coefficient is approximately 3.8 times smaller and the energy coefficient is approximately two times larger that than for the solution obtained with the Tikhonov regularization method shown in Fig. 5.

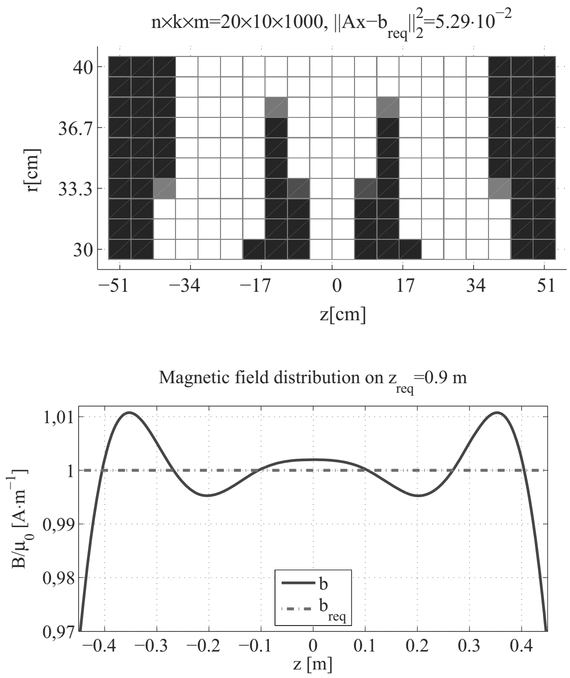

Fig. 9. Constrained QP solution for $n \times k = 20 \times 10$ (shades of gray represent values of currents, $x_i = 0$: white, $x_i = 0.01$: black)

4.5. Comparison of the methods. Table 2 presents results of a number of design tests carried out using all four optimization methods considered in this work. Four design problems are taken into account. These are combinations of one and two-dimensional versions of the coil, and linear and circular target areas. Problems with the dimensions $w = n \cdot k \leq 500$ have been considered. As expected, in all cases the best solution in terms of the field quality coefficient is the one found using the unconstrained LSQ method. However, as mentioned before for large w, these solutions are characterized by a very large maximum current, a very large energy coefficient, the lack of symmetry and opposite currents in neighboring coils. Also note that in some cases the coefficients oscillate when the dimension of the problem is increased. This is an indication of the lack of numerical stability of the computational algorithm. All these properties make the LSQ solutions obtained for large w useless from the application point of view.

The other three methods work by imposing some constraints on solutions. As a consequence, algorithms become numerically stable and lead to symmetric solutions. In all cases, the evaluated coefficients change monotonically with the problem dimension. Out of the three methods, the best solutions in terms of the quality of the magnetic field are the ones obtained using the non-negative

Table 2. Comparison of performance of four optimization methods for different coil design problems, $f(x) = \|Ax - b\|_2^2$: field quality coefficient, $\|x\|_\infty$: maximum current, $\|x\|_2^2$: energy coefficient.

$n\times k$	LSQ			Tikhonov				NNLSQ			constrained QP		
	$f(x)$	$\|x\|_\infty$	$\|x\|_2^2$	$f(x)$	$\|x\|_\infty$	$\|x\|_2^2$	λ_{opt}	$f(x)$	$\|x\|_\infty$	$\|x\|_2^2$	$f(x)$	$\|x\|_\infty$	$\|x\|_2^2$
1D, linear target area													
10×1	$3.93\cdot 10^{-3}$	0.566	1.126	0.027	0.420	0.433	0.458	$2.46\cdot 10^{-2}$	0.428	0.462	0.026	0.420	0.443
25×1	$1.27\cdot 10^{-9}$	50.036	$4.38\cdot 10^{4}$	0.034	0.191	0.158	0.956	$3.03\cdot 10^{-3}$	0.418	0.465	0.019	0.191	0.236
50×1	$5.16\cdot 10^{-18}$	76842	$4.04\cdot 10^{10}$	0.035	0.098	0.078	1.404	$1.50\cdot 10^{-3}$	0.413	0.437	0.016	0.098	0.134
150×1	$2.34\cdot 10^{-19}$	12828	$1.41\cdot 10^{9}$	0.035	0.033	0.026	2.422	$9.15\cdot 10^{-4}$	0.410	0.444	0.015	0.033	0.050
200×1	$3.78\cdot 10^{-19}$	18514	$1.53\cdot 10^{9}$	0.035	0.025	0.019	2.807	$8.58\cdot 10^{-4}$	0.410	0.436	0.015	0.025	0.037
250×1	$1.13\cdot 10^{-19}$	15351	$1.06\cdot 10^{9}$	0.035	0.020	0.016	3.138	$8.26\cdot 10^{-4}$	0.409	0.423	0.015	0.020	0.030
500×1	$1.07\cdot 10^{-19}$	15747	$1.49\cdot 10^{9}$	0.035	0.010	0.008	4.438	$7.64\cdot 10^{-4}$	0.409	0.425	0.014	0.010	0.015
1D, circular target area													
10×1	$1.19\cdot 10^{-2}$	0.351	0.3245	$1.19\cdot 10^{-2}$	0.348	0.320	0.010	$1.19\cdot 10^{-2}$	0.351	0.324	$1.19\cdot 10^{-2}$	0.348	0.321
25×1	$6.48\cdot 10^{-9}$	22.937	$2.33\cdot 10^{3}$	$8.25\cdot 10^{-7}$	0.132	0.115	0.010	$7.93\cdot 10^{-9}$	0.357	0.293	$2.56\cdot 10^{-7}$	0.132	0.129
50×1	$2.55\cdot 10^{-19}$	43653	$5.98\cdot 10^{9}$	$4.68\cdot 10^{-7}$	0.070	0.060	0.010	$6.35\cdot 10^{-11}$	0.362	0.302	$8.43\cdot 10^{-8}$	0.070	0.080
150×1	$1.18\cdot 10^{-26}$	10.493	656.0	$3.80\cdot 10^{-7}$	0.024	0.020	0.015	$1.54\cdot 10^{-11}$	0.363	0.302	$6.09\cdot 10^{-8}$	0.024	0.032
200×1	$7.46\cdot 10^{-27}$	9.733	551.2	$3.77\cdot 10^{-7}$	0.018	0.015	0.018	$1.33\cdot 10^{-11}$	0.363	0.299	$5.97\cdot 10^{-8}$	0.018	0.025
250×1	$9.11\cdot 10^{-27}$	10.716	559.3	$3.74\cdot 10^{-7}$	0.014	0.012	0.020	$1.19\cdot 10^{-11}$	0.787	1.303	$5.95\cdot 10^{-8}$	0.014	0.020
500×1	$9.65\cdot 10^{-27}$	8.047	472.2	$3.79\cdot 10^{-7}$	0.007	0.006	0.028	$1.02\cdot 10^{-11}$	0.669	0.970	$6.01\cdot 10^{-8}$	0.007	0.010
2D, linear target area													
5×2	$3.95\cdot 10^{-5}$	2.602	26.35	0.2255	0.400	0.454	0.373	0.1961	0.582	0.803	0.2190	0.400	0.527
10×4	$8.22\cdot 10^{-23}$	319.29	$5.42\cdot 10^{5}$	0.1624	0.097	0.085	3.089	$1.86\cdot 10^{-2}$	0.401	0.446	$6.93\cdot 10^{-2}$	0.097	0.140
15×10	$1.77\cdot 10^{-24}$	21.797	$3.68\cdot 10^{3}$	0.1646	0.028	0.022	6.169	$7.52\cdot 10^{-3}$	0.363	0.387	$4.75\cdot 10^{-2}$	0.028	0.043
20×10	$2.61\cdot 10^{-24}$	12.829	$2.21\cdot 10^{3}$	0.2018	0.020	0.016	8.261	$4.43\cdot 10^{-3}$	0.527	0.633	$5.28\cdot 10^{-2}$	0.020	0.031
25×5	$1.97\cdot 10^{-24}$	11.665	$1.89\cdot 10^{3}$	0.1680	0.033	0.027	5.758	$3.18\cdot 10^{-3}$	0.275	0.379	$4.57\cdot 10^{-2}$	0.033	0.051
25×10	$3.06\cdot 10^{-24}$	12.243	$1.17\cdot 10^{3}$	0.1977	0.016	0.013	9.132	$2.89\cdot 10^{-3}$	0.269	0.384	$4.88\cdot 10^{-2}$	0.016	0.025
50×5	$1.13\cdot 10^{-24}$	7.174	520.5	0.1642	0.017	0.013	8.039	$1.60\cdot 10^{-3}$	0.262	0.362	$4.33\cdot 10^{-2}$	0.017	0.026
50×10	$4.14\cdot 10^{-24}$	4.171	365.0	0.2002	0.008	0.006	13.059	$1.48\cdot 10^{-3}$	0.305	0.373	$4.84\cdot 10^{-2}$	0.008	0.013
2D, circular target area													
5×2	$1.05\cdot 10^{-4}$	0.700	1.757	$3.81\cdot 10^{-2}$	0.240	0.238	0.828	$1.99\cdot 10^{-2}$	0.311	0.380	$2.29\cdot 10^{-2}$	0.240	0.284
10×4	$1.03\cdot 10^{-21}$	69.562	$1.67\cdot 10^{4}$	$5.72\cdot 10^{-5}$	0.097	0.078	0.068	$2.52\cdot 10^{-7}$	0.234	0.247	$4.25\cdot 10^{-6}$	0.097	0.137
15×10	$8.11\cdot 10^{-27}$	3.265	107.1	$3.10\cdot 10^{-4}$	0.019	0.016	0.508	$3.97\cdot 10^{-8}$	0.356	0.338	$1.22\cdot 10^{-5}$	0.019	0.028
20×10	$4.77\cdot 10^{-27}$	2.984	52.36	$1.89\cdot 10^{-4}$	0.015	0.013	0.468	$5.96\cdot 10^{-9}$	0.354	0.294	$9.66\cdot 10^{-6}$	0.015	0.022
25×5	$3.13\cdot 10^{-27}$	4.044	81.75	$3.35\cdot 10^{-4}$	0.022	0.020	0.511	$1.56\cdot 10^{-9}$	0.371	0.323	$1.80\cdot 10^{-5}$	0.022	0.032
25×10	$3.01\cdot 10^{-27}$	2.518	41.19	$5.66\cdot 10^{-4}$	0.011	0.009	0.978	$1.09\cdot 10^{-9}$	0.364	0.306	$1.93\cdot 10^{-5}$	0.011	0.016
50×5	$1.38\cdot 10^{-27}$	1.621	18.72	$3.08\cdot 10^{-4}$	0.011	0.010	0.688	$2.27\cdot 10^{-10}$	0.375	0.316	$1.51\cdot 10^{-5}$	0.011	0.017
50×10	$1.93\cdot 10^{-27}$	0.994	14.73	$6.54\cdot 10^{-4}$	0.005	0.005	1.501	$1.12\cdot 10^{-10}$	0.369	0.313	$2.18\cdot 10^{-5}$	0.005	0.008

least squares method.

For this method, the maximum current and the energy coefficient are considerably smaller than for the unconstrained LSQ problem. However, as mentioned before, the NNLSQ approach usually produces solutions where only a few of the simple coils have non-zero currents. This leads to a much larger maximum current and the energy coefficient than for the other two methods.

The Tikhonov regularization method works by smoothing the solution. The method of choosing the optimal value of the regularization parameter proposed in this work ensures that all the currents are non-negative. This approach provides a compromise between the field quality coefficient, the energy coefficient, and the maximum value of the current. From the results presented in Table 2, it follows that for the same value of the maximum current, the constrained QP method outperforms the Tikhonov regularization method in terms of the field quality coefficient.

This is achieved at the expense of a slight increase in the energy coefficient. It should be also pointed out that the constrained QP method is the most expensive one of the four tested methods in terms of the computational time. For instance, the solution of the 1D problem with a linear target area for $n = 500$ took approximately 8 minutes (62741 cost function evaluations). The non-negative least squares method and the regularization method solved the same problem in 0.04 seconds and 0.75 seconds, respectively.

5. Conclusions

Several methods for the linear coil design problem have been compared. This includes the least squares method, the Tikhonov regularization method, the non-negative least squares method and the quadratic programming method with box constrains. It has been shown that for a large dimension of the problem the LSQ solutions are unusable from the application point of view. The regularization method provides useful solutions. However, the constrained quadratic programming method produces better results in terms of the quality of the magnetic field for the same maximum value of the current.

Acknowledgment

This work was sponsored in part by the AGH University of Science and Technology (grant no. 11.11.120.343).

References

Bro, R. and Jong, S.D. (1997). A fast non-negativity-constrained least squares algorithm, *Journal of Chemometrics* **11**(5): 393–401.

Fisher, B.J., Dillon, N., Carpenter, T.A. and Hall, L.D. (1997). Design of a biplanar gradient coil using a genetic algorithm, *Magnetic Resonance Imaging* **15**(3): 369–376.

Garda, B. (2012). Linear algebra approach and the quasi-Newton algorithm for the optimal coil design problem, *Przegląd Elektrotechniczny* (7a): 261–264.

Garda, B. and Galias, Z. (2010). Comparison of the linear algebra approach and the evolutionary computing for magnetic field shaping in linear coils, *Nonlinear Theory and Its Applications, NOLTA 2010, Cracow, Poland*, pp. 508–511.

Garda, B. and Galias, Z. (2012). Non-negative least squares and the Tikhonov regularization methods for coil design problems, *Proceedings of the International Conference on Signals and Electronic Systems, ICSES 2012, Wrocław, Poland*.

Hansen, P. (1998). *Regularization Tools: A Matlab Package for Analysis and Solution of Discrete Ill-posed Problems. Version 3.0 for Matlab 5.2*, IMM-REP, Institut for Matematisk Modellering, Danmarks Tekniske Universitet, Kongens Lyngby.

Jin, J. (1999). *Electromagnetic Analysis and Design in Magnetic Resonance Imaging*, Biomedical Engineering Series, CRC Press, Boca Raton, FL.

Lawson, C. and Hanson, R. (1987). *Solving Least Squares Problems*, Classics in Applied Mathematics, Society for Industrial and Applied Mathematics, Philadelphia, PA.

Macovski, A., Xu, H., Conolly, S. and Scott, G. (2000). Homogeneous magnet design using linear programing, *IEEE Transactions on Magnetics* **36**(2): 476–483.

Prasath, V.B.S. (2011). A well-posed multiscale regularization scheme for digital image denoising, *International Journal of Applied Mathematics and Computer Science* **21**(4): 769–777, DOI: 10.2478/v10006-011-0061-7.

Sikora, R., Krasoń, P. and Gramz, M. (1980). Magnetic field synthesis at the plane perpendicular to the axis of solenoid, *Archiv fur elektrotechnik* **62**(3): 135–156.

Szynkiewicz, W. and Błaszczyk, J. (2011). Optimization-based approach to path planning for closed chain robot systems, *International Journal of Applied Mathematics and Computer Science* **21**(4): 659–670, DOI: 10.2478/v10006-011-0052-8.

Tikhonov, A. and Arsenin, V. (1977). *Solutions of Ill-posed Problems*, Scripta Series in Mathematics, John Wiley&Sons, Washington, DC.

Turner, R. (1986). A target field approach to optimal coil design, *Journal of Physics D: Applied Physics* **19**(8): 147–151.

Voglis, C. and Lagaris, I. (2004). BOXCQP: An algorithm for bound constrained convex quadratic problems, *Proceedings of the 1st International Conference: From Scientific Computing to Computational Engineering, IC-SCCE, Athens, Greece*.

Xu, H., Conolly, S., Scott, G. and Macovski, A. (1999). Fundamental scaling relations for homogeneous magnets, *ISMRM 7th Scientific Meeting, Philadelphia, PA, USA*, p. 475.

Zhu, M., Xia, L., Liu, F., Zhu, J., L.Kang and Crozier, S. (2012). Finite difference method for the design of gradient coils in MRI—Initial framework, *IEEE Transactions on Biomedical Engineering* **59**(9): 2412–2421.

Bartłomiej Garda received his M.Sc. degree in electrical engineering in 1999, and the Ph.D. degree in electrical engineering in 2013 from the Faculty of Electrical Engineering, Automatics, Computer Science and Electronics of the AGH University in Cracow, Poland. His research is now mainly focused on applications of numerical methods in electromagnetic field synthesis. He is a member of a research group investigating dynamical phenomena in nonlinear electric circuits.

Zbigniew Galias received the M.Sc. degree in electronics from the AGH University of Science and Technology, Cracow, Poland, in 1990, the M.Sc. degree in mathematics from the Jagiellonian University, Cracow, in 1992, as well as the Ph.D. degree and the senior doctorate degree in electrical engineering from the AGH University of Science and Technology, in 1996 and 2004, respectively. He became an assistant professor at the AGH University in 1992, and was promoted to an associate professor in 1996. Since 2005, he has been a professor of electrical engineering at the AGH University. His research interests include simulation, analysis, and design of nonlinear systems, chaos, interval arithmetic and computer assisted theorem proving. He has published more than 100 research papers. He has held visiting research positions at the Technical University of Munich, the University of California at Berkeley, the University of California at San Diego, and the RMIT University, Melbourne. Doctor Galias was awarded the Polish Research Foundation scholarship in 1994, received the Polish Prime Minister award for the Ph.D. thesis in 1997, and was awarded the Fulbright fellowship in 1999.

Received: 20 January 2013
Revised: 18 August 2013