Appl. Math. and Comp. Sci., 1996, vol.6, No.2, 245–261

SHAPE OPTIMAL DESIGN FOR A FLUID-HEAT COUPLED SYSTEM

DENISE CHENAIS*, JÉRÔME MONNIER*, **
JEAN-PAUL VILA***

This paper deals with the shape optimal design problem for a fluid-heat coupled system used in the car industry. For modelling, we assume that the flow is stationary, potential and incompressible, and we consider the thermal transfer by convection, diffusion and radiation with multiple reflexions. The whole model is a non-linear integro-differential system of two partial differential equations and one integral equation. These three equations are coupled. We present the mathematical analysis of this model (the existence, uniqueness and regularity of the solution) as well as its numerical analysis. Then we present the shape optimal-design problem: we seek to minimize, with respect to the domain in which the equations are defined, a cost function which depends on the fluid temperature. This control problem is solved by a descent algorithm. We prove that, under some physical assumption, the solution of the system is differentiable with respect to the domain. We introduce the adjoint state equation and we give an expression for the differential of the exact cost function.

1. Introduction

The industrial problem we want to solve is the following: we model the air flow and heat transfer under a car bonnet (see Fig. 1). Then we seek to optimize the hose shape in order to minimize the cost function dependent on the hose temperature.

The paper is organized as follows. In Section 2, we describe the physical model and the boundary conditions. In Section 3, we give a proof of the existence and uniqueness of the solution, and we discuss its regularity. Then, in Section 4, we state our results on the existence and uniqueness of the discrete solution and our results of convergence. We present the shape optimal-design problem in Section 5 and give an expression for the design gradient. Finally, numerical results are presented in Section 6.

* Université de Nice-Sophia Antipolis, CNRS LA168, Parc Valrose, F-06108 Nice Cedex 2, France, e-mail: chenais@math.unice.fr
** C.M.A.F., Av Prof. Gama Pinto 2, 1699 Lisboa codex, Portugal, e-mail: jmonnier@ptmat.lmc.fc.ul.pt
*** INSA, Complexe Scientifique de Rangeuil, F-31077 Toulouse Cedex, France, e-mail: vila@cict.fr

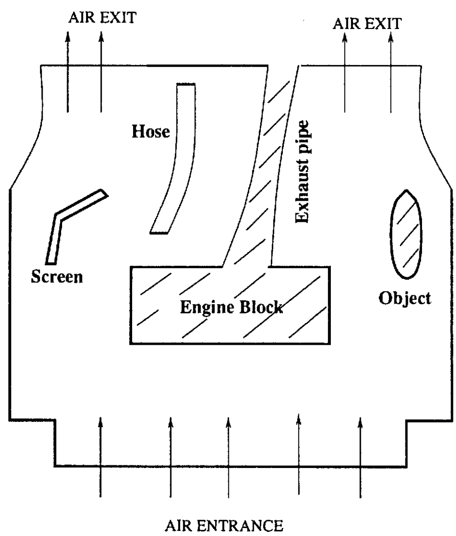

Fig. 1. Engine scheme.

2. Physical Model

We assume that the fluid flow is stationary, incompressible and potential, and we consider the heat transfer by convection, diffusion and radiation. We denote by u the fluid velocity and by θ the temperature. The radiant energy which flows away from a surface per unit area is called the radiosity and is denoted by w.

Let ω be a Lipschitz bounded open set in $\mathbb{R}^n$ (n=2 or 3) and $\partial\omega$ be its boundary. The fluid velocity u is derived from a potential ψ, the solution of the Laplace equation

$$\Delta\psi = 0 \qquad \text{in} \quad \omega \tag{1}$$

The velocity satisfies $u = \nabla\psi$.

The conservation of energy gives

$$-\lambda\Delta\theta + \rho C_p u \cdot \nabla\theta = 0 \quad \text{in} \quad \omega \tag{2}$$

where λ, ρ and C_p are respectively the thermal conductivity, density and specific heat (at a constant pressure) of the fluid.

Boundary conditions. The fluid boundary conditions are as follows:

- ψ is imposed on the air exit. We denote by γ_d^ψ the part of the boundary where the potential is known (Dirichlet's condition),
- $\nabla\psi \cdot n = u \cdot n = \psi_n \leq 0$ at the air entrance, where ψ_n is given, and n is the outwards unit normal vector,

- and $\nabla\psi \cdot n = u \cdot n = 0$ elsewhere. We denote by γ_n^ψ the part of the boundary where $\nabla\psi \cdot n$ is known (Neumann's condition).

We have $\gamma_d^\psi \cup \gamma_n^\psi = \partial\omega$ and $\operatorname{meas}(\gamma_d^\psi) > 0$.

The thermal boundary conditions are as follows:

- The temperature is imposed on the engine-block boundary, on the exhaust pipe and on the air entrance. (We denote by γ_d^θ the part of the boundary where the temperature is known.)
- The thermal flow, $\nabla\theta \cdot n$, equals 0 on the air exit (denoted by γ_n^θ, Neumann's conditions).
- Elsewhere, we consider the heat transfer by radiation, which is proportional to the difference between the "outside" temperature and the boundary temperature. We denote by γ_f^θ this part of the boundary (Fourier's conditions).

As regards the radiation, we consider multiple reflection effects and we assume that surfaces are gray, opaque and separated by a radiatively non-participating media. In addition to that, the emitted and reflected radiation is diffusely distributed (see e.g. (Sparrow and Cess, 1978)).

The radiosity w satisfies the following Fredholm integral equation of the second order:

$$w(x) = \Big(1 - \varepsilon(x)\Big) \int_{\partial\omega} \phi(x,y) w(y)\, \mathrm{d}s(y) + \varepsilon(x)\sigma\theta^4(x) \quad \text{on} \quad \partial\omega \tag{3}$$

where σ is the Stephan-Boltzmann constant and ε is the surface emittance, $\varepsilon \in]0,1[$. The kernel $\phi \in L^1(\partial\omega \times \partial\omega)$ is the angle factor (see e.g. (Sparrow and Cess, 1978)); it is positive and symmetric. It satisfies

$$\int_{\partial\omega} \phi(x,y)\, \mathrm{d}s(x) = 1 \tag{4}$$

We deduce from Fourier's law that

$$-\lambda\nabla\theta \cdot n = h(\theta - \theta_0) + \frac{\varepsilon}{1-\varepsilon}(\sigma\theta^4 - w) \qquad \text{on} \quad \gamma_f^\theta$$

where w is the solution to (3), h is the thermal transfer coefficient and θ_0 is the outside temperature.

We have $\gamma_d^\theta \cup \gamma_n^\theta \cup \gamma_f^\theta = \partial\omega$ and $\operatorname{meas}(\gamma_d^\theta) > 0$.

To prove the existence and uniqueness of the solution, we will need the following assumption.

Assumption 1. a) The velocity u belongs to $(L^p(\omega))^n$ with $p > n$.
b) The parts of the boundary of ω denoted by γ_n^θ and γ_f^θ are such that $u \cdot n$ is given and positive on $\gamma_n^\theta \cup \gamma_f^\theta$.

The whole model is as follows:

$$(P^\psi)\begin{cases} \text{Find } \psi \in H^1(\omega) & \text{such that} \\ -\Delta\psi = 0 & \text{in } \omega \\ \psi = \psi_d & \text{on } \gamma_d^\psi \\ \dfrac{\partial\psi}{\partial n} = \psi_n & \text{on } \gamma_n^\psi \end{cases}$$

$$(P^w)\begin{cases} \text{Let } \vec{u} = \nabla\psi \text{ satisfy Assumption 1.} & \\ \text{Find } (\theta, w) \in H^1(\omega) \times L^2(\partial\omega) \text{ such that} & \\ -\lambda\Delta\theta + \rho C_p \vec{u} \cdot \vec{\nabla}\theta = 0 & \text{in } \omega \\ \theta = \theta_d & \text{on } \gamma_d^\theta \\ \dfrac{\partial\theta}{\partial n} = 0 & \text{on } \gamma_n^\theta \\ -\lambda\dfrac{\partial\theta}{\partial n} = h(\theta - \theta_0) + \dfrac{\varepsilon}{(1-\varepsilon)}(\sigma\,\theta^4 - w) & \text{on } \gamma_f^\theta \\ w(x) = \Big(1 - \varepsilon(x)\Big)\displaystyle\int_{\partial\omega} \phi(x,y) w(y)\,\mathrm{d}s(y) & \\ \qquad\qquad + \varepsilon(x)\sigma\theta^4(x) & \text{on } \partial\omega \end{cases}$$

We suppose that ψ_d and ψ_n are given functions in $H^{\frac{3}{2}}(\gamma_d^\psi)$ and $H^{\frac{1}{2}}(\gamma_n^\psi)$, respectively. The temperatures θ_d and θ_0 are given positive functions respectively in $H^{\frac{3}{2}}(\gamma_d^\theta) \cap L^\infty(\gamma_d^\theta)$ and $H^{\frac{1}{2}}(\gamma_f^\theta) \cap L^\infty(\gamma_f^\theta)$.

3. Mathematical Analysis of the Model

In this section, we prove that the problems (P^ψ) and (P^w) have a unique solution.

The existence and uniqueness of the fluid velocity follows from the Lax-Milgram theorem.

3.1. Resolution of the Integral Equation

In order to prove the existence and uniqueness of the thermal model solution, we first solve the integral equation (Perret and Witomski, 1991).

We define the operator A by

$$Aw(x) = \Big(1 - \varepsilon(x)\Big)\int_{\partial\omega} \phi(x,y) w(y)\,\mathrm{d}s(y) \tag{5}$$

and consider the problem

$$\begin{cases} \text{Let } \theta \in L^p(\partial\omega),\ 1 \le p \le \infty. \text{ Find } w \in L^p(\partial\omega) \text{ such that} \\ (Id - A)w = \varepsilon\sigma\theta^4 \end{cases}$$

The operator A is a contraction one in $L^p(\partial\omega)$ for all $p \in [1,\infty]$, so there exists a unique solution $w \in L^p(\partial\omega)$. We prove that this solution is the sum of a series and can be written as follows (Perret and Witomski, 1991):

$$w(x) = \int_{\partial\omega} K(x,y)\varepsilon(y)\sigma\theta^4(y)\,\mathrm{d}s(y) + \varepsilon(x)\sigma\theta^4(x) \tag{6}$$

where the kernel K belongs to $L^1(\partial\omega \times \partial\omega)$ and is positive. In addition to that, this kernel K satisfies the following properties (Perret and Witomski, 1991; Monnier, 1995):

Proposition 1.

i) $$\int_{\partial\omega} K(x,y)\varepsilon(y)\,\mathrm{d}s(y) = 1 - \varepsilon(x)$$

ii) $$\int_{\partial\omega} K(x,y)\frac{\varepsilon(x)}{1-\varepsilon(x)}\,\mathrm{d}s(x) = 1$$

iii) $$\int_{\gamma_d^\theta} K(x,y)\frac{\varepsilon(x)}{1-\varepsilon(x)}\,\mathrm{d}s(x) > 0$$

3.2. Existence of the Temperature

We deduce from (6) that the problem (P^w) can be written down as follows:

$$(P^Q)\begin{cases} \text{Let } u \text{ satisfy Assumption 1.} & \\ \text{Find } \theta \in H^1(\omega) \text{ such that} & \\ -\lambda\Delta\theta + \rho C_p\, u\cdot\nabla\theta = 0 & \text{in } \omega \\ \theta = \theta_d & \text{on } \gamma_d^\theta \\ \dfrac{\partial\theta}{\partial n} = 0 & \text{on } \gamma_n^\theta \\ -\lambda\dfrac{\partial\theta}{\partial n} = Q(\theta) & \text{on } \gamma_f^\theta \end{cases}$$

where the operator Q is defined by

$$\begin{aligned} \forall x \in \partial\omega,\ Q(\theta)(x) &= h(\theta-\theta_0)(x) + \varepsilon(x)\sigma\theta^4(x) \\ &\quad - \frac{\varepsilon(x)}{1-\varepsilon(x)}\int_{\partial\omega} K(x,y)\varepsilon(y)\sigma\,\theta^4(y)\,\mathrm{d}s(y) \end{aligned}$$

We truncate the operator Q and we first prove the existence of the solution to the truncated problem (Perret and Witomski, 1991).

Let $\theta_{\inf} = \min(\inf_{\gamma_d^\theta} \theta_d, \inf_{\gamma_f^\theta} \theta_0)$ and $\theta_{\sup} = \max(\sup_{\gamma_d^\theta} \theta_d, \sup_{\gamma_f^\theta} \theta_0)$. We define

$$\bar{\theta}(x) = \begin{cases} \theta_{\inf} & \text{if} \quad \theta(x) \le \theta_{\inf} \\ \theta(x) & \text{if} \quad \theta_{\inf} \le \theta(x) \le \theta_{\sup} \\ \theta_{\sup} & \text{if} \quad \theta(x) \ge \theta_{\sup} \end{cases} \tag{7}$$

and the truncated operator $\bar{Q}$ as follows:

$$\begin{aligned} \forall x \in \partial\omega, \ \bar{Q}(\theta)(x) &= h(\bar{\theta} - \theta_0)(x) + \Big[\varepsilon(x)\sigma(\bar{\theta})^4(x) \\ &\quad - \frac{\varepsilon(x)}{1-\varepsilon(x)} \int_{\partial\omega} K(x,y)\varepsilon(y)\sigma(\bar{\theta})^4(y)\,\mathrm{d}s(y)\Big] \end{aligned} \tag{8}$$

From Proposition 1 (i) and (ii), we deduce that the operator $\bar{Q}$ has the following properties (Perret and Witomski, 1991; Monnier, 1995):

Proposition 2.

i) *$\bar{Q}$ is a Lipschitz operator from $L^2(\partial\omega)$ into $L^2(\partial\omega)$.*

ii) *The image of $L^2(\partial\omega)$ by $\bar{Q}$ is bounded in $L^\infty(\partial\omega)$.*

iii) *If $\theta(x) \ge \theta_{\sup}$ (resp. $\theta(x) \le \theta_{\inf}$), then $\bar{Q}(\theta)(x) \ge 0$ (resp. $\bar{Q}(\theta)(x) \le 0$).*

We denote by $(P^{\bar{Q}})$ the truncated problem which is the same as the problem (P^Q) but with the following boundary condition on γ_f^θ:

$$-\lambda \frac{\partial \theta}{\partial n} = \bar{Q}(\theta) \qquad \text{on} \quad \gamma_f^\theta$$

Owing to truncation, we now can prove, with the help of Schauder's fixed-point theorem, the existence of solutions to the truncated problem $(P^{\bar{Q}})$.

Theorem 1. *The problem $(P^{\bar{Q}})$ has a solution in $H^1(\omega)$.*

Proof. We define the operator N by

$$\begin{aligned} N : H^{-\frac{1}{2}}(\partial\omega) &\to H^1(\omega) \\ v &\mapsto \theta \end{aligned}$$

such that

$$\begin{cases} -\lambda\, \Delta\theta + \rho C_p\, \vec{u}\cdot\vec{\nabla}\theta = 0 & \text{in} \quad \omega \\ \theta = \theta_d & \text{on} \quad \gamma_d^\theta \\ \dfrac{\partial\theta}{\partial n} = 0 & \text{on} \quad \gamma_n^\theta \\ -\dfrac{\partial\theta}{\partial n} = v & \text{on} \quad \gamma_f^\theta \end{cases} \tag{9}$$

We deduce from the property $\text{div}(u) = 0$ and Assumption 1 that the bilinear mapping associated with this operator is coercive in $H^1(\omega)$ (Monnier, 1995). It follows from the Lax-Milgram theorem that this problem is well-posed and N is continuous.

We now define the operator M as follows:

$$\begin{aligned} M : H^1(\omega) &\rightarrow H^1(\omega) \\ t &\mapsto \theta \end{aligned}$$

$$\begin{array}{lcr} t \in H^1(\omega) & \xrightarrow{M} & \theta \in H^1(\omega) \\ \downarrow\gamma : \text{trace} & & \uparrow N \\ \gamma t \in H^{\frac{1}{2}}(\partial\omega) & & \bar{Q}(\gamma t) \in H^{-\frac{1}{2}}(\partial\omega) \\ \downarrow i : \text{injection} & & \uparrow i^* : \text{injection} \\ \gamma t \in L^2(\partial\omega) & \xrightarrow{\bar{Q}} & \bar{Q}(\gamma t) \in L^2(\partial\omega) \end{array}$$

We have $M = N \circ i^* \circ \bar{Q} \circ i \circ \gamma$. We know that $\bar{Q}$ is continuous (Proposition 2 (i)). Hence the operator M is continuous from $H^1(\omega)$ into $H^1(\omega)$.

From the compactness of the injection i^* and Proposition 2 (ii), the image of $H^1(\omega)$ through M is relatively compact in $H^1(\omega)$. It follows from Schauder's fixed-point theorem that M has a fixed-point in $H^1(\omega)$ and the truncated problem $(P^{\bar{Q}})$ has a solution. ■

We now prove that the solutions of the truncated problem $(P^{\bar{Q}})$ satisfy the weak maximum principle.

Theorem 2. *Each solution of the problem* $(P^{\bar{Q}})$ *satisfies* $\theta_{\text{inf}} \leq \theta \leq \theta_{\text{sup}}$.

Proof. We write a variational formulation of the problem $(P^{\bar{Q}})$ (with test functions in $H^1(\omega)$) and we choose $t = (\theta - \theta_{\text{sup}})^+$ as a test function, where $t^+ = \max(t, 0)$ (see e.g. (Gilbarg and Trudinger, 1977)). Then we deduce from the property $\text{div}(u) = 0$, Assumption 1 and Proposition 2 (iii), that (Monnier, 1995)

$$\int_\omega |\nabla t|^2 \, dx \;\leq\; 0$$

This gives $\theta \leq \theta_{\text{sup}}$ a.e. in ω.

The inequality $\theta_{\text{inf}} \leq \theta$ is proved in a similar way with $t = (\theta - \theta_{\text{inf}})^-$ where $t^- = \max(-t, 0)$. ■

We have just proved the existence of solutions to the truncated problem $(P^{\bar{Q}})$. Moreover, these solutions belong to the interval $[\theta_{\text{inf}}, \theta_{\text{sup}}]$. Then we have $\forall x \in \partial\omega,\ \bar{Q}(\theta)(x) = Q(\theta)(x)$, and the solutions to the problem $(P^{\bar{Q}})$ are also solutions to the problem (P^Q). Therefore we have proved that the problem (P^Q) has solutions and these solutions satisfy the weak maximum principle.

3.3. Uniqueness of the Temperature

We now establish the uniqueness of the solution.

Theorem 3. *The solution to the problem (P^Q) is unique.*

Proof. We define the operator $q : L^4(\partial\omega) \to L^2(\partial\omega)$ by

$$q(\theta)(x) = \varepsilon(x)\sigma\theta^4(x) - \frac{\varepsilon(x)}{1-\varepsilon(x)} \int_{\partial\omega} K(x,y)\varepsilon(y)\sigma\theta^4(y)\,\mathrm{d}s(y)$$

It follows from Proposition 1 (i) and (ii) that

$$\int_{\partial\omega} q(\theta)\,\mathrm{d}s = 0 \quad \forall\theta \in L^4(\partial\omega)$$

(This relation expresses the conservation of the radiative energy flow through $\partial\omega$.)

We denote by θ_1 and θ_2 two solutions to the problem (P^Q) and we define $\tilde{\theta} = \theta_1 - \theta_2$. From Proposition 1 (iii), we deduce (Monnier, 1995) that

$$\int_{\gamma_f^\theta \cup \gamma_n^\theta} \varepsilon\sigma\tilde{\theta}^4\,\mathrm{d}s = 0$$

This implies $\theta_1 = \theta_2$ a.e. on $\partial\omega$. We thus conclude that the solution is unique. ■

Regularity of the solutions. We now present results on regularity in the case where ω is an open set in $\mathbb{R}^2$.

We suppose that the domain ω has reentering corners and boundary conditions on each side of such corners are of the same kind. In these conditions, we have (Grisvard, 1985)

$$\forall\varepsilon > 0, \quad u \in \left(H^{\alpha_1}(\omega)\right)^2$$

where $\alpha_1 = \frac{1}{2} - \varepsilon$. Then we deduce by the bootstrap method that (Monnier, 1995)

$$\forall\varepsilon > 0, \quad \theta \in H^{1+\alpha_2}(\omega)$$

where $\alpha_2 = \frac{1}{2} - \varepsilon$.

4. Numerical Analysis

4.1. Introduction

We assume that ω is a polygonal domain of $\mathbb{R}^2$. We discretize the equations by means of a finite-element method. We denote by $(\mathcal{T}_h)$ a regular and quasi-uniform family of triangulation, $\bar{\Omega} = \cup_{(T\in\mathcal{T}_h)}T$. We associate this family of triangulation with a single reference finite element of Lagrange of class $\mathcal{C}^0$.

Let k be an integer greater than or equal to 1. We denote by P_k the set of polynomials of degree less than or equal to k. We define the following discrete spaces:

$$V_{0h}^{\psi}(\omega) = \left\{\varphi_h \in \mathcal{C}^0(\bar{\omega}); \forall T \in \mathcal{T}_h,\ \varphi_h|_T \in P_k; \varphi_h|_{\gamma_d^{\psi}} = 0\right\}$$

$$V_{th}^{\psi}(\omega) = \left\{\varphi_h \in \mathcal{C}^0(\bar{\omega}); \forall T \in \mathcal{T}_h,\ \varphi_h|_T \in P_k; \varphi_h|_{\gamma_d^{\psi}} = \psi_d\right\}$$

$$V_{0h}^{\theta}(\omega) = \left\{t_h \in \mathcal{C}^0(\bar{\omega}); \forall T \in \mathcal{T}_h,\ t_h|_T \in P_k; t_h|_{\gamma_d^{\theta}} = 0\right\} \tag{10}$$

$$V_{th}^{\theta}(\omega) = \left\{t_h \in \mathcal{C}^0(\bar{\omega}); \forall T \in \mathcal{T}_h,\ t_h|_T \in P_k; t_h|_{\gamma_d^{\theta}} = \theta_d\right\} \tag{11}$$

$$k \geq 2 \Longrightarrow \mathcal{W}_h = \Big\{v \in \mathcal{C}^0(\partial\omega); \text{for every } T \in \mathcal{T}_h \text{ such that the boundary } \partial T \in \partial\omega, \text{ we have } v|_{\partial T} \in P_{k-1}\Big\} \tag{12}$$

$$k = 1 \Longrightarrow \mathcal{W}_h = \left\{v : \partial\omega \to \mathbb{R},\ v \text{ is piecewise constant}\right\} \tag{13}$$

These spaces are subspaces of $H^1(\omega)$ and $L^2(\partial\omega)$, respectively.

The discrete fluid model is as follows:

$$\begin{cases} \text{Find } \psi_h \in V_{th}^{\psi} \text{ such that} \\ \forall \varphi_h \in V_{0h}^{\psi},\quad (\nabla\psi_h, \nabla\varphi_h) = (\psi_n, \varphi_h) \end{cases} \tag{14}$$

We write $\vec{u}_h = \vec{\nabla}\psi_h$ and obtain the discrete thermal model:

$$(P_h^w)\begin{cases} \text{Let } u_h. \text{ Find } (\theta_h, w_h) \in V_{th}^{\theta}(\omega) \times \mathcal{W}_h(\partial\omega) \text{ such that} \\ \forall t_h \in V_{0h}^{\theta}(\omega),\quad \lambda(\nabla\theta_h, \nabla t_h) + \rho C_p(u_h\nabla\theta_h, t_h) \\ \qquad +\Big(\big[h\ (\theta_h - \theta_0) + \dfrac{\varepsilon}{(1-\varepsilon)}(\sigma\theta_h^4 - w_h)\big], t_h\Big) = 0 \\ \forall v_h \in \mathcal{W}_h(\partial\omega),\quad \Big((I-A)w_h, v_h\Big) = (\varepsilon\sigma\theta_h^4, v_h) \end{cases}$$

where A is the operator defined by (5).

The existence and uniqueness of ψ_h follows from the Lax-Milgram theorem. We obtain the following error estimate (Monnier, 1995):

$$\|u - u_h\|_{0,\partial\omega} = O(h^{\alpha_1})$$

4.2. Thermal Model

We now prove that if the thermal conductivity λ is large enough, then there exists a unique solution to the discrete thermal model (P_h^w). Moreover, this solution converges to the exact solution and we present an error estimate. This analysis is based on the notion of branches of non-singular solutions and makes use of the extended implicit-function theorem (see e.g. (Brezzi *et al.*, 1980)).

4.2.1. Continuous Fixed-Point Problem

We rewrite the problem (P^w) as a fixed-point one. Let $X = H^1(\omega) \times L^2(\partial\omega)$, $Y = L^2(\gamma_f^\theta) \times L^2(\partial\omega)$ and Λ be a compact interval of $\mathbb{R}^{*+}$. We define the linear operator T as follows:

$$\begin{aligned} T :\ & Y \to X \\ & (h, g) \mapsto -(\theta, w) \end{aligned} \tag{15}$$

where $x = (\theta, w) \in X$ is the solution of the following uncoupled linear problem:

$$\begin{cases} -\lambda\Delta\theta + \rho C_p \vec{u} \cdot \vec{\nabla}\theta = 0 & \text{in} \quad \omega \\ \theta = \theta_d & \text{on} \quad \gamma_d^\theta \\ \dfrac{\partial\theta}{\partial n} = 0 & \text{on} \quad \gamma_n^\theta \\ -\dfrac{\partial\theta}{\partial n} = h & \text{on} \quad \gamma_f^\theta \\ w(x) = \Big(1 - \varepsilon(x)\Big) \displaystyle\int_{\partial\omega} \phi(x,y) w(y)\, \mathrm{d}s(y) + g & \text{on} \quad \partial\omega \end{cases}$$

The non-linear operator G is defined by

$$\begin{aligned} G : \Lambda \times X \quad & \rightarrow \quad Y \\ (\lambda; t, v) \quad & \mapsto \quad \Big(h(t - \theta_0) + \frac{\varepsilon}{(1-\varepsilon)}(\sigma t^4 - v), \varepsilon\sigma t^4\Big) \end{aligned} \tag{16}$$

where λ is the thermal conductivity.

We define $F(\lambda; x) = x + T\, G(\lambda; x)$. Then the problem (P^w) is equivalent to the following one:

$$(P^F) \begin{cases} \text{Let } \lambda \in \Lambda. \text{ Find } x = (\theta, w) \in X \text{ such that} \\ F(\lambda; x) = 0 \end{cases} \tag{17}$$

We define the linearized problem as follows:

$$\begin{cases} \text{Let } \lambda \in \Lambda \text{ and } x(\lambda) \text{ be the solution of (17).} \\ \text{Find } l \in X \text{ such that} \\ \dfrac{\partial F}{\partial x}\Big(\lambda; x(\lambda)\Big) \cdot l = 0 \end{cases} \tag{18}$$

We also introduce $\lambda_{\min}$ as

$$\lambda_{\min} = \sqrt{4\ (C_\omega + 1)\varepsilon_1\sigma\theta_{\sup}^3\rho\ C_p} \tag{19}$$

where C_ω is Poincaré's constant.

We proved in (Monnier, 1995) employing the Lax-Milgram theorem that if $\lambda > \lambda_{\min}$, then the linearized problem (18) is well-posed. By definition, the solution $x(\lambda)$ of (17) is non-singular if the linearized problem (18) is well-posed. Therefore the branch of solutions $\{(\lambda, x(\lambda)); \lambda > \lambda_{\min}\}$ is a branch of non-singular solutions.

4.2.2. Discrete Fixed-Point Problem

We now write the discrete thermal problem as a fixed point one.

We define the discrete space $X_h = V_{th}^\theta(\omega) \times \mathcal{W}_h(\omega)$ and the operator T_h as follows:

$$T_h : Y \to X_h; (h, g) \mapsto -(\theta_h, w_h)$$

where $x_h = (\theta_h, w_h) \in X_h$ is the solution of the system

$$\lambda(\nabla\theta_h, \nabla t_h) + \rho C_p\ (\vec{u}_h \cdot \vec{\nabla}\theta_h, t_h) = (h, t_h) \qquad \forall t_h \in V_{0h}^\theta(\omega) \tag{20}$$

$$\Big((I - A)w_h, v_h\Big) = (g, v_h) \qquad \forall v_h \in \mathcal{W}_h(\omega) \tag{21}$$

We proved in (Monnier, 1995) the existence and uniqueness of the solution of eqn. (20) if h is small enough.

We define $F_h(\lambda; x_h) = x_h + T_h\, G(\lambda; x_h)$. The discrete problem (P_h^w) is equivalent to the problem

$$(P_h^F) \begin{cases} \text{Let } \lambda \in \Lambda. \text{ Find } x_h = (\theta_h, w_h) \in X_h \text{ such that} \\ F_h(\lambda; x_h) = 0 \end{cases} \tag{22}$$

We have the following result (Monnier, 1995).

Theorem 4. *We suppose that Assumption 1 is satisfied and we denote by β a real such that $w \in H^\beta(\partial\omega)$, $\beta > 0$. Then, for h small enough, there exists a unique branch of non-singular solutions $\{(\lambda, (\theta_h(\lambda), w_h(\lambda)));\ \lambda > \lambda_{\min}\}$ to the problem (P_h^F). Moreover, there exists a constant C independent of h such that for all $\varepsilon > 0$*

$$\|\theta(\lambda) - \theta_h(\lambda)\|_{1,\omega} + \|w(\lambda) - w_h(\lambda)\|_{0,\partial\omega} \le Ch^{\alpha_\varepsilon}$$

with $\alpha_\varepsilon = \min(\alpha_1 - \varepsilon, \alpha_2 - \varepsilon, \beta)$.

Proof. This result is proved for a non-linear abstract problem in (Brezzi *et al.*, 1980). The assumptions of this theorem are satisfied in the case of our thermal model (Monnier, 1995). The main assumptions we have to satisfy are the following:

- The operator G is C^2 from $]\lambda_{\min}, +\infty[\times X$ into Y.

- For all $(\lambda, x) \in]\lambda_{\min}, +\infty[\times X$, the operator $TD_xG(\lambda; x)$ is compact.
- $\lim_{h\to 0} \|(T - T_h)(y)\|_X = 0 \quad \forall y \in Y$.

(Let us notice that the error estimates are only stated for the uncoupled linear problem (20)–(21), and they are written in fractional-order Sobolev spaces.) ■

5. Shape Optimal-Design Problem

We define the spaces

$$V_0^{\psi}(\omega) = \left\{\psi \in H^1(\omega);\ \psi|_{\gamma_d^{\psi}} = 0\right\},\quad V_0^{\theta}(\omega) = \left\{\theta \in H^1(\omega);\ \theta|_{\gamma_d^{\theta}} = 0\right\}$$

and their translated counterparts

$$V_t^{\psi}(\omega) = \left\{\psi \in H^1(\omega);\ \psi|_{\gamma_d^{\psi}} = \psi_d\right\},\quad V_t^{\theta}(\omega) = \left\{\theta \in H^1(\omega);\ \theta|_{\gamma_d^{\theta}} = \theta_d\right\}$$

Furthermore, we set

$$V_t(\omega) = V_t^{\psi}(\omega) \times V_t^{\theta}(\omega) \times L^2(\partial\omega),\quad V_0(\omega) = V_0^{\psi}(\omega) \times V_0^{\theta}(\omega) \times L^2(\partial\omega)$$

The variational formulation of the coupled problem (called the *state* problem) is as follows:

$$\left\{\begin{array}{l} \text{Let } \omega \text{ be a given domain. Find } y^{\omega} = (\psi^{\omega}, \theta^{\omega}, w^{\omega}) \in V_t(\omega) \text{ such that} \\ \forall z = (\varphi, t, v) \in V_0(\omega), E_{\omega}(y^{\omega}, z) = 0 \end{array}\right. \tag{23}$$

The state problem is a system of two partial differential equations and an integral equation, which can be written as

$$\left\{\begin{array}{l} \text{Find } \psi^{\omega} \in V_t^{\psi}(\omega) \text{ such that} \\ \forall \varphi \in V_0^{\psi}(\omega), \displaystyle\int_{\omega} \nabla\psi\nabla\varphi\,\mathrm{d}x = \int_{\gamma_n^{\psi}} \psi_n \varphi\,\mathrm{d}s \end{array}\right.$$

$$\left\{\begin{array}{l} \text{Let } u^{\omega} = \nabla\psi^{\omega}.\ \text{Find } (\theta^{\omega}, w^{\omega}) \in V_t^{\theta}(\omega) \times L^2(\partial\omega) \text{ such that} \\ \forall t \in V_0^{\theta}(\omega),\ \lambda \displaystyle\int_{\omega} \nabla\theta\nabla t\,\mathrm{d}x + \rho C_p \int_{\omega} u\nabla\theta t\,\mathrm{d}x + h\int_{\gamma_f^{\theta}} (\theta - \theta_0)t\,\mathrm{d}s \\ \qquad\qquad + \displaystyle\int_{\gamma_f^{\theta}} \frac{\varepsilon}{1-\varepsilon}(\sigma\theta^4 - w)t\,\mathrm{d}s = \int_{\gamma_n^{\theta}} \phi_n t\,\mathrm{d}s \\ \forall v \in L^2(\partial\omega), \displaystyle\int_{\partial\omega} wv\,\mathrm{d}s - \int_{\partial\omega} (1-\varepsilon(x))\Big[\int_{\partial\omega} \phi(x,y)w(y)\,\mathrm{d}s(y)\Big]v(x)\,\mathrm{d}s(x) \\ \qquad = \sigma \displaystyle\int_{\partial\omega} \varepsilon\theta^4 v\,\mathrm{d}s \end{array}\right.$$

We define the observation $J_{\omega} : V_t^{\theta}(\omega) \to \mathbb{R};\ \theta \mapsto \frac{1}{2}\int_{\gamma_h} \theta^2\,\mathrm{d}s$, where γ_h is a part of γ_f^{θ}.

We denote by $\mathcal{D}$ the space of admissible domains and consider the cost function

$$j : \mathcal{D} \to \mathbb{R} : \omega \mapsto j(\omega) = J_\omega(y^\omega)$$

where y^ω is the state of the system.

The minimization problem we seek to solve is

$$\left| \begin{array}{l} \text{Find } \omega^\star \in \mathcal{D} \text{ such that} \\ j(\omega^\star) = \min_{\omega \in \mathcal{D}} j(\omega) \end{array} \right.$$

We use a gradient method to minimize the cost function. Therefore we have to differentiate the functional $J_\omega(y^\omega)$, $y^\omega \in V_t(\omega)$, with respect to the domain. We use a method of deformation domain which consists in transporting functionals on a reference domain Ω. Then we differentiate with respect to the transformation T, where T is such that $\omega = T(\Omega)$ (see e.g. (Murat and Simon, 1976; Céa, 1981; Rousselet, 1982; Sokołowski and Zolésio, 1992)).

We consider Lipschitz transformations (see e.g. Monnier, 1995) and we set $V = T - I$, where I is the identity of $\mathbb{R}^n$.

Let

$$\bar{j} : \mathcal{T} \to \mathbb{R} : T \mapsto \bar{j}(T) = j(T(\Omega))$$

Then we define the derivative of j with respect to the domain in the following way:

$$\forall V \text{ Lipschitz}, \quad \frac{\mathrm{d}j}{\mathrm{d}\omega}(\Omega) \cdot V = \frac{\mathrm{d}\bar{j}}{\mathrm{d}T}(I) \cdot V$$

5.1. Differentiability of the Solution with Respect to the Domain

To compute the sensitivity gradient, we have to prove the differentiability of the cost function with respect to the domain. Therefore we have to prove the differentiability of the solution y^ω with respect to the domain ω.

We have already noticed in the numerical analysis (Section 4) that if the thermal conductivity λ is greater than $\lambda_{\min}$, then the linearized problem (18) is well-posed. Thus we deduce from the implicit-function theorem that in this case the solution y^ω is differentiable with respect to ω (Monnier, 1995).

5.2. Sensitivity Gradient

Proposition 3. *If the thermal conductivity λ is greater than $\lambda_{\min}$, then the cost function $j(\omega)$ is differentiable with respect to ω. Moreover, for all Lipschitz V, we have*

$$\frac{\mathrm{d}j}{\mathrm{d}\omega}(\Omega) \cdot V = \frac{\partial J_\Omega}{\partial \omega}(y^\Omega) \cdot V - \frac{\partial E_\Omega}{\partial \omega}(y^\Omega, p^\Omega) \cdot V \tag{24}$$

where $y^\Omega \in V(\Omega)$ *is the solution of the state problem (23) defined in* Ω *and* $p^\Omega \in V(\Omega)$ *is the solution of the following adjoint equation:*

$$\forall z \in V(\Omega), \qquad \frac{\partial E_\Omega}{\partial y}(y^\Omega, p^\Omega)\cdot z = \frac{\partial J_\Omega}{\partial y}(y^\Omega)\cdot z \tag{25}$$

Furthermore, p^Ω *exists and is unique.* ■

We refer the reader to the works (Chenais *et al.*, 1995; Monnier, 1995) for the proof of this proposition and for a more detailed expression for the adjoint state equation. We only notice that this adjoint problem is an integro-differential system of two linear partial differential equations and one integral equation. The three equations are coupled. To solve this system, we first have to solve the adjoint thermal problem and then the adjoint fluid one.

6. Numerical Results

We recall that we seek to optimize the hose shape in order to minimize the cost function $j(\omega) = \frac{1}{2}\int_{\gamma_h} \theta^2 \,\mathrm{d}s$. The cost function depends on both the hose temperature and its length.

The hose shape is modelled by cubic splines, its thickness is constant and its extremities are fixed. A piecewise linear approximation is used to solve numerically all the partial differential equations and a piecewise constant one is used to solve two integral equations.

The optimization process is as follows. We use a descent algorithm which builds a sequence of domains (Ω_n). The algorithm computes a perturbation V which decreases the cost function. The new domain Ω_{n+1} is the image of Ω_n through the function $T = (I + V)$. The algorithm of minimization we used was a Quasi-Newton one from the library Basile (the company Simulog S.A.).

In this example, we have the Peclet number $P_e = \frac{\rho C_p U^* L^*}{\lambda} = 10^3$ (where U^* and L^* are respectively a characteristic velocity and a characteristic length of the flow) and $h = 20$. The engine-block temperature is $420K$, the exhaust-pipe one is $520K$ and that of the air entrance is $280K$. The emittance ε depends on x and is such that $0.5 < \varepsilon(x) < 0.8$.

Between the initial shape (Fig. 2 and Fig. 3) and the best shape we get (Fig. 4 and Fig. 5), the cost function decreases by 20%.

7. Conclusion

In this paper, we solved a shape optimal-design problem for a fluid-heat coupled system; the cost function in this case was differentiable. We presented in detail the mathematical analysis of the state problem as well as the corresponding control problem analysis. We proved that if the thermal conductivity of the fluid is large enough, the linearized problem is well-posed. Then we obtained, using the implicit-function theorem, the solution differentiability with respect to the domain. We deduced that, under this condition, the adjoint problem is well-posed. This analysis

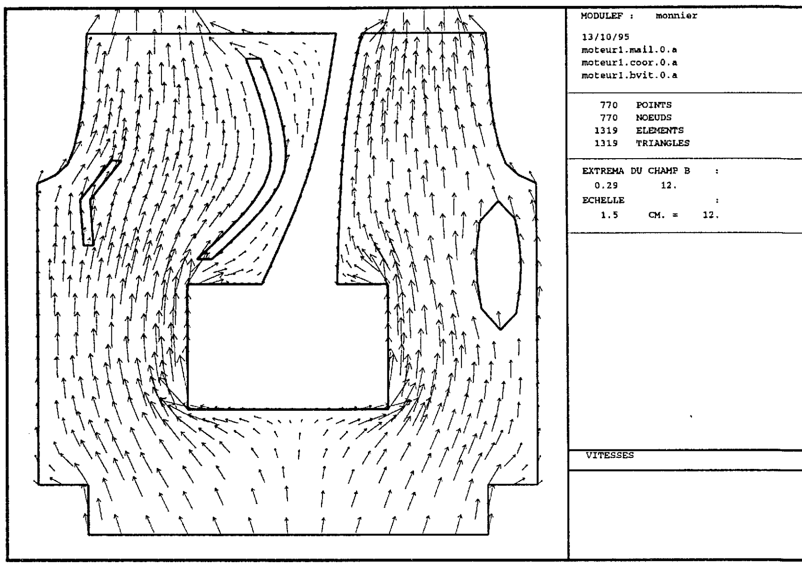

Fig. 2. Initial shape. Fluid velocity.

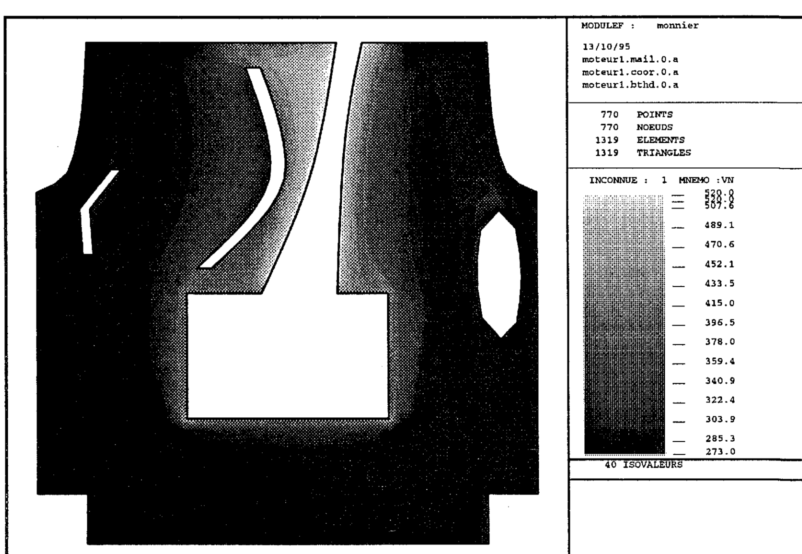

Fig. 3. Initial shape. Isotherms.

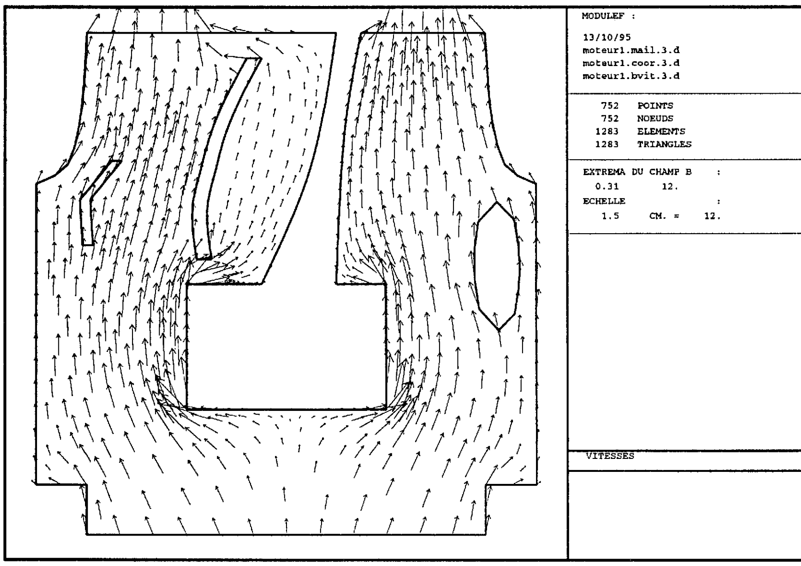

Fig. 4. Optimal shape. Fluid velocity.

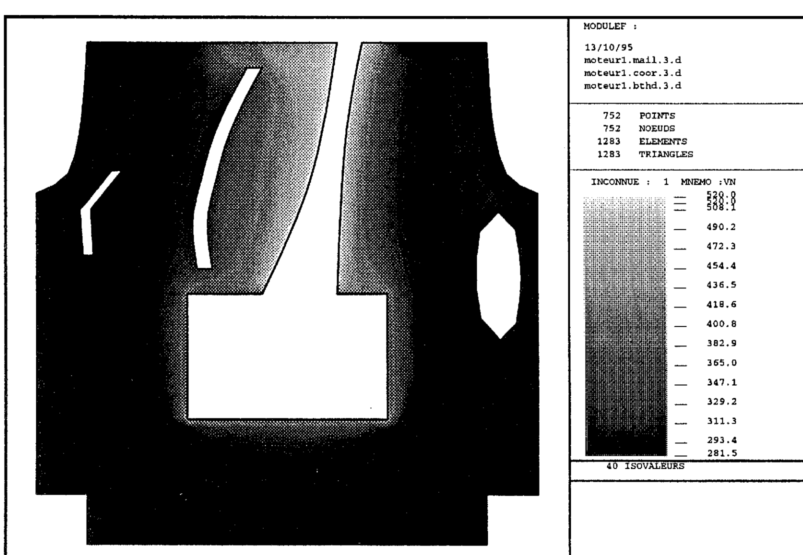

Fig. 5. Optimal shape. Isotherms.

justifies the resolution of this shape optimal-design problem by a gradient method. We solved numerically these equations by the finite-element method. We proved, under the same condition on the thermal conductivity, the existence and uniqueness of the discrete solution and the convergence of the numerical scheme. The proof was based on the notion of branches of non-singular solutions. Finally, we obtained, for our example, the "optimal" shape which made the cost function decrease.

References

Brezzi F., Rappaz J. and Raviart P.A. (1980): — *Finite-dimensionnal approximation of non-linear problems. Part I: Branches of nonsingular solutions.* — Num. Math., v.36, pp.1–25.

Céa J. (1981): In: Optimization of Distributed Parameter Structures (E.J. Haug and J. Céa Eds). — v.II, Sijthoff and Noordhoff, Alphen aan den Rijn, The Nertherlands.

Chenais D., Monnier J. and Vila J.P. (1995): — *Shape optimization for an aerothermic problem.* — Proc. WCSMO Conf. in Goslar (Germany), Pergamon, (Eds. N. Olhoff and G. Rozvany), pp.757–764.

Gilbarg D. and Trudinger N. (1977): *Elliptic Partial Differential Equations of Second Order.* — Berlin: Springer Verlag.

Grisvard P. (1985): *Elliptic Problems in Non-Smooth Domains.* — Monograph and Studies in Mathematics, v.24, Boston: Pitman.

Monnier J. (1995): *Conception optimale de forme pour un système couplé fluide-thermique. Application à l'aérothermique d'un véhicule.* — Thesis, University of Nice-Sophia Antipolis.

Murat F. and Simon J. (1976): *Sur le contrôle par un domaine géométrique.* — Publication du L.A. 189, Université P. et M. Curie Paris VI, Paris.

Perret C. and Witomski P. (1991): *Equation de la chaleur et réflections multiples.* — Annales Institut Poincaré, v.8, No.6, pp.677–689.

Rousselet B. (1982): *Quelques résultats en optimisation de domaine.* — Thèse d'état, Université de Nice.

Sokołowski J. and Zolésio J.P. (1992): *Introduction to Shape Optimization. Shape Sensitivity Analysis.* — Berlin: Springer Verlag.

Sparrow E.M. and Cess R.D. (1978): *Radiation Heat Transfer.* — New-York: Mc Graw-Hill Book Company.