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ON THE NUMBER OF ACTUATORS IN PARABOLIC SYSTEMS

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Given a parabolic system defined on a bounded domain Ω and excited by a finite number of actuators, the purpose of this paper is to show that one actuator can achieve the controllability of the system. The method is based on a perturbation of the boundary of Ω.

1. Introduction

Real systems are very often excited by a finite number of actuators. The actuators may be of zone, pointwise or boundary type. If we consider the problem of steering the system to desired given states (controllability problem), it can be shown that for zone actuators, their number and structures can play an important role to achieve controllability (El Jai and Berrahmoune, 1983a; 1983b; 1984; El Jai and Pritchard, 1987; 1988; Sakawa, 1974; Triggiani, 1975; 1976). This result was extended for the pointwise and boundary cases by El Jai and Berrahmoune (1983a; 1983b; 1984), El Jai and Pritchard (1987; 1988). In these cases it was shown that the number of actuators must be greater than the supremum of the multiplicity of the eigenvalues of the associated eigenvalue problem.

In this paper we shall point out the fact that there is a link between the geometry of the domain Ω and the multiplicity of the eigenvalues. Then we focussed on a result which shows that one actuator may be sufficient to achieve controllability if the domain Ω is replaced by Ω^* such that:

$$d(\Omega, \Omega^*) \leq \delta$$

where d is an appropriate distance and δ–a sufficiently small number.

2. Rank Condition

Let Ω be an open bounded subset of $\mathbb{R}^n$ with smooth boundary $\partial\Omega$ and consider the following class of linear parabolic systems described by

$$(S_p) \quad \begin{cases} \dfrac{\partial y}{\partial t} - \Delta y = \sum\limits_{i=1}^{p} g_i(x) u_i(t) & \Omega \times]0, T[\\ y(x, 0) = 0 & \Omega \\ y = 0 & \partial\Omega \times]0, T[\end{cases}$$

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with

$$g_i \in L^2(\Omega_i),\ \Omega_i \subset \Omega \quad \text{and} \quad \Omega_i \cap \Omega_j = \emptyset \quad \text{if } i \neq j, \quad \text{int}(\Omega_i) \neq \emptyset \tag{1}$$

and

$$u_i \in \mathcal{U} = L^2(0,T), \qquad i = 1, 2, ..., p \tag{2}$$

The system (S_p) is assumed to be excited by p strategic zone actuators. We recall that a zone type actuator is defined by a couple (Ω_i, g_i) where $\Omega_i \subset \Omega$ is the support of the actuator and $g_i \in L^2(\Omega_i)$ is the spatial distribution of the action on Ω_i. A strategic actuator is such that the excited system is weakly controllable. For more details of these concepts see (El Jai and Pritchard, 1987; 1988).

Let (φ_{nj}) be the sequence of eigenfunctions of the operator $-\Delta$ with homogeneous Dirichlet conditions associated to the eigenvalues λ_n, and r_n the multiplicity of λ_n.

$$(P) \quad \begin{cases} -\Delta\varphi_{nj} = \lambda_n \varphi_{nj} & \Omega \\ \varphi_{nj} = 0 & \partial\Omega \\ \|\varphi_{nj}\|^2 = 1 & \forall\ n,\ j \end{cases} \tag{3}$$

then we have the following characterization:

Proposition 1. *The sequence of actuators* $(\Omega_i, g_i)_{1 \leq i \leq p}$ *is strategic for the system* (S_p) *if and only if*

(i) $$p \geq \sup_n r_n \tag{4}$$

(ii) $$\text{rank}\,(G_n) = r_n \quad \forall n \in \mathbb{N} \tag{5}$$

where G_n *is the matrix of order* $(p \times r_n)$ *defined by*

$$(G_n)_{i,j} = \langle g_i, \varphi_{nj} \rangle_{L^2(\Omega_i)}; \quad i = 1, 2, ..., p \quad \textit{and} \quad j = 1, 2, ..., r_n \tag{6}$$

This result is proved in (Curtain and Pritchard, 1978; El Jai and Pritchard, 1987; 1988). In the case of pointwise and boundary controls, the same characterization is valid but one needs the controls to be more regular to achieve controllability in $L^2(\Omega)$ (El Jai and Berrahmoune, 1983a; 1983b; 1984).

Remark 1. It is well known that for every domain Ω such that the eigenvalues of $-\Delta_\Omega$ are simple we can achieve controllability by **one** actuator.

Remark 2. Let us consider the system (S_p) in a bidimensional case with $\Omega =]\,0, a[\ \times\ ]\,0, b[$. The eigenvalues of the problem (P) are given by

$$\lambda_{mn} = -\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)\pi^2 \tag{7}$$

We notice that $\lambda_{mn} \longrightarrow \infty$ when $m, n \longrightarrow \infty$. From algebra theory we know that the multiplicity s of the solutions of the equation

$$X^2 + Y^2 = A^2$$

is such that $s \longrightarrow \infty$ when $A \longrightarrow \infty$.
So from (7) if $a^2/b^2 \notin \mathbb{Q}$, then obviously the multiplicity of λ_{mn} is 1, but if $a^2/b^2 \in \mathbb{Q}$, then the multiplicity r_n of λ_{mn} is such that $r_n \longrightarrow \infty$ when $m, n \longrightarrow \infty$.
From this remark, the previous proposition shows that the actuators $(\Omega_i, g_i)_{1 \leq i \leq p}$ may be non strategic ((S_p) weakly controllable) if p is finite for a certain domain Ω.

Example. We consider again the system (S_p) with $\Omega =]0, a[\times]0, a[$. By the previous remark, the system is not weakly controllable for any integer p. But if we consider a "small" perturbation of the geometry of Ω, with $\Omega_\varepsilon =]0, a[\times]0, a+\varepsilon[$ and $\varepsilon = \dfrac{a}{n + \sqrt{2}}$ with n sufficiently large, then we have

$$\frac{a^2}{(a+\varepsilon)^2} = \frac{(n+\sqrt{2})^2}{(n+\sqrt{2}+1)^2} \notin \mathbb{Q}$$

and on Ω_ε, **one** $(p = 1)$ actuator may be strategic.
The aim of the following sections is to generalize the result of the previous example to the case of more general but regular domains.

3. Actuators Number and Domain Deformation

In this section we shall see that by a "small" change of the geometry of Ω it is possible to ensure the weak controllability of the system (S_p) with only **one** actuator.
Let Ω be an open bounded simply connected set in $\mathbb{R}^n$ with smooth boundary $\partial\Omega$.
Let $\psi \in C^3(\mathbb{R}^n)$, where $C^3(\mathbb{R}^n) = \{\psi : \mathbb{R}^n \longrightarrow \mathbb{R}^n / D^\alpha \psi$ is continuous and bounded on $\mathbb{R}^n$ for $|\alpha| \leq 3\}$.
$C^3(\mathbb{R}^n)$ is a Banach space normed by

$$||\psi||_{C^3} = \sup_x \max \left\{ ||\psi^{(i)}(x)||, \ 0 \leq i \leq 3 \right\} \tag{8}$$

where $\psi^{(i)}$ is the i-th derivative of ψ.
Let us consider the problem (S_p) with the Laplace operator Δ_Ω depending on Ω and let λ_0 be the first isolated eigenvalue of $-\Delta_\Omega$ with multiplicity equal to $m > 1$. It is always possible to consider an interval U such that $\overline{U} \cap \sigma(-\Delta_\Omega) = \{\lambda_0\}$ where $\sigma(-\Delta_\Omega)$ is the spectrum of $-\Delta_\Omega$.

3.1. Splitting of the Eigenvalue λ_0

3.1.1. Definition of the Perturbed Domain Ω_ε

In this section we show that it is possible to define, for $\varepsilon > 0$ sufficiently small, a new domain $\Omega_\varepsilon = \Omega + \varepsilon\psi(\Omega)$ such that

$$\overline{U} \cap \sigma(-\Delta_{\Omega_\varepsilon}) = \left\{ \lambda_1^{\Omega_\varepsilon}, \ldots, \lambda_i^{\Omega_\varepsilon} \right\}$$

where $\lambda_j^{\Omega_\varepsilon}$ is an eigenvalue of $-\Delta_{\Omega_\varepsilon}$ with multiplicity equal to $m_j < m$.
Consider the map $F_\varepsilon : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined by

$$F_\varepsilon = I + \varepsilon\psi \tag{9}$$

Then for ψ with compact support and ε small enough

$$|\varepsilon| < \varepsilon_0 = \frac{1}{\sup_x \|\psi'(x)\|}$$

F_ε is a diffeomorphism of class C^3 and then $\Omega_\varepsilon = F_\varepsilon(\Omega)$ is a domain obtained by a perturbation of Ω.

3.1.2. Definition of the Operator $T(\varepsilon)$

Let us consider the map:
$\gamma : u \in L^2(\Omega_\varepsilon) \longrightarrow \gamma(u) = \widehat{u} \in L^2(\Omega)$ with

$$\widehat{u}(x) = \sqrt{J(x)}\,u(F_\varepsilon(x)) \tag{10}$$

where J denotes the determinant of the Jacobian matrix of F_ε.
γ is a topological and algebraic isometry. Denote by $T(\varepsilon)$ a selfadjoint operator in $L^2(\Omega)$ such that

$$T(\varepsilon)\gamma = \gamma(-\Delta_{\Omega_\varepsilon}) \tag{11}$$

hence $(-\Delta_{\Omega_\varepsilon})$ and $T(\varepsilon) = \gamma(-\Delta_{\Omega_\varepsilon})\gamma^{-1}$ are spectrally equivalent, i.e. they have the same eigenvalues with the same multiplicities for $|\varepsilon|$ sufficiently small.

Calculation of $T(\varepsilon)$

We calcultate explicitely $T(\varepsilon)$ defined in (11).
The scalar products and the corresponding norms in $L^2(\Omega_\varepsilon)$ and $H_\varepsilon = H_0^1(\Omega_\varepsilon)$ will be denoted by

$$(u,v)_{L^2(\Omega_\varepsilon)} = \int_{\Omega_\varepsilon} u(x)v(x)\mathrm{d}x, \qquad |u|^2 = \int_{\Omega_\varepsilon} u^2(x)\mathrm{d}x \tag{12}$$

$$((u,v))_{H_\varepsilon} = \int_{\Omega_\varepsilon} \sum_k \frac{\partial u}{\partial x_k}\frac{\partial v}{\partial x_k}\mathrm{d}x, \qquad \|u\|^2 = \int_{\Omega_\varepsilon} \sum_k \left[\frac{\partial u}{\partial x_k}\right]^2 \mathrm{d}x \tag{13}$$

We have

$$((u,v))_{H_\varepsilon} = (-\Delta_{\Omega_\varepsilon} u, v)_{L^2(\Omega_\varepsilon)} \tag{14}$$

Since

$$(\gamma u, \gamma v)_{L^2(\Omega)} = (u,v)_{L^2(\Omega_\varepsilon)} \tag{15}$$

and from (11) and (14) we deduce

$$((u,v))_{H_\varepsilon} = (T(\varepsilon)\widehat{u}, \widehat{v})_{L^2(\Omega)} \tag{16}$$

with

$$\widehat{u} = \sqrt{J} u \circ F_\varepsilon \quad \text{and} \quad \widehat{v} = \sqrt{J} v \circ F_\varepsilon \tag{17}$$

$$F_\varepsilon = I + \varepsilon\psi \quad \text{and} \quad F_\varepsilon^{-1} = I + \varepsilon\chi \tag{18}$$

and then

$$((u,v))_{H_\varepsilon} = \int_\Omega \sum_k \sum_{i,j} \left(\sqrt{J}\frac{\partial}{\partial y_i}\frac{\widehat{u}}{\sqrt{J}}\right)\left(\delta_{ik} + \varepsilon\frac{\partial \chi_i}{\partial x_k}\right)\left(\sqrt{J}\frac{\partial}{\partial y_j}\frac{\widehat{v}}{\sqrt{J}}\right)\left(\delta_{jk} + \varepsilon\frac{\partial \chi_j}{\partial x_k}\right)\mathrm{d}y \tag{19}$$

Consider the functions $S_{ij}(\varepsilon)$, $K_i(\varepsilon)$ and the operator $D_i(\varepsilon)$ defined by

$$\begin{cases} S_{ij}(\varepsilon) = \sum_k \left[\delta_{ik} + \varepsilon\dfrac{\partial \chi_i}{\partial x_k}\right]\left[\delta_{jk} + \varepsilon\dfrac{\partial \chi_j}{\partial x_k}\right] \\ K_i(\varepsilon) = -\dfrac{1}{2}\dfrac{\partial \log J}{\partial y_i} \\ D_i(\varepsilon) = \dfrac{\partial}{\partial y_i} + K_i(\varepsilon) \end{cases} \tag{20}$$

Using (16) and (19) we have

$$(T(\varepsilon)\widehat{u},\widehat{v})_{L^2(\Omega)} = \int_\Omega \sum_{i,j} S_{ij}(\varepsilon)\left(\sqrt{J}\frac{\partial}{\partial y_i}\frac{\widehat{u}}{\sqrt{J}}\right)\left(\sqrt{J}\frac{\partial}{\partial y_j}\frac{\widehat{v}}{\sqrt{J}}\right)$$

and, on the other hand, we have

$$\sqrt{J}\frac{\partial}{\partial y_i}\frac{\widehat{u}}{\sqrt{J}} = \sqrt{J}\left(\frac{\partial \widehat{u}}{\partial y_i}\sqrt{J} - \widehat{u}\frac{\partial\sqrt{J}}{\partial y_i}\right)\frac{1}{J} = \frac{\partial \widehat{u}}{\partial y_i} - \widehat{u}\frac{\frac{\partial\sqrt{J}}{\partial y_i}}{\sqrt{J}}$$

$$= \frac{\partial \widehat{u}}{\partial y_i} - \widehat{u}\frac{\partial \log\sqrt{J}}{\partial y_i} = \frac{\partial \widehat{u}}{\partial y_i} - \frac{1}{2}\widehat{u}\frac{\partial \log J}{\partial y_i} = \left(\frac{\partial}{\partial y_i} - \frac{1}{2}\frac{\partial \log J}{\partial y_i}\right)\widehat{u} = D_i(\varepsilon)\widehat{u}$$

which implies

$$(T(\varepsilon)\widehat{u},\widehat{v})_{L^2(\Omega)} = \int_\Omega \sum_{i,j} S_{ij}(\varepsilon)D_i(\varepsilon)\widehat{u}D_j(\varepsilon)\widehat{v} = \left(\sum_{i,j} S_{ij}(\varepsilon)D_i(\varepsilon)\widehat{u}, D_j(\varepsilon)\widehat{v}\right)_{L^2(\Omega)}$$

This result can also be written in the form

$$(T(\varepsilon)\widehat{u},\widehat{v})_{L^2(\Omega)} = \left(\sum_{i,j} S_{ij}(\varepsilon)D_j^*(\varepsilon)D_i(\varepsilon)\widehat{u}, \widehat{v}\right)_{L^2(\Omega)} \tag{21}$$

where D_j^* denotes the adjoint operator of D_j with respect to the inner product in $L^2(\Omega)$. Finally, we obtain

$$T(\varepsilon)\widehat{u} = \sum_{i,j} S_{ij}(\varepsilon)D_j^*(\varepsilon)D_i(\varepsilon)\widehat{u} \tag{22}$$

Analyticity of $T(\varepsilon)$

We shall study the analyticity of the map $\varepsilon \longrightarrow T(\varepsilon)$. We can proove easily that the functions $S_{ij}(\varepsilon)$ and $K_i(\varepsilon)$ are analytical with respect to ε for $|\varepsilon| < \varepsilon_0$. Then we can write

$$S_{ij}(\varepsilon) = \sum_k S_{ij}^{(k)} \varepsilon^k \tag{23}$$

$$K_i(\varepsilon) = \sum_k K_i^{(k)} \varepsilon^k \tag{24}$$

with

$$\begin{cases} S_{ij}^{(0)} = \delta_{ij} & K_i^{(0)} = 0 \\ S_{ij}^{(1)} = -\dfrac{\partial \psi_i}{\partial y_j} - \dfrac{\partial \psi_j}{\partial y_i} & K_i^{(1)} = -\dfrac{1}{2}\dfrac{\partial}{\partial y_i}(\mathrm{div}\psi) \end{cases} \tag{25}$$

Using (22)–(23) and (24) we obtain

$$T(\varepsilon) = \sum_k T_k \varepsilon^k$$

with

$$T_0 \widehat{u} = (-\Delta_\Omega)\widehat{u} \tag{26}$$

and

$$T_1 \widehat{u} = \frac{1}{2}\Delta_\Omega(\mathrm{div}\psi)\widehat{u} + \sum_{i,j} \frac{\partial}{\partial y_j}\left[\left(\frac{\partial \psi_j}{\partial y_i} + \frac{\partial \psi_i}{\partial y_j}\right)\frac{\partial \widehat{u}}{\partial y_i}\right] \tag{27}$$

Now the operator $T(\varepsilon)$ given by (22) satisfies the hypothesis of a classical theorem of Rellich and Nagy (Nagy, 1946) given by

Theorem 1. *Let T_i be a sequence of self-adjoint operators on the Hibert space H, all having the same domain D. Let λ_0 be an isolated eigenvalue of T_0 with multiplicity $m > 1$ and let U be an interval with $\overline{U} \cap \sigma(T_0) = \{\lambda_0\}$. If we assume that there exist two positive numbers M and r such that*

$$||T_i f|| \leq \frac{M}{r^{i-1}}\left[||f||_H + ||T_0 f||_H\right] \qquad \forall f \in \mathcal{D} \text{ and } i = 1, 2, \cdots \tag{28}$$

which ensures the existence of the sum $T(\varepsilon)$ of the series

$$T(\varepsilon) = T_0 + \varepsilon T_1 + \varepsilon^2 T_2 + \varepsilon^3 T_3 + \cdots, \tag{29}$$

then for $|\varepsilon|$ sufficiently small there exist m real values

$$\lambda_j(\varepsilon) = \lambda_j^{(0)} + \varepsilon \lambda_j^{(1)} + \cdots, \qquad 1 \leq j \leq m \tag{30}$$

with $\lambda_j^{(0)} = \lambda_0$ *and* m *elements of* H

$$\omega_j(\varepsilon) = \omega_j^{(0)} + \varepsilon\omega_j^{(1)} + \cdots, \qquad 1 \le j \le m \tag{31}$$

where $\left(\omega_1^{(0)}, \omega_2^{(0)}, \cdots, \omega_m^{(0)}\right)$ *is an orthonormal basis of the eigenspace associated to* λ_0 *such that*

$$\overline{U} \cap \sigma(T(\varepsilon)) = \{\lambda_j(\varepsilon),\ j = 1, 2, \cdots, m\} \tag{32}$$

Remark 3. Since $T(\varepsilon)$ satisfies the hypothesis of the previous theorem we obtain the following formula

$$\langle \omega_j(\varepsilon), \omega_k(\varepsilon) \rangle_H = \delta_{jk} \tag{33}$$

$$T(\varepsilon)\omega_j(\varepsilon) = \lambda_j(\varepsilon)\omega_j(\varepsilon) \tag{34}$$

and

$$\langle T(\varepsilon)\omega_j(\varepsilon), \omega_k(\varepsilon) \rangle_H = \delta_{jk}\lambda_j(\varepsilon) \tag{35}$$

where $\langle .,. \rangle$ is the scalar product in $H = H_0^1(\Omega)$.

Spectrum of $T(\varepsilon)$

Consider the orthonormal system $\{\omega_1, \omega_2, ..., \omega_m\}$ of the corresponding eigenspace of λ_0 and let $A = (\mu_{ij})_{1 \le i,j \le m}$ be defined by:

$$\mu_{ij} = \langle -T_1\omega_i, \omega_j \rangle_H \tag{36}$$

With the previous theorem, we have the following relations:

1. Using formulae (31) and (33) for $j = k$, we obtain

$$\sum_{p+q=r} (\omega_j^{(p)}, \omega_j^{(q)})_H = 0 \qquad \text{for} \quad r \ge 1$$

and for $j \ne k$ we have

$$\sum_{p+q=r} (\omega_j^{(p)}, \omega_k^{(q)})_H = 0 \qquad \text{for} \quad r \ge 1$$

2. From (34) we obtain

$$\sum_{p+q=r} T_p\omega_j^{(q)} = \sum_{p+q=r} \lambda_j^{(p)}\omega_j^{(q)} \qquad \text{for} \quad r \ge 0$$

3. From (35) we have

$$\sum_{p+q+s=r} \left(T_s\omega_j^{(p)}, \omega_k^{(q)}\right)_H = \delta_{jk}\lambda_j^{(r)} \quad \text{for} \quad r \ge 0$$

and for $r = 1$ this relation becomes

$$\sum_{p+q+s=1} \left(T_s\omega_j^{(p)}, \omega_k^{(q)}\right)_H = \delta_{jk}\lambda_j^{(1)}$$

$$\iff \left(T_1\omega_j^{(0)}, \omega_k^{(0)}\right)_H + \left(T_0\omega_j^{(0)}, \omega_k^{(1)}\right)_H + \left(T_0\omega_j^{(1)}, \omega_k^{(0)}\right)_H = \delta_{jk}\lambda_j^{(1)}$$

$$\iff \left(T_1\omega_j^{(0)}, \omega_k^{(0)}\right)_H + \lambda_0\left(\left(\omega_j^{(0)}, \omega_k^{(1)}\right)_H + \left(\omega_j^{(1)}, \omega_k^{(0)}\right)_H\right) = \delta_{jk}\lambda_j^{(1)}$$

Finally with this formula we obtain

$$\mu_{ij} = (-T_1\omega_i, \omega_j) = \left(-T_1\omega_i^{(0)}, \omega_j^{(0)}\right) = -\delta_{ij}\lambda_i^{(1)}$$

It is then clear that the eigenvalues of A are equal to $(-\lambda_j^{(1)})$ with $1 \le j \le m$. Then we have the following result (Micheletti, 1976):

Proposition 2. *If the eigenvalues of $T(\varepsilon)$ are all equal, then the eigenvalues of A are also all equal. Consequently, if the eigenvalues of A are of multiplicity equal to 1, then the $\lambda_j(\varepsilon)$ are also of multiplicity equal to 1.*

Proof. If we assume that all the eigenvalues $(\lambda_j(\varepsilon))_{1\le j\le m}$ of $T(\varepsilon)$ (then of $(-\Delta_{\Omega_\varepsilon})$), defined in (30) are equal, then $\left(\lambda_j^{(r)}\right)_{1\le j\le m}$ are also equal for r $\ge$ 1. Particularly for r = 1, all the eigenvalues of the matrix A, i.e. $\left(-\lambda_j^{(1)}\right)_{1\le j\le m}$, are equal. ■

3.1.3. Theorem of Splitting of λ_0

First, let us calculate the matrix elements μ_{ij} and let $\dfrac{\partial\omega_i}{\partial\nu}$ be the normal derivative of ω_i

$$\frac{\partial\omega_i}{\partial\nu} = \operatorname{grad}\omega_i.\nu = \sum_k \frac{\partial\omega_i}{\partial y_k}\nu_k$$

and

$$\Delta(\omega_i\omega_j) = \omega_i\Delta\omega_j + \omega_j\Delta\omega_i + 2\sum_k \frac{\partial\omega_i}{\partial y_k}\frac{\partial\omega_j}{\partial y_k} = -2\lambda_0\omega_i\omega_j + \sum_k \frac{\partial\omega_i}{\partial y_k}\frac{\partial\omega_j}{\partial y_k}$$

$$\mu_{ij} = (-T_1\omega_i, \omega_j)_{L^2(\Omega)}$$

$$= -\int_\Omega \sum_{k,l} \frac{\partial}{\partial y_l}\left[\left(\frac{\partial\psi_k}{\partial y_l} + \frac{\partial\psi_l}{\partial y_k}\right)\frac{\partial\omega_i}{\partial y_k}\right]\omega_j \mathrm{d}y - \frac{1}{2}\int_\Omega \Delta_\Omega(\mathrm{div}\psi)\omega_i\omega_j\,\mathrm{d}y$$

$$= \int_\Omega \sum_{k,l} \frac{\partial\psi_k}{\partial y_l}\left(\frac{\partial\omega_i}{\partial y_k}\frac{\partial\omega_j}{\partial y_l} + \frac{\partial\omega_i}{\partial y_l}\frac{\partial\omega_j}{\partial y_k}\right)\mathrm{d}y - \frac{1}{2}\int_\Omega \Delta_\Omega(\mathrm{div}\psi)\omega_i\omega_j\mathrm{d}y$$

$$= \int_{\partial\Omega} \sum_{k,l} \psi_k \nu_l \left(\frac{\partial \omega_i}{\partial y_k} \frac{\partial \omega_j}{\partial y_l} + \frac{\partial \omega_i}{\partial y_l} \frac{\partial \omega_j}{\partial y_k} \right) d\sigma - \int_\Omega \sum_{k,l} \psi_k \left(\frac{\partial^2 \omega_i}{\partial y_l^2} \frac{\partial \omega_j}{\partial y_k} + \frac{\partial^2 \omega_j}{\partial y_l^2} \frac{\partial \omega_i}{\partial y_k} \right)$$

$$+ \psi_k \left(\frac{\partial \omega_i}{\partial y_l} \frac{\partial^2 \omega_j}{\partial y_l \partial y_k} + \frac{\partial \omega_j}{\partial y_l} \frac{\partial^2 \omega_i}{\partial y_l \partial y_k} \right) dy - \frac{1}{2} \int_\Omega \Delta_\Omega (\mathrm{div}\psi) \omega_i \omega_j dy$$

$$= \int_{\partial\Omega} \sum_{k,l} \psi_k \nu_l \left(\frac{\partial \omega_i}{\partial y_k} \frac{\partial \omega_j}{\partial y_l} + \frac{\partial \omega_i}{\partial y_l} \frac{\partial \omega_j}{\partial y_k} \right) d\sigma$$

$$+ \lambda_0 \int_\Omega \sum_k \psi_k \left(\omega_i \frac{\partial \omega_j}{\partial y_k} + \omega_j \frac{\partial \omega_i}{\partial y_k} \right) - \int_{\partial\Omega} \sum_{k,l} \psi_k \nu_k \left(\frac{\partial \omega_i}{\partial y_l} \frac{\partial \omega_j}{\partial y_l} \right) d\sigma$$

$$+ \int_\Omega \sum_{k,l} \frac{\partial \psi_k}{\partial y_k} \left(\frac{\partial \omega_i}{\partial y_l} \frac{\partial \omega_j}{\partial y_l} \right) dy - \frac{1}{2} \int_\Omega \Delta_\Omega (\mathrm{div}\psi) \omega_i \omega_j dy$$

Hence we obtain

$$\mu_{ij} = \int_{\partial\Omega} \left\{ \sum_k \psi_k \left[\frac{\partial \omega_i}{\partial \nu} \frac{\partial \omega_j}{\partial y_k} + \frac{\partial \omega_i}{\partial y_k} \frac{\partial \omega_j}{\partial \nu} \right] - \sum_l \left[\sum_k \psi_k \nu_k \right] \frac{\partial \omega_i}{\partial y_l} \frac{\partial \omega_j}{\partial y_l} \right\} d\sigma$$

$$+ \int_\Omega (\mathrm{div}\psi) \sum_l \frac{\partial \omega_i}{\partial y_l} \frac{\partial \omega_j}{\partial y_l} dy + \lambda_0 \int_\Omega \sum_k \left[\omega_i \frac{\partial \omega_j}{\partial y_k} + \omega_j \frac{\partial \omega_i}{\partial y_k} \right] dy$$

$$- \frac{1}{2} \int_\Omega \Delta_\Omega (\mathrm{div}\psi) \omega_i \omega_j dy$$

If ψ_ν is the component of ψ with respect to the vector ν, we have:

$$\psi_\nu = \vec{\psi} \cdot \vec{\nu} = \sum_k \psi_k \nu_k$$

and then

$$\mu_{ij} = \int_{\partial\Omega} \left\{ \sum_k \psi_k \left[\frac{\partial \omega_i}{\partial \nu} \frac{\partial \omega_j}{\partial y_k} + \frac{\partial \omega_i}{\partial y_k} \frac{\partial \omega_j}{\partial \nu} \right] - \sum_l \psi_\nu \frac{\partial \omega_i}{\partial y_l} \frac{\partial \omega_j}{\partial y_l} \right\} d\sigma \tag{37}$$

We can observe that the term

$$\sum_k \psi_k \frac{\partial \omega_i}{\partial y_k}$$

is proportional to the derivative of ω_i with respect to the vector ψ. Thus, in the case where ψ is tangent to $\partial\Omega$ and with the condition $\omega_i|_{\partial\Omega} = 0$ we obtain $\mu_{ij} = 0$ and in the other case we can consider a function $\alpha(\cdot)$ defined by

$$\begin{cases} \psi_k(\xi) = \alpha(\xi) \nu_k \,, & \xi \in \partial\Omega \\ \psi_\nu(\xi) = \alpha(\xi) \end{cases} \tag{38}$$

then we can reduce expression (37) to

$$\mu_{ij} = \int_{\partial\Omega} \alpha(\xi) \frac{\partial \omega_i}{\partial \nu} \frac{\partial \omega_j}{\partial \nu} \mathrm{d}\xi \tag{39}$$

Remark 4. If λ_0 is an eigenvalue of multiplicity $m > 1$ of $T_0 = -\Delta_\Omega$ and $\{\omega_1, \omega_2, ..., \omega_m\}$ an orthonormal system of the associated eigenspace to λ_0 and if all the eigenvalues of $T(\varepsilon)$, $\lambda_j(\varepsilon) = \sum_k \lambda_j^{(k)} \varepsilon^k$ with $\lambda_j(0) = \lambda_0$ are equal for every deformation of the domain Ω, then for every $\psi \in \mathcal{C}^3(\mathbb{R}^n)$ we have the matrix $A = (\mu_{ij})_{1 \leq i,j \leq n} = \rho I$.
If $\alpha(\xi)$ denotes the component of $\psi(\xi)$ with respect to the normal vector ν to $\partial\Omega$, there exists at least one $i \neq j$ such that $\mu_{ij} \neq 0$ and then $A \neq \rho I$. The reason for this comes from the fact that $\dfrac{\partial \omega_i}{\partial \nu} \dfrac{\partial \omega_j}{\partial \nu}$ is a continuous nonzero function on $\partial\Omega$.
We have the following theorem of splitting of the eigenvalues:

Theorem 2. *Let*

- Ω *be a bounded domain of class* C^3 *in* $\mathbb{R}^n$,
- λ_0 *an eigenvalue of the operator* $-\Delta_\Omega$ *of multiplicity* $m > 1$,

and

- U *be an open bounded interval such that* $\overline{U} \cap \sigma(-\Delta_\Omega) = \{\lambda_0\}$,

then there exists, a function $\psi \in \mathcal{C}^3(\mathbb{R}^n)$ *and* $\varepsilon_0 > 0$ *such that, setting* $F_\varepsilon = I + \varepsilon\psi$ *and* $\Omega_\varepsilon = F_\varepsilon(\Omega)$ *for* $|\varepsilon| < \varepsilon_0$, *the set* $\overline{U} \cap \sigma(-\Delta_{\Omega_\varepsilon})$ *is given by a finite number of distinct eigenvalues* $\{\lambda_1^\varepsilon, \lambda_2^\varepsilon, ..., \lambda_i^\varepsilon\}$ *where* $i > 1$ *and each of these eigenvalues has multiplicity* $r_j \geq 1$ *and* $\sum_{j=1}^{i} r_j = m$

3.2. Extension to All the Eigenvalues of $-\Delta_\Omega$

Let us consider now the space $\mathcal{F}^3$ defined by

$$\mathcal{F}^3 = \left\{ \begin{array}{l} \text{diffeomorphism} \quad f/f = I + \psi \quad \text{with} \quad \psi \in \mathcal{C}^3(\mathbb{R}^m) \\ \text{and} \\ \|\psi^{(i)}(x)\| \longrightarrow 0 \ \text{ when } \ \|x\| \longrightarrow \infty, \quad i = 1,2,3 \end{array} \right\} \tag{40}$$

where $\psi^{(i)}$ denotes the i–th derivative of ψ.
Then we have the following result (Courant and Hilbert, 1962).

Definition 1. Consider the decomposition in the group $(\mathcal{F}^3, \circ)$ of F and F^{-1}

$$F = (I + f_n) \circ \cdots \circ (I + f_1) \qquad \text{where} \quad n \in \mathbb{N} \ \text{is given} \tag{41}$$

$$F^{-1} = (I + g_n) \circ \cdots \circ (I + g_1) \quad \text{where} \quad n \in \mathbb{N} \ \text{is given} \tag{42}$$

and let

$$d(I,F) = \inf_{f_i} \sum_{i=1}^{n} ||f_i||_{C^3} + \inf_{g_j} \sum_{j=1}^{m} ||g_j||_{C^3} \tag{43}$$

$$d(F,G) = d(I, G \circ F^{-1}) \tag{44}$$

then formula (43) defines a metric in $\mathcal{F}^3$ and $(\mathcal{F}^3, d)$ is a complete metric space.

Consider now

$$\mathcal{G} = \{F \in \mathcal{F}^3 / F(\Omega) = \Omega\} \tag{45}$$

and

$$\Lambda = \{\Omega' \subset \mathbb{R}^m / \exists F \in \mathcal{F}^3 \text{ with } \Omega' = F(\Omega)\} \tag{46}$$

(Λ, d) is a complete metric space where d is the Courant metric defined by

$$d(\Omega_1, \Omega_2) = \inf_{G, \widetilde{G} \in \mathcal{G}} d(F_1 \circ G, F_2 \circ \widetilde{G}) \tag{47}$$

where

$$\Omega_1 = F_1(\Omega) \quad \text{and} \quad \Omega_2 = F_2(\Omega) \tag{48}$$

3.2.1. Construction of the Sequence (Ω_k)

Proceeding now as in the last paragraph in order to find a domain Ω^* such that all the eigenvalues of $-\Delta_{\Omega^*}$ are simple, we obtain (Micheletti, 1976).

Lemma 1. *Let Ω be an open bounded subset of $\mathbb{R}^n$ with a smooth boundary $\partial\Omega$ and let $\{\varepsilon_k\}_{k\geq 0}$ be a sequence of positive numbers. Then there exist:*

(i) *a sequence $\{F_k\}_{k\geq 0}$ of diffeomorphisms in $C^3(\mathbb{R}^n)$,*

(ii) *a sequence $\{\Omega_k\}_{k\geq 0}$ of open bounded subsets of $\mathbb{R}^n$ with $\Omega_0 = \Omega$,*

(iii) *a sequence $\{U_k\}_{k\geq 0}$ of open bounded intervals with $\overline{U}_p \cap \overline{U}_q = \emptyset$ if $p \neq q$ such that*

1. $\Omega_k = F_k(\Omega_{k-1})$,
2. $||F_k - I||_{C^3} \leq \varepsilon_k$,
3. $\lambda_i^{\Omega_k} \in \overline{U}_k \sigma(-\Delta_{\Omega_k})$ *for* $i \geq 0$.

Let us consider the sequence $\{\Omega_k\}_{k\geq 0}$ defined in the last lemma and consider now the sequence

$$\mathcal{F}_k = F_k \circ F_{k-1} \circ \cdots \circ F_1$$

We have then

$$\Omega_k = \mathcal{F}_k(\Omega)$$

3.2.2. Convergence of the Sequence (Ω_k)

Lemma 2. *The sequence* $\{\mathcal{F}_k\}_{k\geq 0}$ *converges to* $\mathcal{F}^*$ *in the complete metric space* $\{\mathcal{F}^3, d\}$ *if we take*

$$\varepsilon_k = r^k \quad \text{with} \quad 0 < r < \frac{1}{7}$$

The above lemma is proved in (El Yacoubi, 1990).
We consider now the domain $\Omega^* = \mathcal{F}^*(\Omega)$, then we have

Theorem 3. *If* Ω *is an open bounded domain of* $\mathbb{R}^n$, *then there exists a domain* $\Omega^* \in \Lambda$ *such that all the eigenvalues of the operator* $-\Delta_{\Omega^*}$ *are of multiplicity* $r_n = 1 \; \forall \, n$.

Proof. Since the map

$$\Lambda \longrightarrow \mathbb{R}$$
$$\Omega \longrightarrow \lambda_n^{\Omega}$$

is continuous, we have

$$\Omega_k \stackrel{(\Lambda,d)}{\longrightarrow} \Omega^* \text{ when } k \longrightarrow \infty \;\Rightarrow\; \lambda_i^{\Omega_k} \longrightarrow \lambda_i^{\Omega^*} \text{ when } k \longrightarrow \infty$$

If we assume that there exists $i \neq j$ such that

$$\lambda_i^{\Omega^*} = \lambda_j^{\Omega^*}$$

then

$$\overline{U}_i \cap \overline{U}_j \neq \emptyset$$

and this is not possible by the previous lemma. ■

3.3. Controllability Problem

Let us consider the system (S_p) defined on the new domain Ω^* obtained by a small perturbation of Ω as defined in the previous section.

$$(S_p') \begin{cases} \frac{\partial y}{\partial t} - \Delta y = \sum_{i=1}^{p} g_i(x) u_i(t) & \Omega^* \times]0,T[\\ y(x,0) = 0 & \Omega^* \\ y = 0 & \partial\Omega^* \times]0,T[\end{cases}$$

Then we have the following result (El Yacoubi, 1990):

Proposition 3. *The system* (S_p') *is weakly controllable by* $p = 1$ *actuator* (Ω_0, g_0) *with* $\Omega_0 \subset \Omega^*$ *if and only if*

$$\langle g_0, \varphi_n \rangle_{L^2(\Omega_0)} \neq 0 \;\; \forall n$$

where (φ_n) *are the eigenfunctions associated to the eigenvalues* (λ_n) *with homogeneous Dirichlet boundary conditions on* Ω^* *(solutions of (3) on* Ω^**).*

For the proof we use Proposition 1 and Theorem 3.

Remarks

- If we consider the problem (P) with Neumann boundary condition, then the same result can be achieved.
- For more general elliptic operators than $(-\Delta_\Omega)$ the results can also be extended.
- No numerical results have been developed for this problem but it is clear that for the case of a simple geometry we can decide easily for a modified domain which ensures the result.
- These results are also valid in the case of pointwise actuators or boundary actuators.
- The previous results can be applied to the observability of the system (S_p) augmented with an output

$$z(t) = Cy(.,t) \tag{49}$$

where the operator C designates the sensors structure. As the observability rank condition is of the same type as the controllability one (El Jai and Pritchard, 1987; 1988), the system (S_p) – (49) can be made observable by means of *one* sensor.

4. Conclusion

In this paper we have used theoretical results of domain deformations for the controllability problem of a parabolic system. From these results we obtain a new domain on which the spectral problem (P) gives eigenvalues of multiplicity equal to one. Then, the characterization result of controllability leads to a possible controllability by means of one actuator.

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