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ORDER OPTIMAL REAL–TIME OBSERVERS FOR COMPLETELY OBSERVABLE CONTROL SYSTEMS

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For a complety observable control system governed by a nonlinear ordinary differential equation, we consider dynamical (real–time) observation operators stable with respect to small input perturbations. We treat them as regularizing operators on an appropriate set of well–posedness. Two solutions order optimal with respect to the perturbation accuracy are described. The tool is based on the idea of guided models from the theory of closed–loop differential games, and the lower bounds for the unimprovable observation accuracy.

1. Introduction

The observation theory for control systems described by linear ordinary differential equations involves many ultimate results. The foundation of the theory was systematically presented by Krasovskii (1968). The general structure of time–depending sets of all (unobserved) state coordinates compatible with current observations was described by Kurzhanskii (1977). The problem of designing an approximation to an unperturbed observation result stable with respect to small input perturbations was considered by Gusev and Kurzhanskii (1987). For some particular classes of nonlinear completely observable systems, the above problem was solved in (Osipov and Kryazhimskii, 1981; Kryazhimskii, 1985). The solution operators proposed in these papers are finite–step and satisfy the rather sharp Volterra condition whose practical aspect is feasibility in real time. This approach was extended in (Kryazhimskii and Osipov, 1990) to nonlinear systems that are in general not completely observable. However, the system's being not completely observable implies usually that there does not exist an approximation bound uniformly small with respect to all admissible inputs.

In this paper we deal with uniform approximation bounds for a completely observable system affine in control. Following Kryazhimskii (1985) where the observed state vector $z(t)$ was supposed to satisfy the simplest evolution equation $\dot{z}(t) = v(t)$ ($v(t)$ is a control), we extend the results to the general case. The solution operators are based on the method of guided models (Krasovskii and Subbotin, 1988). For the analysis, the tool of lower approximation bounds (Ivanov *et al.*, 1978) is used.

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2. Problem Formulation

Consider a control system

$$\dot{y}(t) = f_1(t, y(t), z(t)) + f_2(t, y(t), z(t))v(t) \tag{1}$$

$$\dot{z}(t) = g_1(t, y(t), z(t)) + g_2(t, y(t), z(t))v(t) \tag{2}$$

on a bounded time interval $I = [t_0, \vartheta_0]$. Here, $y(t) \in \mathbb{R}^n$, $z(t) \in \mathbb{R}^m$ are observed state vectors and $v(t) \in Q \subset \mathbb{R}^1$ is a control vector. The vector functions $f_1(\cdot)$ and $g_1(\cdot)$, and the matrix functions $f_2(\cdot)$ and $g_2(\cdot)$ are continuous. Any measurable function $v(\cdot) : I \to Q$ is called *a control*. A motion from an initial state (y_0, z_0) generated by a control $v(t)$ is a Caratheodory solution $(y(\cdot), z(\cdot))$ to (1), (2) on I such that $y(t_0) = y_0$, $z(t_0) = z_0$. Fix a compactum $X_0 \subset \mathbb{R}^n \times \mathbb{R}^m$ and denote $\mathcal{X}$ the set of all motions from initial states belonging to X_0 generated by all controls. Suppose that $\mathcal{X}$ is nonempty. Further, $|\cdot|$ stands both for the Euclidean vector norm and the matrix norm corresponding to the latter, and $\|\cdot\|$ denotes the standard functional sup–norm.

The problem in question consists in designing an approximation to the unobserved component $y(\cdot)$ of a motion $(y(\cdot), z(\cdot)) \in \mathcal{X}$ on the basis of a (perturbed) input $\xi = (y_{0*}, z_{0*}, \zeta(\cdot))$ here (y_{0*}, z_{0*}) is a perturbed initial state $(y_0, z_0) = (y(t_0), z(t_0))$, $|y_{0*} - y_0| \le s_y h$, $|z_{0*} - z_0| \le s_z h$, and $\zeta(\cdot) : I \to \mathbb{R}^m$ is a perturbed observed component $z(\cdot)$ of the motion, $\|\zeta(\cdot) - z(\cdot)\| \le h$. A triple ξ satisfying the above conditions will be called *an input $h-$accurate for* $(y(\cdot), z(\cdot))$; s_y and s_z are fixed. The set of all such inputs will be denoted by $\Xi(h|y(\cdot), z(\cdot))$. For the set of all *inputs* (i.e. the triples ξ not necessarily satisfying the above conditions), introduce the notation Ξ. Let $\mathcal{W}$ be the set of all bounded functions from I into $\mathbb{R}^n$. Any mapping $D : \xi \to w(\cdot|D, \xi)$ from Ξ into $\mathcal{W}$ will be called *an observation operator*. An observation operator D is called to be *dynamical*, if for each two inputs $\xi_1 = (y_1, z_1, \zeta_1(\cdot))$ and $\xi_2 = (y_2, z_2, \zeta_2(\cdot))$ such that $y_1 = y_2$, $z_1 = z_2$ and $\zeta_1(\tau) = \zeta_2(\tau)$ for all $\tau \in [t_0, t]$, it holds $w(\tau|D, \xi_1) = w(\tau|D, \xi_2)$ for all $\tau \in [t_0, t]$ (the Volterra property). The uniform accuracy of an observation D (for an input accuracy h) is given by

$$\nu(h, D) = \sup \{\|w(\cdot|D, \xi) - y(\cdot)\| : \xi \in \Xi(h|y(\cdot), z(\cdot)),\ (y(\cdot), z(\cdot)) \in \mathcal{X}\}$$

and the *unimprovable observation accuracy* $\nu_0(h)$ is defined to be the infimum of $\nu(h, D)$ over all observation operators D. A family (D_h), $h > 0$, of observation operators will be said to be *uniformly stable*, if $\nu(h, D_h) \to 0$ as $h \to 0$; it will be called to be *order optimal*, if there exsist $c > 0$ and $h_0 > 0$ such that $\nu(h, D_h) \le c\nu_0(h)$ for all $h \le h_0$.

The accurate problem formulation is as follows: find a uniformly stable family (D_h) of dynamical observation operators. We will also be interested in providing explicit upper bounds for $\nu(h, D_h)$ and proving (D_h)'s order optimality.

3. Assumptions

For a vector or a matrix function $b(\cdot)$ on $I \times \mathbb{R}^n \times \mathbb{R}^m$, introduce the growth condition: $|b(t,y,z)| \leq K(1+|y|+|z|)$; the Lipschitz condition for $b(\cdot)$ will be considered with respect to the norm $(t,y,z) \to |t|+|y|+|z|$ on the space of the arguments. We assume the functions $f_1(\cdot), f_2(\cdot), g_1(\cdot)$ and $g_2(\cdot)$ satisfy the growth condition and to be Lipschitz on every bounded set.

Let $\mathcal{Z} = \{(y(t_0), z(t_0), z(\cdot)) : (y(\cdot), z(\cdot)) \in \mathcal{X}\}$ and $Z : (y(\cdot), z(\cdot)) \to (y(t_0), z(t_0), z(\cdot)) : \mathcal{X} \to \mathcal{Z}$. The system (1), (2) is called to be ***completely observable***, if the operator Z is invertible. If it is so we will consider the ***reconstruction operator*** $Y : (y_0, z_0, z(\cdot)) \to y(\cdot) : \mathcal{Z} \to \mathcal{W}$ where $y(\cdot)$ is determined by $(y(\cdot), z(\cdot)) = Z^{-1}(y_0, z_0, z(\cdot))$. Following the terminology of the theory of ill–posed problems, we will say that the observation problem is *well–posed* ($\mathcal{X}$ is the *set of well–posedness*), if the system (1), (2) is completely observable and the operator Y is uniformly continuous on $\mathcal{Z}$, with respect to the metrics on $\mathcal{W}$ and $\mathcal{Z}$ induced, respectively, by the norms $\|\cdot\|$ and $|\cdot|_\star : (y_0, z_0, z(\cdot)) \to |y_0|+|z_0|+\|z(\cdot)\|$. According to Ivanov *et al.* (1978), well posedness is a necessary and sufficient condition for existence of a uniformly stable family (D_h) of observation operators; in particular, D_h can be provided by the method of quasi–solutions: $w(\cdot|D_h, \xi) = Y\eta_\star$, where $\eta_\star \in \mathcal{Z}$ is such that $|\eta_\star - \xi|_\star \leq \inf\{|\eta - \xi|_\star : \eta \in \mathcal{Z}\} + h$. However, the question whether well–posedness implies existence of a uniformly stable family of dynamical observation operators, is nontrivial.

It is assumed the following condition ensuring well–posedness: rank $g_2(t,y,z)$ is equal to the dimension 1 of a control vector. Indeed, let $g_2^+(t,y,z)$ be the matrix pseudoinverse to $g_2(t,y,z)$, and

$$f_1^\star(t,y,z) = f_1(t,y,z) - f_2(t,y,z) g_2^+(t,y,z) g_1(t,y,z)$$

$$f_2^\star(t,y,z) = f_2(t,y,z) g_2^+(t,y,z)$$

One can easily prove that $f_1^\star(\cdot)$ and $f_2^\star(\cdot)$ satisfy the conditions imposed above on $f_1(\cdot)$ and $f_2(\cdot)$. Therefore, for any $(y_0, z_0, z(\cdot)) \in \mathcal{Z}$, there exists the unique Caratheodory solution to the Couchy problem

$$\dot{y}(t) = f_1^\star(t, y(t), z(t)) + f_2^\star(t, y(t), z(t)) \dot{z}(t)$$

$$y(t_0) = y_0 \tag{3}$$

on I. That, due to the definition of $f_1^\star(\cdot)$ and $f_2^\star(\cdot)$, implies immediately that $(y(\cdot), z(\cdot)) \in \mathcal{X}$ if and only if $y(\cdot)$ is the solution of (3) for $y_0 = y(t_0)$. Consequently, the system (1), (2) is completely observable. The uniform continuity of the operator Y can be proved by using the above properties of $f_1^\star(\cdot)$ and $f_2^\star(\cdot)$.

4. Uniformly Stable Families

The conditions imposed on the right hand sides of the equations (1), (2) imply that the set $\mathcal{X}$ is uniformly bounded. Thus, we can fix a compactum $Q^\star \subset \mathbb{R}^m$ such that $\dot{z}(t) \in Q^\star$ and $t \in I$ for any $(y(\cdot), z(\cdot)) \in \mathcal{X}$. Without loss of generality, put it to be a parallelepiped: $Q^\star = \{u \in \mathbb{R}^m : q_k^- \leq u^{(k)} \leq q_k^+,\ 1 \leq k \leq m\}$; here $u^{(k)}$ stands for the k–th coordinate of a vector u. Introduce the following observation operator D^δ determined by a positive parametr δ. Let $(t_i)_{i=0}^N$ be the δ–net on I, i.e. $t_{i+1} - t_i = \delta$ for $0 \leq i \leq N-1$, $t_N = \vartheta_0$, and $t_N - t_{N-1} \leq \delta$. Define the $w(\cdot) = w(\cdot|D^\delta, \xi)$ where $\xi = (y_{0\star}, z_{0\star}, \zeta(\cdot))$ by

$$w(t_0) = pr_y(y_{0\star}|X_0) \tag{4}$$

$$w(t) = w(t_i), \quad t_i < t \leq t_{i+1} \tag{5}$$

$$w(t_{i+1}) = w(t_i) + [f_1^\star(t_i, w(t_i), \zeta(t_i)) + f_2^\star(t_i, w(t_i), \zeta(t_i))u(t_i)]\,\delta \tag{6}$$

$$u^{(k)}(t_i) = (q_k^+ - q_k^- \left[1 + \mathrm{sign}(p^k(t_i) - \zeta^{(k)}(t_i))\right] /2, \quad 1 \leq k \leq m \tag{7}$$

$$p(t_0) = pr_z(z_{0\star}|X_0) \tag{8}$$

$$p(t_{i+1}) = p(t_i) + u(t_i)\delta \tag{9}$$

the right hand sides (4) and (8) are projections of $y_{\star 0}$ and $z_{\star 0}$ on the sets $X_{0y} = \{y : (y,z) \in X_0\}$ and $X_{0z} = \{z : (y,z) \in X_0\}$. It is easely seen that the observation operator D^δ is dynamical. According to the terminology of the theory of closed–loop game control (Krasovskii and Subbotin, 1988), equations (5) and (9) provide a discrete–time *guided model*, and (7) describes a closed–loop *control law* for the model; the particular form of (7) is a modified *extremal shift* control law.

Theorem 1. *Let* $c > 0$, $\delta(h) \leq ch$, *and* $D_h = D^{\delta(h)}$ $(h > 0)$. *Then*

(i) *for every* $h_\star > 0$ *there exists a* $K > 0$ *such that* $\nu(h, D_h) \leq Kh$ *for all* $h \leq h_\star$,

(ii) *the family* (D_h) *of dynamical observation operators is uniformly stable.*

Statement (ii) follows directly from (i). The proof of (i) has two parts. First, the estimation $\|p(\cdot) - z(\cdot)\| \leq c_1 h$ is shown; here c_1 is a constant, and $p(\cdot)$ is the function (7), (8) extended to I by $p(t) = p(t_i)$, $t_i < t \leq t_{i+1}$. This is the basis to prove the bound $\|w(\cdot) - y(\cdot)\| \leq Kh\ (h \leq h_\star)$ that, due to the arbitrarity of a motion $(y(\cdot), z(\cdot)) \in \mathcal{X}$ and an input ξ h–accurate for $y(\cdot), z(\cdot))$, yelds (i).

Remark. In Theorem 1, a constant K is expressed for any $h_\star > 0$ explicitly via a growth and a Lipschitz constants of $f_1^\star(\cdot)$ and $f_2^\star(\cdot)$.

Considering D^δ as a finite–step real–time numerical algorithm, note that it depends on the parameters q_k^+ and q_k^- (see (7)) of the set $Q^\star$; the latter must consequently be found *apriori*. Let us give a modified operator Δ^δ that does not use the set $Q^\star$ and does not imply any kind of *apriori* analysis of the set $\mathcal{X}$. Define the $w(\cdot) = w(\cdot|\Delta^\delta, \xi)$ by (4) through (9) with replacing q_k^+ and q_k^- in (7) respectively, by $q_{k,i}^+$ and $q_{k,i}^-$ determined as follows: $Q_{k,i}^\star = \{u \in \mathbb{R}^m : q_{k,i}^- \leq u^{(k)} \leq q_{k,i}^+,\ 1 \leq k \leq m\}$ is the minimal m–dimensional parallelepiped containing the a_0–neighborhood of the set $\{g_1(t_i, w(t_i), \zeta(t_i)) + g_2(t_i, w(t_i), \zeta(t_i))v : v \in Q\}$; here a_0 is a fixed parameter.

Theorem 2. *Theorem 1 remains true after replacing* $D^{\delta(h)}$ *by* $\Delta^{\delta(h)}$.

5. Order Optimality

Introduce the modulo of continuity of the reconstruction operator Y

$$\Omega(h) = \sup \{\|Y\eta_1 - Y\eta_2\| : \eta_1 = (y_{01}, z_{01}, z_1(\cdot)),\ \eta_2 = (y_{02}, z_{02}, z_2(\cdot)) \in \mathcal{Z},$$
$$|y_{01} - y_{02}| \le s_y h,\ |z_{01} - z_{02}| \le s_z h,\ \|z_1(\cdot) - z_2(\cdot)\| \le h\}$$

The order optimality analysis is based on the following lower bound for the unimprovable observation accuracy $\nu(h)$ (Ivanov *et al.*, 1978).

Lemma. $\nu_o(h) \ge \Omega(2h)/2$.

Now we give a condition implying an explicit lower bound for $\Omega(h)$. It is called that the equation (1) is *nondegenerate*, if there exist a $(y(\cdot), z(\cdot)) \in \mathcal{X}$, a $t \in I$, and $v_1, v_2 \in Q$ such that $f_2(t, y(t), z(t))v_1 \ne f_2(t, y(t), z(t))v_2$.

Lemma. *If the equation* (1) *is nondegenerate, then there exist* $c_0 > 0$ *and* $h_0 > 0$ *such that* $\Omega(h) \ge c_0 h$ *for all* $h \ge h_0$.

Remark. Constants c_0 and h_0 can be written out explicitly. Taking into account Theorems 1 and 2, and Lemmas, the following theorem can be formulated.

Theorem 3. *Let the equation* (1) *be nondegenerate and the conditions of Theorem 1 or Theorem 2 be fulfilled. Then the family* (D_h) *is order optimal.*

6. Conclusions

For a dynamical system affine in controls, the problem to construct a uniformly stable family of dynamical observation operators is considered. Along with standard growth and Lipschitz conditions for the right hand side of the motion equations, two extra ones concerned with special types of nondegenerancy of the equations (2) and (1) for the observed and unobserved state coordinates are formulated. In case the equation (2) is nondegenerate, two desired families of operators are provided. If in addition the equation (1) is nondegenerate, the families are order optimal.

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