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NEW TRANSITIVITY OF ATANASSOV'S INTUITIONISTIC FUZZY SETS IN A DECISION MAKING MODEL

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Atanassov's intuitionistic fuzzy sets and especially his intuitionistic fuzzy relations are tools that make it possible to model effectively imperfect information that we meet in many real-life situations. In this paper, we discuss the new concepts of the transitivity problem of Atanassov's intuitionistic fuzzy relations in an epistemic aspect. The transitivity property reflects the consistency of a preference relation. Therefore, transitivity is important from the point of view of real problems appearing, e.g., in group decision making in preference procedures. We propose a new type of optimistic and pessimistic transitivity among the alternatives (options) considered and their use in the procedure of ranking the alternatives in a group decision making problem.

Keywords: optimistic and pessimistic transitivity, preference relations, optimistic and pessimistic intuitionistic fuzzy negations.

1. Introduction

In this paper, Atanassov's intuitionistic fuzzy relations (A-IFRs) are studied, especially preference relations based on a new epistemic concept. These Atanassov's intuitionistic fuzzy relations were introduced by Atanassov (1999) as a generalization of the concept of a fuzzy relation (FR) defined by Zadeh (1965). Fuzzy sets and fuzzy relations found applications in diverse types of areas, for example, in databases, pattern recognition, neural networks, fuzzy modeling, economy, medicine, multicriteria decision making, etc.

Atanassov's intuitionistic fuzzy relations can be interpreted as a tool that may help to model in a better way imperfect information, particularly with imperfectly defined facts and imprecise knowledge, especially in optimistic or pessimistic aspects. Diverse properties of Atanassov's intuitionistic fuzzy relations or other extensions of fuzzy sets theory have been studied by a range of authors (Atanassov, 1999; 2021; 2016; Burillo and Bustince, 1995; Pradhan and Pal, 2017; Xu and Yager, 2009; Pękala, 2009; Pękala *et al.*, 2015; 2018; 2016; Dudziak and Pękala, 2011; Xu *et al.*, 2014).

Among other things, research concerns the properties

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of transitivity or consistency and their impact on the preferences relations. This allows Atanassov's intuitionistic fuzzy relations to be applied in group decision making problems in a situation when a solution from the individual preferences over some set of options should be derived. We will consider here group decision making while each option fulfills a set of criteria to some extent and, on the other hand, it does not fulfill the same set of criteria to some extent. This clearly suggests that the alternatives can be conveniently expressed via Atanassov's intuitionistic fuzzy sets (intuitionistic fuzzy alternatives).

Transitivity is a fundamental notion in decision theory. It is most universally assumed in disciplines of decision theory and generally accepted in the principle of rationality. Transitivity of a fuzzy preference relation has received great attention in the past decades.

In this paper, we concentrate on the new idea of transitivity, in particular, we consider the following objectives:

1. To introduce transitivity in the optimistic and pessimistic issues of Atanassov's intuitionistic fuzzy relations.
2. To introduce intuitionistic negation in the optimistic and pessimistic issues.
3. To study properties and dependences of new transitivity with aggregation and negation intuitionistic functions.
4. To use in the preference structure aggregation and negation intuitionistic functions, especially in the optimistic and pessimistic issues.
5. To show an application where the introduction of the new concept of transitivity is justified.

The introduced notions of transitivity reflect uncertainty, expressed by a degree of hesitation. Hence, these notions are compatible with the new model of Atanassov's intuitionistic fuzzy sets where we consider intervals that can be formally employed to represent Atanassov's intuitionistic fuzzy sets (A-IFSs). Another approach is proposed by Deschrijver and Kerre (2003), for whom the representations of IFSs (assumptions leading to their conclusion) are different—only two terms, i.e., memberships values and non-membership values, are taken into account. We consider a different model, i.e., taking into account all three terms: membership values, non-membership values and the hesitation margins. Both models are correct but they are not the same. As a result, our conclusion is different—IVFSs and IFSs are not equivalent (Szmidt and Kacprzyk, 2017). Moreover, the above-mentioned new transitivity and intuitionistic fuzzy negation also initiated in optimistic and pessimistic aspects are used together with the adequate aggregation functions for the preference model and, in consequence, for the decision making model.

The paper is organized as follows. First, some concepts and results useful in further considerations are recalled (Section 2). Then, results concerning comparability relations are given (Section 3). Next, intuitionistic operations, such as aggregation and negations are presented and later proposed for use in the preference model (Sections 4 and 5). Section 6 consists of the new concept of transitivity and its connection with intuitionistic aggregation and negation functions. Finally, an example of an application using the algorithm with the new transitivity in a group decision making problem is presented (Section 7).

2. Basic notions

At the beginning, let us recall the concept of a fuzzy relation. A fuzzy relation in $X \neq \emptyset$ (Zadeh, 1965) is given by

$$\rho' = \{((x,y), R'(x,y)) | x,y \in X\}, \tag{1}$$

where $R'(x,y) \in [0,1]$ is the membership function of the fuzzy relation ρ'.

One of the possible extensions of the fuzzy relation (1) is Atanassov's intuitionistic fuzzy relation (Atanassov, 1999) ρ given by

$$\rho = \{((x,y), R(x,y), R^d(x,y)) | x,y \in X\}, \tag{2}$$

where: $R : X \times X \to [0,1]$ and $R^d : X \times X \to [0,1]$ such that

$$0 \leq R(x,y) + R^d(x,y) \leq 1, \ \ (x,y) \in (X \times X) \tag{3}$$

and $R(x,y), R^d(x,y) \in [0,1]$ denote the degree of membership and the degree of nonmembership of $(x,y) \in \rho$, respectively. The value $\pi_\rho \colon X \times Y \to [0,1]$ is associated with each of Atanassov's intuitionistic fuzzy relations ρ, where

$$\pi_\rho(x,y) = 1 - R(x,y) - R^d(x,y), \quad x \in X, \quad y \in Y.$$

Obviously, each fuzzy relation may be represented by the following intuitionistic fuzzy relation:

$$\rho' = \{((x,y), R'(x,y), 1 - R'(x,y)) | x,y \in X\}.$$

The family of all Atanassov's intuitionistic fuzzy relations in a set X is denoted by AIFR(X), where we may represent intuitionistic fuzzy relations ρ in short as (R, R^d, π_ρ).

The value $\pi_\rho(x,y)$ is called an intuitionistic fuzzy index of a pair (x,y) in Atanassov's intuitionistic fuzzy relation ρ. It is also described as an index (a degree) of hesitation whether or not x and y are in relation ρ. This value is also regarded as a measure of the lack of knowledge and is useful in applications.

2.1. Representation of A-IFSs by intervals. We will consider here the consequences of the fact that A-IFSs can be represented in the form of intervals (Szmidt and Kacprzyk, 2017; Szmidt, 2014). Using all three terms we may build two-intervals which represent the intuitionistic fuzzy relation $\rho = (R, R^d, \pi_\rho)$ in the following way:

$$[R, R + \alpha\pi_\rho], \quad [R^d, R^d + (1-\alpha)\pi_\rho], \tag{4}$$

where $\alpha \in [0, 1]$.

Example 1. Jon and Bob (x and y) can cooperate well, i.e., $R(x, y) = 0.7$; sometimes they disagree so $R^d(x, y) = 0.2$, and as we are not sure of their behaviour concerning a new project, the hesitation margin $\pi_\rho(x, y) = 0.1$ concerning their cooperation. Such data can be important from the point of view of building a well cooperating team working on an important project. In the best situation ($\alpha = 1$) we have $R(x, y) = 0.7 + 0.1 = 0.8$, and $R^d(x, y) = 0.2$. In the worst situation ($\alpha = 0$) we have $R(x, y) = 0.7$ and $R^d(x, y) = 0.2 + 0.1 = 0.3$. Other scenarios analyzing the same time $R(x, y)$ and $R^d(x, y)$ are also possible.

For example, for $\alpha = 0.4$, we have $R(x, y) = 0.7 + 0.04 = 0.74$, and $R^d(x, y) = 0.2 + 0.06 = 0.26$. Similar analyses are done in real life. Taking into account the two term representation of AIFSs does not make it possible (Szmidt and Kacprzyk, 2017) (only one interval can be built then). ♦

Considering such scenarios might be useful when decisions are to be made about some future events which can be described only to some extent, such as, e.g., in voting processes, introducing a new product into a market, looking for a new job, or buying a house. The success or failure then strongly depends on the values of the hesitation margins. It is worth stressing that three-term representation is different from the representation used in IVFSs (Szmidt, 2014; Szmidt and Kacprzyk, 2017). In other words, the IFSs expressed by the two-term representation and IVFSs are, like Atanassov and Gargov stated in 1989, "equipollent", that is, deducible from each other, but still not "the same" (e.g., because of different operators), whereas the A-IFSs expressed via a three-term representation are equivalent to considering two intervals which certainly means that they are not "the same" as IVFSs (Szmidt and Kacprzyk, 2017).

3. Intuitionistic fuzzy setting

3.1. Comparability relations. Note that Atanassov's intuitionistic fuzzy sets can be represented by the L^*-fuzzy sets in the sense of Goguen, where

$$L^* = \{x = (x_\mu, x_\nu) : \; x_\mu, x_\nu \in [0, 1], \; x_\mu + x_\nu \leq 1\},$$

with the uncertainty value for each intuitionistic element x:

$$\pi_x = 1 - x_\mu - x_\nu,$$

and the following partial order using shortly presentations of $x = (x_1, x_2), y = (y_1, y_2) \in L^*$:

$$(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1 \text{ and } x_2 \geq y_2.$$

A partial order in the family $(L^*, \leq_{L^*})$ means that we can have incomparability intuitionistic fuzzy values. Moreover, uncertainty is not included. This was the inspiration for considerations of the issue of comparing the intuitionistic fuzzy elements and searching for new methods of comparability.

Due to the imprecise or incomplete information presented by intuitionistic values we have a problem with the comparability of the above-mentioned values. We may use the above-mentioned comparability relations, for example, in the decision making model to represent the uncertainty or fuzziness of the trust relationship among a group of experts. In this case, decision making involves individuals generating problems, providing potential solutions, voting for solutions, and the software aggregating individual votes ultimately derives the final decision. Many decision making processes take place in an environment in which the information is not precisely known. As a consequence, experts may feel more comfortable using an interval number, rather than a precise numerical value to represent their preference. Therefore, intuitionistic fuzzy preference relations can be considered as an appropriate representation format to capture experts' uncertain preference information, hereby optimistic and pessimistic comparability relations.

3.1.1. Optimistic and pessimistic relations. In the sequel we shall focus on comparability relations used for intuitionistic values and intuitionistic fuzzy relations connected with epistemic and ontic settings (Dubois and Prade, 2012; Dubois *et al*., 2014). An epistemic (disjunctive) set S contains an ill-known actual value of a point-valued quantity x, so we can write $x \in S$. It represents the epistemic state of an agent, hence does not exist *per se*. Sets representing collections of elements forming composite objects are called ontic (conjunctive). A conjunctive set is the precise representation of an objective entity. An ontic set S is the value of a set-valued variable X, so we can write $X = S$. These relations are realized by optimistic and pessimistic comparability relations, respectively. Next to the standard relation $\leq_{L^*}$, which does not respect the uncertainty put in intuitionistic fuzzy values, we will consider more general, i.e., optimistic and more restrictive (pessimistic comparability relations). Representations of the above-mentioned optimistic and pessimistic comparability relations in interval interpretations and their dependences with partial order were considered and partially presented by Pękala *et al.* (2016) and Pękala (2019).

Optimistic comparability relation. An optimistic relation describes a more general situation, which we may write for $x = (x_1, x_2), y = (y_1, y_2) \in L^*$ and we define the comparability measure in an optimistic idea in the following way:

$$x \leq_O y \Leftrightarrow x_1 \leq y_1 + \pi_y.$$

Remark 1. By analogy, for Atanassov's intuitionistic fuzzy relations $\rho = (R, R^d), \sigma = (S, S^d)$ we have

$$\rho \leq_O \sigma \Leftrightarrow R \leq S + \pi_\sigma.$$

The relation $\leq_O$ is more suitable for the epistemic (disjunctive) setting of the intuitionistic values. Consequently, if (x_1, x_2) is an unprecise description of a variable x and (y_1, y_2) is an unprecise description of a variable y, then $x \leq_O y$ means that it is possible that the true value of x is less than or equal to the true value of y. Thus the relation $\leq_O$ has a possibility interpretation (Dubois and Prade, 1988) which we call optimistic.

Pessimistic comparability relation. We define the following restricted case of comparability intervals, i.e., a necessary relation, which we may interpret as a conjunctive (ontic) relation and show that one interval contains a collection of true values of each variable less than or equal to all true values from the second interval.

For $x = (x_1, x_2), y = (y_1, y_2) \in L^*$ we define the comparability measure in a pessimistic idea in the following way:

$$x \leq_P y \Leftrightarrow x_1 + \pi_x \leq y_1.$$

Remark 2. Similarly to the optimistic relation, by analogy to Atanassov's intuitionistic fuzzy relations $\rho = (R, R^d), \sigma = (S, S^d)$, we have

$$\rho \leq_P \sigma \Leftrightarrow R + \pi_\rho \leq S.$$

Moreover, we observe the following connections between the mentioned comparability relations:

Proposition 1. *Let $x, y \in L^*$. Then*

$$x \leq_P y \Rightarrow x \leq_{L^*} y \Rightarrow x \leq_O y.$$

Thus, considerations of necessary and possible comparability relations give a wider outlook than the description of the situation by classical order. In each aspect, we will use the following notation respectively for the largest and smallest element in L^*:

$$\mathbf{1} = (1, 0), \quad \mathbf{0} = (0, 1).$$

4. Intuitionistic fuzzy operations

4.1. Intuitionistic fuzzy aggregation functions. We recall the concept of an aggregation function on L^*, which is a crucial definition for this paper, because it is important in decision making problems and other applications.

Definition 1. (*Beliakov* et al.*, 2021; Zapata* et al.*, 2017*) Let $n \in \mathbb{N}, n \geq 2$. An operation $\mathcal{A} : (L^*)^n \to L^*$ is called an *Atanassov's intuitionistic fuzzy aggregation function* if it is increasing with respect to the order $\leq$, i.e.,

$$\forall_{x_i, y_i \in L^*} x_i \leq y_i \Rightarrow \mathcal{A}(x_1, \ldots, x_n) \leq \mathcal{A}(y_1, \ldots, y_n) \tag{5}$$

and

$$\mathcal{A}(\underbrace{\mathbf{0}, \ldots, \mathbf{0}}_{n\times}) = \mathbf{0}, \quad \mathcal{A}(\underbrace{\mathbf{1}, \ldots, \mathbf{1}}_{n\times}) = \mathbf{1}.$$

Note that the special case of Atanassov intuitionistic fuzzy aggregation operation is a representable Atanassov's intuitionistic fuzzy aggregation function with respect to $\leq_{L^*}$.

Definition 2. (Drygaś, 2011; Deschrijver *et al.*, 2004) Atanassov's intuitionistic fuzzy aggregation function $\mathcal{A} : (L^*)^n \to L^*$ is called *representable* if there exist aggregation functions $A_1, A_2 : [0,1]^n \to [0,1]$ such that

$$\mathbf{A}(x_1, \ldots, x_n) = (A_1(x_{\mu 1}, \ldots, x_{\mu n}), A_2(x_{\nu 1}, \ldots, x_{\nu n}))$$

for all $x_1, \ldots, x_n \in L^*$.

Moreover, we have the following characterization.

Theorem 1. (Drygaś, 2011; Deschrijver *et al.*, 2004) *An operation $\mathcal{A} : (L^*)^n \to L^*$ is a representable Atanassov's intuitionistic fuzzy aggregation function with respect to $\leq_{L^*}$ if and only if there exist aggregation functions $A_1, A_2 : [0,1]^n \to [0,1]$ such that for all $x_1, \ldots, x_n \in L^*$ and $A_1 \leq A_2^N$, i.e., $A_2^N(a, b) = 1 - A_2(1 - a, 1 - b)$,*

$$\mathcal{A}(x_1, \ldots, x_n) = (A_1(x_{\mu 1}, \ldots, x_{\mu n}), A_2(x_{\nu 1}, \ldots, x_{\nu n})).$$

Example 2. Examples of representable aggregation functions with respect to $\leq_{L^*}$ are

- $\mathcal{A}_\vee(x, y) = (\max(x_\mu, y_\mu), \min(x_\nu, y_\nu))$,
- $\mathcal{A}_\wedge(x, y) = (\min(x_\mu, y_\mu), \max(x_\nu, y_\nu))$,
- $\mathcal{A}_p(x, y) = (x_\mu y_\mu, x_\nu + y_\nu - x_\nu y_\nu)$,
- $\mathcal{A}_{\text{mean}}(x, y) = (x_\mu + y_\mu/2, x_\nu + y_\nu/2)$,
- $\mathcal{A}_{\text{gmean}}(x, y) = (\sqrt{x_\mu y_\mu}, 1 - \sqrt{(1 - x_\nu)(1 - y_\nu)})$,

for $x, y \in L^*$. ♦

4.1.1. Optimistic and pessimistic aggregation functions. We present here new types of aggregation functions on L^*. In the monotonicity condition in Definition 1 we replace the partial order with the relations $\preceq_O$ and $\preceq_P$. Note that the obtained aggregation functions are not special cases of aggregation functions on lattices (well described in the literature), since relations $\preceq_O$ and $\preceq_P$ may be not partial orders and represent the optimistic and pessimistic point of view.

Definition 3. (*Bentkowska, 2018*) Let $n \geq 2,\ n \in \mathbb{N}$. An operation $\mathcal{A} : (L^*)^n \to L^*$ is called an *optimistic aggregation function* (we shall write the O-aggregation function for short) if for $x_i, y_i \in L^*, i = 1, \dots, n$

$$x_i \preceq_O y_i \Rightarrow \mathcal{A}(x_1, \dots, x_n) \preceq_O \mathcal{A}(y_1, \dots, y_n), \quad (6)$$

$$\mathcal{A}(\underbrace{\mathbf{0}, \dots, \mathbf{0}}_{n\times}) = \mathbf{0}, \ \ \mathcal{A}(\underbrace{\mathbf{1}, \dots, \mathbf{1}}_{n\times}) = \mathbf{1}.$$

Definition 4. (*Bentkowska, 2018*) Let $n \geq 2,\ n \in \mathbb{N}$. An operation $\mathcal{A} : (L^*)^n \to L^*$ is called a *pessimistic aggregation function* (we shall write the P-aggregation function for short) if for $x_i, y_i \in L^*, i = 1, \dots, n$

$$x_i \preceq_P y_i \Rightarrow \mathcal{A}(x_1, \dots, x_n) \preceq_P \mathcal{A}(y_1, \dots, y_n), \quad (7)$$

$$\mathcal{A}(\underbrace{\mathbf{0}, \dots, \mathbf{0}}_{n\times}) = \mathbf{0}, \quad \mathcal{A}(\underbrace{\mathbf{1}, \dots, \mathbf{1}}_{n\times}) = \mathbf{1}.$$

The family of all O-aggregation functions will be denoted by $\mathcal{A}_O$ and the family of all P-aggregation functions will be denoted by $\mathcal{A}_P$. For the simplicity of notation, we include all results for two argument functions. We will present dependencies between the families of known aggregation functions on L^* and O- and P-aggregation functions.

Example 3. (*Bentkowska, 2018; Pękala, 2019*) The intuitionistic fuzzy representable aggregation function is an O-aggregation function. For example, by aggregation $A : [0,1]^2 \to [0,1]$ we have

$$\mathcal{A}_{O_1}(x, y) = \begin{cases} \mathbf{0} & \text{if } (x, y) = (\mathbf{0}, \mathbf{0}), \\ (A(x_\mu, y_\mu), 1) & \text{otherwise}, \end{cases}$$

$$\mathcal{A}_{O_2}(x, y) = \begin{cases} \mathbf{1} & \text{if } \quad (x, y) = (\mathbf{1}, \mathbf{1}), \\ (0, A(x_\nu, y_\nu)) & \text{otherwise}. \end{cases}$$

are also O-aggregation function (but they are not P-aggregation functions).

Moreover, the following functions on L^* are O-aggregation functions but they are neither aggregation functions nor P-aggregation functions (in $\mathcal{A}_{O_3}$ we use the convention $0/0 = 0$):

$$\mathcal{A}_{O_3}(x, y) = \begin{cases} \mathbf{0} & \text{if } \quad (x, y) = (\mathbf{0}, \mathbf{0}), \\ \left(\frac{{x_\mu}^2 + {y_\mu}^2}{x_\mu + y_\mu}, 0\right) & \text{otherwise}, \end{cases}$$

$$\mathcal{A}_{O_4}(x, y) = (x_\mu \cdot |2y_\mu - 1|, x_\nu),$$

$$\mathcal{A}_{O_5}(x, y) = (\sqrt{x_\mu y_\mu}, \frac{x_\nu + y_\nu}{2}).$$

Moreover, $\mathcal{A}_P$ and $\mathcal{A}_{\text{mean}}$ are IV aggregation and O- and P-aggregation functions what is more; the following aggregations are also P-aggregation functions:

$$\mathcal{A}_{P_1}(x, y)$$
$$= (A(x_\mu, y_\mu), \min(A(1 - x_\mu, y_\nu), A(x_\nu, 1 - y_\mu)),$$
$$\mathcal{A}_{P_2}(x, y)$$
$$= (\min(A(x_\mu, 1 - y_\nu), A(1 - x_\nu, y_\mu)), A^N(x_\nu, y_\nu)),$$

where A is the aggregation function and

$$A^N(a, b) = 1 - A(1 - a, 1 - b) \quad \text{for} \quad a, b \in [0, 1],$$

$$\mathcal{A}_{P_3}(x, y) = \Big(\frac{y_\mu + \frac{x_\mu + 1 - x_\nu}{2}}{2}, \frac{x_\nu + y_\nu}{2}\Big)$$

for $x = (x_\mu, x_\nu, \pi_x)$, $y = (y_\mu, y_\nu, \pi_y) \in L^*$. ♦

New types of aggregation functions have possible applications in practical models, where the process of aggregation of interval data is involved. It would be interesting to check the effectiveness of applying O- and P-aggregation functions and this will be presented in Section 7. For notational simplicity we include all results for two-argument functions. We will present dependencies between the families of known aggregation functions on L^* and O- and P-aggregation functions. Inspired by a study on an interval-valued setting (cf. Bentkowska, 2018) we observe the following properties.

Theorem 2. *Let $A_1, A_2 : [0,1]^2 \to [0,1]$. If $\mathcal{A} : (L^*)^2 \to L^*$ is a representable aggregation function, $\mathcal{A} = (A_1, A_2)$, $A_1 \leq A_2^N$, then $\mathcal{A}$ is a (representable) O-aggregation function.*

We observed (in the above examples) representable aggregation functions, $\mathcal{A}$, which are O-aggregation functions but may not be aggregation functions. That is not the case for P-aggregation functions which is shown in the next theorem.

Theorem 3. *Let $\mathcal{A} : (L^*)^2 \to L^*$ be a representable aggregation function. $\mathcal{A}$ is a P-aggregation function if and only if $A_1 = A_2^N$.*

Remark 3. By the assumption of $A_1 = A_2^N$ we also observe that aggregation functions A_{P_2} and A_{P_3} are P-aggregation functions.

Remark 4. Let us pay attention to the fact that the three classes of aggregation functions mentioned above have a common part, i.e., there are optimistic aggregation functions that are not classic and, on the other hand similarly, there are classic aggregation functions that are not optimistic. A similar situation may be observed for a pair of the pessimistic and classic aggregation functions or optimistic and pessimistic aggregation functions.

4.2. Intuitionistic fuzzy negation. Now we shall analyse the notion of an intuitionistic fuzzy negation. Firstly, we recall the definition of intuitionistic fuzzy negation with respect to partial order.

Definition 5. (*Atanassov, 2008; Asiain* et al.*, 2018*) An *intuitionistic fuzzy negation* is a function $N : L^* \to L^*$ that decreases with respect to $\leq_{L^*}$ with $N(\mathbf{1}) = \mathbf{0}$ and $N(\mathbf{0}) = \mathbf{1}$. An intuitionistic fuzzy negation is said to be *involutive* if it fulfills $N(N(x)) = x$ for any $x \in L^*$.

We will refer to on intuitionist fuzzy negation if it fulfills Definition 5 in a classic sense.

Example 4. (*Atanassov, 2008; Zapata* et al.*, 2017*) The functions $N_1, N_2 : L^* \to L^*$ defined by

- $N_1(x) = N_1((x_\mu, x_\nu)) = (x_\nu, x_\mu)$,
- $N_2(x) = N_2((x_\mu, x_\nu)) = (2x_\nu + x_\nu^2, x_\mu^2)$

are interval-valued fuzzy negations with respect to the order $\leq_{L^*}$. ♦

Proposition 2. (Deschrijiver *et al.*, 2004) *Let N be an involutive intuitionistic fuzzy negation. Then there exist an involutive fuzzy negations n_1, n_2 ($n_1 \leq n_2$) that satisfy $N(x) = (n_1(1 - x_\nu), 1 - n_2(x_\mu))$, $x \in L^*$ and N is said to be a representable involutive intuitionistic fuzzy negation with respect to $\leq_{L^*}$.*

For example, N_1 is an involutive intuitionistic fuzzy negation, called the standard intuitionistic fuzzy negation for standard fuzzy negations n_1 and n_2, i.e., $n_1(a) = 1-a$ and $n_2(a) = 1 - a$, $a \in [0, 1]$.

4.2.1. Optimistic and pessimistic intuitionistic fuzzy negation functions.

We present here new types of negation functions on L^*. We replace the partial order in the monotonicity condition in Definition 5 with the relations $\preceq_O$ and $\preceq_P$. Note that the obtained negation functions are connected with optimistic and pessimistic points of view.

Definition 6. An *optimistic intiutionistic fuzzy negation* is a function $N : L^* \to L^*$ that decreases with respect to $\leq_O$ with $N(\mathbf{1}) = \mathbf{0}$ and $N(\mathbf{0}) = \mathbf{1}$.

Definition 7. A *pessimistic intuitionistic fuzzy negation* is a function $N : L^* \to L^*$ that decreases with respect to $\leq_P$ with $N(\mathbf{1}) = \mathbf{0}$ and $N(\mathbf{0}) = \mathbf{1}$.

Example 5. (*New intuitionistic fuzzy negations*) The operations N_1 and N_2 are also intuitionistic fuzzy negations with respect to $\leq_O$ and $\leq_P$ (optimistic and pessimistic). Moreover, the following operations:

$$N_{O_1}(x) = \begin{cases} \mathbf{1} & \text{if } x = \mathbf{0}, \\ \mathbf{0} & \text{if } x = \mathbf{1}, \\ \Big(\dfrac{1 - x_\mu}{2}, 1 - \dfrac{1 - x_\nu}{2}\Big) & \text{otherwise}, \end{cases}$$

$$N_{O_2}(x) = \begin{cases} \mathbf{1} & \text{if } x = \mathbf{0}, \\ (0, 1 - x_\nu) & \text{otherwise}, \end{cases}$$

$$N_{O_3}(x) = \begin{cases} \mathbf{0} & \text{if } x = \mathbf{1}, \\ (1 - x_\mu, 0) & \text{otherwise} \end{cases}$$

are optimistic intuitionistic fuzzy negations (non representable) with respect to the order $\leq_O$ but not with respect to $\leq_P$. This contrasts with the operation

$$N_P(x) = (x_\nu, 1 - x_\nu)$$

for $x \in L^*$ which is an intuitionistic fuzzy negation (non- representable) and a pessimistic intuitionistic fuzzy negation, i.e., with respect to the order $\leq_P$ and $\leq_{L^*}$ but not with respect to $\leq_O$.

Moreover, the operation

$$N_{O_4}(x) = \begin{cases} \mathbf{1} & \text{if } x = \mathbf{0}, \\ \mathbf{0} & \text{if } x = \mathbf{1}, \\ (0, 1 - x_\mu) & \text{otherwise}, \end{cases}$$

is an optimistic intuitionistic fuzzy negation with respect to the order $\leq_O$ but not with respect to $\leq_{L^*}$. ♦

Proposition 3. *Let n_1, n_2 be fuzzy negation functions and $x \in L^*$. The representable intuitionistic fuzzy negation $N(x) = (n_1(1 - x_\nu), 1 - n_2(x_\mu))$*

(i) is a representable (optimistic) intuitionistic fuzzy negation with respect to $\leq_O$;

(ii) is a representable (pessimistic) intuitionistic fuzzy negation with respect to $\leq_P$ if $n_1 = n_2$.

Remark 5. From the above observations, we can conclude that the three classes of intuitionistic fuzzy negations mentioned above are connected in the part between them, so that, e.g., there are intuitionistic fuzzy negations and also optimistic ones. But there are also optimistic negation functions that are not intuitionistic fuzzy negation or vice versa. Similar dependencies may be observed for pairs of pessimistic and classic negation functions and optimistic and pessimistic negation functions.

5. Preference structure

Considering decision making problems in the intuitionistic fuzzy environment, we are dealing with a finite set of alternatives $X = \{x_1, \dots, x_n\}$ ($X \neq \emptyset$) and an expert providing his/her preference information over alternatives. In the sequel, we will consider a preference relation on the set X which makes it possible to represent Atanassov's intuitionistic fuzzy relations by matrices.

Definition 8. (*Xu, 2007*) Let $\mathrm{card}(X) = n$. An *intuitionistic fuzzy preference relation* ρ on the set X is represented by a matrix $\rho = (\rho_{ij})_{n\times n}$ with $\rho_{ij} = (R(i,j), R^d(i,j), \pi_\rho)$, for all $i, j = 1, \dots, n$, where ρ_{ij} is an intuitionistic fuzzy value, composed by the degree $R(i,j)$ to which x_i is preferred to x_j, the degree $R^d(i,j)$ to which x_i is non-preferred to x_j, and the lack of knowledge degree $\pi(i,j)$ concerning both $R(i,j)$ and $R^d(i,j)$, as follows:

- $\rho_{ij} = (0.5, 0.5, 0)$ indicates indifference between x_i and x_j ($x_i \sim x_j$),
- $\rho_{ij} > (0.5, 0.5, 0)$ represents an uncertain preference of x_i over x_j ($x_i \succ x_j$ ($x_i \succeq x_j$ for $\rho_{ij} \geq (0.5, 0.5, 0)$)),
- $\rho_{ij} = (\mathbf{1}, \mathbf{0}, \mathbf{0})$ when x_i is definitely (certainly) preferred to x_j,
- $\rho_{ij} = (\mathbf{0}, \mathbf{1}, \mathbf{0})$ when x_j is definitely (certainly) preferred to x_i.

A preference structure can be characterized by a weak preference relation called the large preference relation. It has been mentioned that it is possible to construct a preference structure from a large preference relation ρ in the classical case and this was also examined in the fuzzy case by Fodor and Roubens in 1994.

We continue these examinations and we propose their generalization to the intuitionistic fuzzy structure. Then for an intuitionistic fuzzy relation, $\rho = (\rho_{ij})$, we build the corresponding intuitionistic fuzzy strict preference (P), intuitionistic fuzzy indifference (I), and intuitionistic fuzzy incomparability (J) by using intuitionistic fuzzy aggregation functions instead of intuitionistic fuzzy t-norms and general negations instead of classic negations.

We propose the following method for building the preference structure by using intuitionistic fuzzy aggregation functions $\mathcal{A}$ and $\mathcal{B}$ and intuitionistic fuzzy negation N (cf. Pękala, 2019):

• intuitionistic fuzzy strict preference

$$P_{ij} = \mathcal{A}(\rho_{ij}, N(\rho_{ji})), \tag{8}$$

• intuitionistic fuzzy indifference

$$I_{ij} = \mathcal{B}(\rho_{ij}, \rho_{ji}), \tag{9}$$

• intuitionistic fuzzy incomparability

$$J_{ij} = \mathcal{B}(N(\rho_{ij}), N(\rho_{ji})) \tag{10}$$

for all $i, j \in \{1, \dots, n\}$.

6. Transitivity properties for Atanassov's intuitionistic fuzzy relations

Now we will consider the transitivity property and its connection with Atanassov's operators and reciprocal property. We observe that for Atanassov's intuitionistic fuzzy relation ρ the condition $R(i,j) \geq 0.5$ implies $R^d(i,j) \leq 0.5$, so that $\rho_{ij} \geq (0.5, 0.5, 0)$. The transitivity property of interval-valued fuzzy relations is now examined. This property is important because of its possible applications in the preference procedures. The accuracy of the final ranking of the alternatives must be based on consistent judgments, as an inconsistent preference relation may lead to incorrect conclusions.

Traditionally, the consistency of a preference relation is characterized by transitivity, in the sense that if an alternative A is preferred to or equivalent to alternative B, and B is preferred to or equivalent to alternative C, then A must be preferred to or equivalent to C. The transitivity assumption can be used to check for the judgmental consistency of the group decision making. Therefore, the study of the consistency of a preference relation is very important. Another detailed discussion on the transitivity of reciprocal relations (for fuzzy setting) was presented by De Baets in 2005 and 2006 or by Switalski in 2003.

Remark 6. The transitivity of $\rho \in \mathrm{AIFR}(X)$ may be characterized by the property involving composition, namely $\rho^2 \leq \rho$. In the context of preference relations, for $X = \{x_1, \dots, x_n\}$, transitivity captures the fact that, if the alternative x_i is preferred to x_k and x_k is preferred to x_j, then x_i should be preferred to x_j.

Here we recall $\mathcal{B}$-transitivity by partial order, but we are concerned with optimistic and pessimistic transitivity. Thus ρ is $\mathcal{B}$-transitive (in the classical point of view, and called a standard transitivity) if

$$\mathcal{B}(\rho(x,z), \rho(z,y)) \leq \rho(x,y). \tag{11}$$

For the order $\leq_{L^*}$ and the representable intuitionistic fuzzy aggregation we may write $\mathcal{B}$-transitivity in the following way:

$$B_1(R(x,z), R(z,y)) \leq R(x,y),$$
$$B_2(R^d(x,z), R^d(z,y)) \geq R^d(x,y)$$

for $\mathcal{B} = (B_1, B_2)$ and $B_1 \leq B_2^N$ (see Theorem 1).

Due to the lack of considerations in the previous definitions of transitivity of uncertainty, i.e., the index π, it seems justified to look at the optimistic and pessimistic points of view of transitivity.

6.1. Optimistic and pessimistic transitivity. For $\rho = (R, R^d, \pi_\rho)$ concerning uncertainty, i.e., by the use of interpretation (4) naturally a new concept of transitivity has emerged, especially in optimistic and pessimistic issues:

- ρ is optimistic B-transitive if
$$B(R(x,z),R(z,y)) \le R(x,y) + \pi_\rho(x,y)$$
and
$$B(R^d(x,z),R^d(z,y)) \le R^d(x,y) + \pi_\rho(x,y);$$
- ρ is pessimistic B-transitive if
$$B(R(x,z)+\pi_\rho(x,z),R(z,y)+\pi_\rho(z,y)) \le R(x,y)$$
and
$$B(R^d(x,z)+\pi_\rho(x,z),R^d(z,y)+\pi_\rho(z,y)) \le R^d(x,y)$$
for the aggregation function $B:[0,1]^2 \to [0,1]$.

These new transitivities are different from others known in the literature, for example, the weak transitivity studied by Pękala *et al*. (2018) or Xu *et al*. (2014) and 0.5-transitivity (Bentkowska *et al*., 2015) or possibly and necessary transitivity created only with the first conditions of both proposed: optimistic and pessimistic but in the interval-valued setting (Pękala *et al*., 2016).

6.2. Interdependence between optimistic, pessimistic or standard properties. Directly by the definitions of optimistic-B-transitivity and pessimistic-B-transitivity, we observe the following implications:

Corollary 1. *Let $R \in AIFR(X)$ and $\mathcal{B}$ be a representable intuitionistic fuzzy aggregation function.*

- *If R is pessimistic-B-transitive, then R is $\mathcal{B}$-transitive, where $\mathcal{B}=(B,B^N)$.*
- *If R is $\mathcal{B}$-transitive, then R is optimistic-B_1-transitive, where $\mathcal{B}=(B,B^N)$.*

Example 6. The relation $\rho=(R,R^d,\pi_\rho)$, where
$$R=\begin{bmatrix}0.5 & 0.5\\ 0.8 & 0.9\end{bmatrix}, \quad R^d=\begin{bmatrix}0.5 & 0.2\\ 0.1 & 0\end{bmatrix},$$
is optimistic-min-transitive, but not $(\min,\min)$-transitive and the relation $\sigma=(S,S^d,\pi_\sigma)$, where
$$S=\begin{bmatrix}0.5 & 0.6\\ 0.5 & 0.5\end{bmatrix}, \quad S^d=\begin{bmatrix}0 & 0.3\\ 0.5 & 0.5\end{bmatrix},$$
is $(\min,\max)$-transitive and optimistic-min-transitive. ♦

Thus the optimistic transitive property is the weakest; hence, from a practical point of view, we would like to use the optimistic or pessimistic transitive property in the decision making model.

For two operations, one less than or equal to the other, transitivity by a larger operation implies transitivity by a smaller operation.

Proposition 4. *Let $B_1,B_2:[0,1]^2 \to [0,1]$ be aggregations and $B_1 \le B_2$. If $R \in AIFR(X)$ is optimistic-B_2-transitive (pessimistic-B_2-transitive), then R is optimistic-B_1-transitive (pessimistic-B_1-transitive).*

6.3. Preservation of optimistic-B-transitivity and pessimistic-B-transitivity properties by intuitionistic fuzzy operations.

6.3.1. Preservation of transitivity by intuitionistic fuzzy aggregation. We will also examine an arbitrary aggregation of intuitionistic fuzzy relations having optimistic-B-transitivity and pessimistic-B-transitivity properties and the problem of the preservation of these properties. We generally intend to consider the same type of property and aggregation function, namely based on the same type of comparability relation $\preceq_O$ or $\preceq_P$. However, to complete the information, we also present the mixture of aggregation type and the type of comparability relation (Bentkowska, 2018).

To preserve transitivity we will need to use the concept of domination (Saminger *et al*., 2002).

Proposition 5. *Let $n \in \mathbb{N}$, B be an aggregation function and $\rho_1,\rho_2,\dots,\rho_n \in AIFR(X)$.*

1. *If $\rho_1,\rho_2,\dots,\rho_n$ are pessimistic-B-transitive relations, then $\mathcal{A}(\rho_1,\rho_2,\dots,\rho_n)$ is pessimistic-B-transitive for the representable pessimistic-aggregation function $\mathcal{A}=(A_1,A_2)$, where $A_2 \gg B$, $A_2^N \gg B$ and $A_1 = A_2^N$.*
2. *If $\rho_1,\rho_2,\dots,\rho_n$ are optimistic-B-transitive relations, then $\mathcal{A}(\rho_1,\rho_2,\dots,\rho_n)$ is optimistic-B-transitive for the representable optimistic-aggregation function $\mathcal{A}=(A_1,A_2)$, where $A_2 \gg B$, $A_1 \gg B$ and $A_1 \le A_2^N$.*

6.3.2. Preservation of transitivity by an intuitionistic fuzzy negation. For the representable intuitionistic fuzzy negation we may observe the following conditions:

Proposition 6. *Let N be a representable intuitionistic fuzzy negation such that $N(x)=(n_1(1-x_\nu),1-n_2(x_\mu))$, where $n_1=n_2$ be a standard fuzzy negation and $x \in L^*$. Then*

1. *N preserves optimistic-B-transitivity,*

2. *N preserves pessimistic-B-transitivity.*

These considerations have possible applications in multi-criteria (or similarly multi-agent) decision making problems with intervals (not just numbers in [0, 1]). By virtue of using all possible approaches of the interpretation of the intervals, we may have applications depending on the presented problem from real-life situations. In such cases, for aggregation of the given data (gathered as interval-valued fuzzy relations) it can be interesting to use an adequate type of aggregation function, which follows from the assumed interpretation.

7. Note of application

The presented "optimistic" and "pessimistic" approach (aggregations, negations, transitivity) can be tested and compared with classical aggregations, negations, and transcendence in decision Algorithm 1.

7.1. Practical example. Consider a group decision making example illustrating some problems which can be overcome by Algorithm 1.

Using data from Taylor (2005), as well as Pękala *et al.* (2018), we have a department with three members of recruitment, one of them being the manager. They are in the process of filling a position in the department and have interviewed three finalists for a job. We need some procedure for passing from the preferences of the individuals in the department to the "preferences" (decision) of the group. A ballot would have several names, intuitively representing either a group that this department member feels is tied for the top, or those candidates that the department member finds acceptable. But such ballots could allow each department member to rank-order the candidates from best to worst, in his or her opinion, perhaps allowing ties (representing indifference $(0.5, 0.5, 0)$) in the individual ballots and perhaps not.

Moreover, we can obtain the voting paradox (also known as Condorcet's paradox), which is a situation in which collective preferences can be cyclic (i.e., not transitive), even if the preferences of individual voters are not cyclic. This is paradoxical because this means that a majority allows the possibility of conflict with others. When this occurs, this is because the conflicting majorities are each made up of different groups of individuals.

Thus an expectation that transitivity on the part of all individuals' preferences should result in the transitivity of societal preferences may be false. For our three candidates, A, B, and C, there are three voters with preferences as follows:

Voter 1: $A \succeq B \succeq C$,

Voter 2: $B \succeq C \succeq A$,

Voter 3: $C \succeq A \succeq B$.

Algorithm 1. Preference_Structure.

Inputs : $X = \{x_1, \dots, x_n\}$ set of alternatives; $\rho_1, \dots, \rho_n \in$ AIFR(X): intuitionistic fuzzy preference relations; method of selection of intuitionistic fuzzy values; intuitionistic fuzzy optimistic (pessimistic or classic) aggregation functions $\mathcal{A}, \mathcal{B}$.

Output : Solution alternative: $x_{\text{selection}}$ of the objects.

Step 1. Aggregation of given relations $\rho_1, \dots, \rho_n \in$ AIFR(X) by the usage of one of the aggregations to obtain $\rho \in$ AIFR(X).

Step 2. Building P, I, J intuitionistic fuzzy relations based on ρ^*.

Step 3. Calculation of

$$M_{ij} = \mathcal{A}(P_{ij}, I_{ij}, J_{ij}).$$

Step 4. Building optimistic (pessimistic) transitive relation ρ^* from ρ.

Step 5. Finding

$$x_i = \mathcal{B}_{1 \le j \le n}(M_{ij}),$$

where $\mathcal{B} \ge \max$.

Step 6. Ordering the alternatives.

If C is chosen as the winner, it can be argued that B should win instead, since two voters, 1 and 2, prefer B to C and only one voter, 3, prefers C to B. However, by the same argument A is preferred to B, and C is preferred to A, by a margin of two to one in each case. Thus the society's preferences show cycling: A is preferred over B which is preferred over C, which is preferred over A. A paradoxical feature of relations between the voters' preferences described above is that although the majority of voters agree that A is preferable to B, B to C and C to A, all three coefficients of rank correlations between the voters' preferences are negative.

If the above preferences of our voters are represented by intuitionistic fuzzy relations and we use the presented algorithm, then we omit Condorcet's paradox in the voting problem and we see a different solution from Pękala *et al.* (2018), cf. Fig. 1.

To solve the problem of selection of a worker with the best relationships in a corporation, we use Algorithm 1 with the following assumptions:

1. We use aggregation and negation functions with the same class, such as the kind of transitivity. Thus we study three classes: optimistic "$\mathcal{A}_O$", pessimistic "$\mathcal{A}_P$" and classic "$\mathcal{A}_C$" to build the preference structure: P, I and J and in Steps 3 and 5 of Algorithm 1.

Voter 1

	A	B	C
A	$(0.5, 0.5, 0)$	$(0.7, 0.1, 0.2)$	$(0.6, 0.3, 0.1)$
B	$(0.1, 0.7, 0.2)$	$(0.5, 0.5, 0)$	$(0.8, 0.2, 0)$
C	$(0.3, 0.6, 0.1)$	$(0.2, 0.8, 0)$	$(0.5, 0.5, 0)$

,

Voter 2

	A	B	C
A	$(0.5, 0.5, 0)$	$(0.2, 0.6, 0.2)$	$(0.1, 0.7, 0.2)$
B	$(0.6, 0.2, 0.2)$	$(0.5, 0.5, 0)$	$(0.8, 0.2, 0)$
C	$(0.7, 0.1, 0.2)$	$(0.2, 0.8, 0)$	$(0.5, 0.5, 0)$

,

Voter 3

	A	B	C
A	$(0.5, 0.5, 0)$	$(0.8, 0.1, 0.1)$	$(0.1, 0.9, 0)$
B	$(0.1, 0.8, 0.1)$	$(0.5, 0.5, 0)$	$(0.2, 0.6, 0.2)$
C	$(0.9, 0.1, 0)$	$(0.6, 0.2, 0.2)$	$(0.5, 0.5, 0)$

.

Fig. 1. Solution to the practical example.

2. We use the following method (Szmidt and Kacprzyk, 2009) for the ranking of the alternatives Y_i:

$$\mathrm{SK}(Y_i) = 0.5(1 + \pi_{Y_i}) d_H(M, Y_i), \tag{12}$$

where M is the ideal positive alternative $(1, 0, 0)$. This equation tells us about the "quality" of an alternative Y_i – the lower the value of $\mathrm{SK}(Y_i)$, the better the alternative Y_i in the sense of the amount of positive information included, and reliability of the information.

In (12) the normalized Hamming distance between the AIFRs (Szmidt and Kacprzak, 2000; 2006) is used.

Definition 9. Let $\rho = (R, R^d, \pi_\rho), \sigma = (S, S^d, \pi_\sigma) \in \mathrm{AIFR}(X)$, $\mathrm{card}(X) = n$, $n \in \mathbb{N}$. Set

$$\begin{aligned} &d_H(\rho, \sigma) \\ &= \frac{1}{2n} \sum_{i,j=1}^{n} |R(i,j) - S(i,j)| \qquad (13) \\ &\quad + |R^d(i,j) - S^d(i,j)| + |\pi_\rho(i,j) - \pi_\sigma(i,j)|. \end{aligned}$$

If we assume equal ranges of each expert and use in Step 1 of Algorithm 1 the arithmetic mean $\mathcal{A}_{\text{mean}}$ (the arithmetic mean preserves reciprocal property) then after aggregation of the above three relations:

$$\mathcal{A}_{\text{mean}}(\rho_i) = \Big(\frac{1}{3} \sum_i R_i, \frac{1}{3} \sum_i R_i^d, \frac{1}{3} \sum_i \pi_i \Big)$$

for $i = 1, 2, 3$ we obtain relation A_{gV} of Table 2.

Then by the above assumptions, we observe the influence kind of transitivity, i.e., optimistic-B-transitivity (I), pessimistic-B-transitivity (II) (Table 1) in the solution.

7.2. Results and a discussion. We will focus on presenting the conclusion of the algorithm analysis in the following aspects:

(i) using different aggregation functions in two classes of transitivity (optimistic (I) and pessimistic (II)) and their influence on the ranking of alternatives;

(ii) using different aggregation functions to build the preference structure (P, I, J) and to aggregate them.

In Table 1 we present the results for different aggregation functions used to create the preference structure and in Step 3 of the algorithm Preference_Structure. Moreover, in Step 5 of the algorithm, we used $\mathcal{B} = \max$ and in the adequate transitivity $B = \min$. We can observe that for a more restrictive pessimistic transitivity we have a unequivocal solution, i.e.,

$$C \succeq B \succeq A,$$

also for optimistic transitivity with optimistic aggregation functions by the majority method the alternative, C, wins.

On the other hand, it is not surprising that there is an ambiguous solution for classical aggregations, which indicates the necessity to use the same class of transitivity and aggregation.

We can also support this solution by analysing the A_{gV} relation. Namely, for each row, i.e., for each alternative, we measure the uncertainty, i.e., the entropy, and we can observe

$$E_A = 0.26 \geq E_B = 0.23 \geq E_C = 0.17,$$

where from (Burillo and Bustince, 1996) we recall

$$E_F = \sum_{i=1,\dots,n} \pi_{F(x_i)} \quad \text{for} \quad F \in \mathrm{AIFS}(X),$$

and $\mathrm{card}(X) = n$. This also suggests alternative C is the best.

The above reasoning explains why the new transitivity, and so the new approach, is a better solution than methods based on weak transitivities, such as in the work of Pękala *et al.* (2018). We may conclude that for pessimistic transitivity we obtain an unequivocal winner from among the candidates.

8. Conclusions

In this article, we discuss the new concepts of the transitivity problem of Atanassov's intuitionistic fuzzy relations, in an epistemic aspect. We propose a new optimistic and pessimistic transitivity among the preference of alternatives (options) considered and their use in the procedure of ranking the alternatives in a group decision making problem.

Table 1. Results of Algorithm 1.

Aggregation/Transitivity	P	I	J	Step 3	Order of alternatives
	$\mathcal{A}_{O_1}$	$\mathcal{A}_{O_2}$	$\mathcal{A}_{O_2}$	$\mathcal{A}_{O_5}$	$C \succeq B \succeq A$
$\mathcal{A}_O$/I	$\mathcal{A}_{O_3}$	$\mathcal{A}_{O_4}$	$\mathcal{A}_{O_4}$	$\mathcal{A}_{O_5}$	$C \succeq B \succeq A$
	$\mathcal{A}_{O_4}$	$\mathcal{A}_{O_5}$	$\mathcal{A}_{O_5}$	$\mathcal{A}_{\text{mean}}$	$B \succeq C \succeq A$
	$\mathcal{A}_{P_3}$	$\mathcal{A}_{\text{mean}}$	$\mathcal{A}_{\text{mean}}$	$\mathcal{A}_{\text{mean}}$	$C \succeq B \succeq A$
$\mathcal{A}_P$/II	$\mathcal{A}_{\text{mean}}$	$\mathcal{A}_{P_3}$	$\mathcal{A}_{P_3}$	$\mathcal{A}_{\text{mean}}$	$C \succeq B \succeq A$
	$\mathcal{A}_P$	$\mathcal{A}_{P_3}$	$\mathcal{A}_{P_3}$	$\mathcal{A}_P$	$C \succeq B \succeq A$
	$\wedge$	$\vee$	$\vee$	$\mathcal{A}_{\text{mean}}$	$C \succeq B \succeq A$
$\mathcal{A}_C$/I	$\mathcal{A}_{\text{mean}}$	$\mathcal{A}_P$	$\mathcal{A}_P$	$\mathcal{A}_{\text{gmean}}$	$B \succeq C \succeq A$
	$\mathcal{A}_{\text{mean}}$	$\mathcal{A}_{\text{gmean}}$	$\mathcal{A}_{\text{gmean}}$	$\mathcal{A}_{\text{mean}}$	$B \succeq C \succeq A$
	$\wedge$	$\vee$	$\vee$	$\mathcal{A}_{mean}$	$C \succeq B \succeq A$
$\mathcal{A}_C$/II	$\mathcal{A}_{\text{mean}}$	$\mathcal{A}_P$	$\mathcal{A}_P$	$\mathcal{A}_{\text{gmean}}$	$C \succeq B \succeq A$
	$\mathcal{A}_{\text{mean}}$	$\mathcal{A}_{\text{gmean}}$	$\mathcal{A}_{\text{gmean}}$	$\mathcal{A}_{\text{mean}}$	$C \succeq B \succeq A$

Table 2. Relation $A_q v$ of the practical example.

A_{gV}	A	B	C
A	$(0.5, 0.5, 0)$	$(0.57, 0.27, 0.16)$	$(0.27, 0.63, 0.1)$
B	$(0.27, 0.57, 0.16)$	$(0.5, 0.5, 0)$	$(0.6, 0.33, 0.07)$
C	$(0.63, 0.27, 0.1)$	$(0.33, 0.6, 0.07)$	$(0.5, 0.5, 0)$

In particular, the mentioned new transitivity and intuitionistic fuzzy negation, also initiated in optimistic and pessimistic aspects, are used together with the adequate aggregation functions for the preference model and, as a consequence, for the decision making model. In the future, we would like to study the effectiveness of the algorithm presented in this paper with a new transitivity for other data. We will study the proposed transitivity properties in other real-world problems, e.g., to construct an equivalence measure that we may use in image processing. Moreover, in the future, we will also consider the possibility of using the transitivity and interval interpretation used in the work, taking into account the uncertainty in such interesting areas as the recommender systems (Rutkowski *et al.*, 2019) or bootstrap methods (Grzegorzewski *et al.*, 2020).

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Appendix

A1. Proof of Proposition 3

Let $x = (x_\mu, x_\nu)$, $y = (y_\mu, y_\nu) \in L^*$ and $\pi_x = 1 - x_\mu - x_\nu$, $\pi_y = 1 - y_\mu - y_\nu$. We prove first Condition 1. If $x \leq_O y$, i.e., $x_\mu \leq 1 - y_\nu$, then for $n_1 \leq n_2$ (see Proposition 2)

$$n_2(x_\mu) \geq n_1(x_\mu) \geq n_1(1 - y_\nu),$$

i.e., $N(y) \leq_O N(x)$. Thus the representable intuitionistic fuzzy negation decreases with respect to $\leq_O$ and the following boundary conditions holds:

$$\begin{aligned} N(\mathbf{1}) &= N((1,0)) \\ &= (n_1(1-0), 1 - n_2(1)) = (0,1) = \mathbf{0}, \\ N(\mathbf{0}) &= N((0,1)) \\ &= (n_1(1-1), 1 - n_2(0)) = (1,0) = \mathbf{1}, \end{aligned}$$

which means that the representable intuitionistic fuzzy negation is an intuitionistic fuzzy negation with respect to $\leq_O$.

Now we consider Condition 2. Let $x \leq_P y$, i.e.

$$1 - x_\nu \leq y_\mu,$$

so

$$n_2(y_\mu) \leq n_2(1 - x_\nu) \leq n_1(1 - x_\nu),$$

i.e., $N(y) \leq_P N(x)$.

Thus the representable intuitionistic fuzzy negation decreases with respect to $\leq_P$ and the boundary conditions hold, which means that the representable intuitionistic fuzzy negation is an intuitionistic fuzzy negation with respect to $\leq_P$.

A2. Proof of Proposition 5

If A_2 and A_2^N dominates B and $(\rho_i) = (R_i, R_i^d, \pi_{\rho_i})$ is a family of pessimistic- B-transitive relations, then using notation $R + \pi_\rho = 1 - R^d$ we have

$$\begin{aligned} & B(\mathcal{A}(\rho_1, \rho_2, \dots, \rho_n)_\mu(x,y) + \pi_\rho(x,y), \\ & \mathcal{A}(\rho_1, \rho_2, \dots, \rho_n)_\mu(y,z) + \pi_\rho(y,z)) \\ & \quad = B(A_1(R_1(x,y), \dots, R_n(x,y)) + \pi_\rho(x,y), \\ & A_1(R_1(y,z), \dots, R_n(y,z))) + \pi_\rho(y,z) \\ & \quad = B(1 - A_2(R_1^d(x,y), \dots, R_n^d(x,y)), \\ & 1 - A_2(R_1^d(y,z), \dots, R_n^d)(y,z)) \\ & \quad = B(A_2^N(1 - R_1^d(x,y), \dots, 1 - R_n^d(x,y)), \\ & A_2^N(1 - R_1^d(y,z), \dots, 1 - R_n^d(y,z))) \\ & \quad \leq A_2^N(B(1 - R_1^d(x,y), 1 - R_1^d(y,z)), \dots, \\ & B(1 - R_n^d(x,y), 1 - R_n^d(y,z))) \\ & \quad \leq A_2^N(R_1(x,z), \dots, R_n(x,z)) \\ & \quad = A_1(R_1(x,z), \dots, R_n(x,z)) \\ & \quad = \mathcal{A}(\rho_1, \rho_2, \dots, \rho_n)_\mu(x,z). \end{aligned}$$

For the second condition of pessimistic transitivity, using the notation $R^d + \pi_\rho = 1 - R$, we obtain

$$\begin{aligned} & B(\mathcal{A}(\rho_1, \rho_2 \dots, \rho_n)_\nu(x,y) + \pi_\rho(x,y), \\ & \mathcal{A}(\rho_1, \rho_2, \dots, \rho_n)_\nu(y,z) + \pi_\rho(y,z)) \\ & \quad = B(1 - A_2^N(R_1(x,y), \dots, R_n(x,y)), \\ & 1 - A_2^N(R_1(y,z), \dots, R_n)(y,z)) \\ & \quad = B(A_2(1 - R_1(x,y), \dots, 1 - R_n(x,y)), \\ & A_2(1 - R_1(y,z), \dots, 1 - R_n)(y,z)) \\ & \quad \leq A_2(B(1 - R_1(x,y), 1 - R_1(y,z)), \dots, \\ & B(1 - R_n(x,y), 1 - R_n(y,z))) \\ & \quad \leq A_2(R_1^d(x,z), \dots, R_n^d(x,z)) \\ & \quad = \mathcal{A}(\rho_1, \rho_2, \dots, \rho_n)_\nu(x,z). \end{aligned}$$

The proof of optimistic-B-transitivity is similar.

A3. Proof of Proposition 6

Let ρ be a optimistic-B-transitive interval-valued fuzzy relation ($\rho \in \mathrm{IVFR}(X)$), i.e.,

$$B(R(x,z), R(z,y)) \leq R(x,y) + \pi_\rho(x,y)$$

and

$$B(R^d(x,z), R^d(z,y)) \leq R^d(x,y) + \pi_\rho(x,y).$$

We check if

$$N(\rho)(x,y) = (n_1(1 - R^d(x,y)), 1 - n_2(R(x,y)))$$

is optimistic-B-transitive for all $x, y, z \in X$. Thus

$$\begin{aligned}
&B(N(\rho)_\mu(x,y), N(\rho)_\mu(y,z)) \\
&\quad = B(n_1(1 - R^d(x,y)), n_1(1 - R^d(y,z))) \\
&\quad = B(R^d(x,y), R^d(y,z)) \\
&\quad \le R^d(x,z) + \pi_\rho \\
&\quad = 1 - R(x,z) \\
&\quad = 1 - (1 - n_2(R(x,z))) \\
&\quad = N(\rho)_\mu(x,z) + \pi_{N_\rho}(x,z);
\end{aligned}$$

By analogy, for the second condition of optimistic B-transitivity we have

$$\begin{aligned}
&B(1 - n_1(R(x,y)), 1 - n_1(R(y,z))) \\
&\quad = B(R(x,y), R(y,x)) \\
&\quad \le 1 - R^d(x,z) \\
&\quad = 1 - n_1(1 - R^d(x,z)) \\
&\quad = N(\rho)_\nu(x,z) + \pi_{N_\rho}(x,z),
\end{aligned}$$

completing the proof for the standard fuzzy negation $n_1 = n_2$. We can prove the second condition in a similar way.

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