Appl. Math. and Comp. Sci., 1996, vol.6, No.2, 303–319

A FULL-STRESS TECHNIQUE FOR STRUCTURAL SHAPE OPTIMIZATION†

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A technique for shape optimization based on a generalization of the Full-Stress Design approach is presented. Full-stress procedures for trusses design are based on the assumption that in an optimal structure each member is subjected to its allowable stress under at least one loading condition. We extend this criterion to shape optimization by assuming that a good shape design is obtained when the boundary is subjected to its allowable stress under at least one of the load cases. Based on this criterion, we present a procedure for plane structures, that, at the same time, avoids stress concentration in many parts of the contour caused by abrupt changes in the geometry and reduces the amount of structural material. Four illustrative test cases are solved and compared with the published results obtained with alternative models. In all the cases, good solutions are obtained in a very efficient way.

1. Introduction

The traditional technique for shape design is based on an optimization model that minimizes the structural weight under response constraints, namely displacements, stress, eigenvalue and technological limitations. One of the pioneering works using modern structural optimization is due to Zienkiewicz and Campbell (1973), where a discrete finite-element model was employed, with nodal points as design variables. The authors evaluated sensitivities of response and employed sequential linear programming. Literature surveys describing previous work can be found e.g. in (Benett and Botkin, 1985; Braibant and Fleury, 1984; Dems and Mróz, 1984; Ding, 1986; Haftka and Grandhi, 1986; Haug, 1981; Haug and Céa, 1981; Haug *et al.*, 1986; Kristensen and Madsen, 1976; Morris, 1982). Recent overviews and related works can be found in (Anido *et al.*, 1991; Arora, 1995; Dems and Mróz, 1993; Haftka and Adelman, 1989; Herskovits, 1995a; Hinton and Sienz, 1994; Olhoff and Lund, 1995; Rasmussen, 1991; Rasmussen *et al.*, 1992).

† This research was partially supported by the CNPq (Brazilian Research Council) and by JNICT (Portuguese Council for Research and Technology)

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The traditional approach requires an iterative procedure, based on a mathematical-programming algorithm, that needs the evaluation of the functions involved, constraints and their respective derivatives. These can be obtained analytically, semianalytically and by global finite differences (Arora, 1995; Olhoff and Lund, 1995).

The objective of this paper is to establish a full-stress criterion for shape optimization and to develop a design technique based on it. A fully stressed truss is such that each member is subjected to its allowable stress under at least one loading condition (Bartholomew and Morris, 1976; Haftka and Gurdal, 1992; Kirsch, 1993; Morris, 1982). We extend this definition to shape optimization by assuming that a solid is fully stressed when the points on the boundary are subjected to its allowable stress under at least one of the load cases. Without proving that such a solid exists, it seems reasonable to suppose that a solid shape is better as the stress distribution becomes closer to a fully stressed one. Based on these ideas, a target stress is defined and the shape with a stress distribution on the boundary closest to the target is obtained. This can be carried out by minimizing the sum of the squared differences between the equivalent stresses at points on the boundary and the target stress, which should not be greater than a specified allowable stress.

In this paper, two-dimensional problems are considered with domain given by a closed geometric shape in $\mathbb{R}^2$. To get a smooth boundary, the geometry is defined by B-spline curves (Mortenson, 1985; Plastock and Kalley, 1986). The design variables will give the position of the control points that determine the B-spline curves. To solve the unconstrained minimization problem included in the present model, a globally convergent Gauss-Newton algorithm is proposed.

To illustrate the validity of the model, standard test problems are optimized. In these cases, we obtain the sensitivities of the stresses with respect to the shape design variables by finite differences.

2. A Full-Stress Model for Shape Optimization

Let us consider now elastic solids submitted to one or more static loadings. In most cases, the equivalent stresses are maximum at points on the boundary of the solid, or near the boundary. Then, in general, there is no stress concentration if the stress distribution on the boundary is smooth. On the other hand, if the stress on the boundary is low, we can reduce the amount of material by moving the boundary inwards. Hence, it can be deduced that we have a "good shape" when the boundary is subjected to the allowable stress, at least under one of the load cases. Based on these ideas, we define the following full-stress criterion.

Definition 1. An elastic solid submitted to several load cases is *fully stressed* if the stress distribution at the points on the boundary is allowable and equal to the maximum allowable stress at least for one of the load cases.

2.1. Full-Stress Design Model

Let us consider a 2D solid in a domain Ω, subjected to p load cases. We introduce the Full-Stress Design Model for shape optimization by the following procedures:

1) Define sections of the boundary to be optimized, called 'moving sections' and denoted by $\partial\Omega_i$, $i \in K_M$. The sections that remain fixed are $\partial\Omega_i$, $i \in K_F$, (Fig. 1).
2) Define a set of m 'test points' in the moving sections. The stress will be evaluated at these points.
3) Define a parametrization (or discretization) of the moving sections:

$$\partial\Omega_i = \partial\Omega_i(x), \qquad x \in \mathbb{R}^n, \quad i \in K_M$$

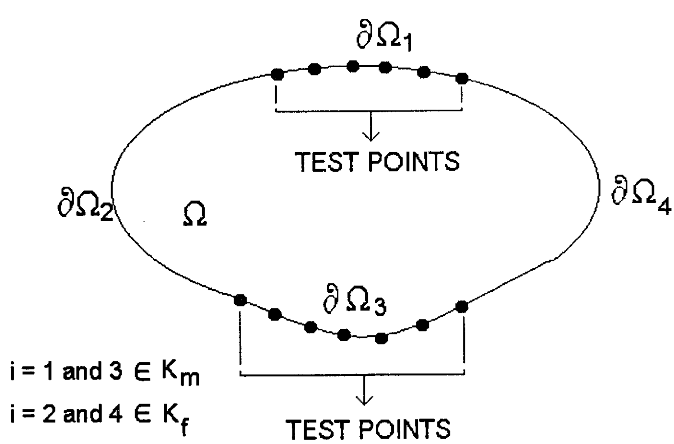


Fig. 1. Moving sections and test points.

4) Find the shape design that minimizes the function

$$F(x) = \sum_{i=1}^{m} \frac{1}{2} \left(\sup_j \left(\sigma_i^j(x)\right) - \overline{\sigma_i} \right)^2, \qquad j = 1, \ldots, p \tag{1}$$

where $\overline{\sigma_i}$ is the target stress at the i-th test point and σ_i^j is the Von Mises stress at the i-th test point due to the load case j.

In this way, if a zero value is obtained for F, then the present Full-Stress Criterion is strictly satisfied. This model has the advantage of being solved as an unconstrained optimization problem. Very quick convergence is obtained with Gauss-Newton method (Dennis and Schnabel, 1983), that requires only first derivatives.

2.2. Discrete Model of the Geometry

The moving contours will be represented by B-spline functions (Mortenson, 1985; Plastock and Kalley, 1986). In this way, smooth shapes are obtained avoiding singularities in structural analysis. Each moving contour is determined by a set of points, called *vertices* or *control points*, $P_1, P_2, \ldots, P_n$, (Fig. 2). These points form a base, said to be the *control polygon*, tangent to the corresponding B-spline curve. The quantities $X_i \equiv (X_{1i}, X_{2i})$ are the coordinates of P_i.

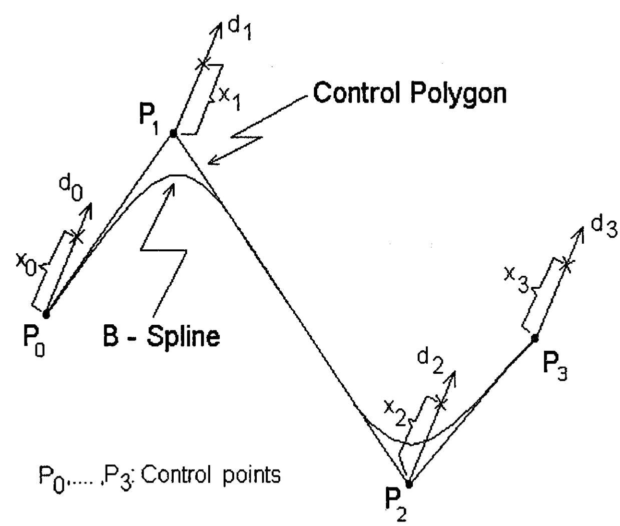


Fig. 2. Discrete model of the geometry.

In this paper, the control points are determined by the design variables. Given an initial configuration, at each control point P_i, the user defines a direction of movement $d_i \equiv (d_{i1}, d_{i2})$, such that $\|d\| = 1$. The design variables x_i determine the steps of the control points along the direction of movement. Then, if X_i^0 are the initial control points' coordinates, we have

$$X_i(x) = X_i^0 + x_i \times d_i, \qquad i = 1, 2, \ldots, n$$

as shown in Fig. 2.

At each iteration, given new design variables, the corresponding control points' coordinates are obtained. For this new shape, a finite-element mesh is generated, using the adaptive mesh generator GAMAT2 (Guimarães and Feijóo, 1990).

3. Globally Convergent Gauss-Newton Algorithm

It should be noticed that $F(x)$ introduced in (1) is not smooth, since the gradient is not defined when the maximum equivalent stress at a test point is obtained simultaneously for more than one of the loadings. Since the objective of the present paper is to obtain a simple and efficient method, $F(x)$ will be minimized by means of smooth optimization techniques, yielding a quick and satisfactory design.

Taking

$$R_i(x) = \sup_j \left(\sigma_i^j(x)\right) - \overline{\sigma_i}\right)$$

we have

$$F(x) = \frac{1}{2} R^t(x) R(x)$$

Then

$$\nabla F(x) = R^t(x) \nabla R(x)$$

and

$$\nabla^2 F(x) = \nabla R^t(x) \nabla R(x) + \beta(x)$$

where

$$\beta(x) = \sum_{i=1}^{m} R_i(x) \nabla^2 R_i(x)$$

The Newton method can be employed to minimize $F(x)$, but it requires calculation of $\nabla^2 F(x)$ at each iteration and $\nabla^2 R_i(x)$ is very hard to compute. In the Gauss-Newton method, an approximation of $\nabla^2 F(x)$ is obtained by removing $\beta(x)$. Note that $\beta(x) = 0$ when $R(x) = 0$, i.e. if the stress distribution on the boundary is equal to the target stress. Then the Gauss-Newton iteration is defined as follows:

$$x^{k+1} = x^k - \left[\nabla R^t(x^k) \nabla R(x^k)\right]^{-1} \nabla R^t(x^k) R(x^k)$$

In (Dennis and Schnabel, 1983) it is proved that, in the case when $R(x) = 0$ in the solution, the Gauss-Newton method has a quadratic rate of convergence, i.e.

$$\frac{\|(x^{k+1} - x^*)\|}{\|(x^k - x^*)\|^2} < \beta < \infty$$

where x^* minimizes $F(x)$. It seems reasonable to assume that the convergence is nearly quadratic when $R(x^*)$ is small.

In the proposed algorithm, we define the search direction as

$$d^k = x^{k+1} - x^k$$

Since $\nabla R^t(x^k)$ is a full-rank matrix, $\nabla R^t(x^k) \nabla R(x^k)$ is positive definite. If we consider

$${d^k}^t \nabla F(x^k) = {d^k}^t \nabla R^t(x^k) R(x^k)$$

then we conclude that

$${d^k}^t \nabla F(x^k) = -{d^k}^t \left[\nabla R^t(x^k) \nabla R(x^k)\right] d^k < 0$$

which proves that d^k is a descent direction of the objective function if x is not a local minimum. To obtain a globally convergent algorithm, we introduce a line search along d^k.

In what follows, we propose a globally convergent algorithm based on the Gauss-Newton method including Armijo's line search procedure (Armijo, 1966; Herskovits, 1995b), that ensures a reasonable decrease in the value of the objective function.

Optimization Algorithm:

Parameters: $\eta \in (0,1)$ and $\nu \in (0,1)$.

Data: Initial design $x^0 \in \mathbb{R}^n$. Set $x = x^0$.

Step 1. Computation of search direction d by solving the linear system

$$\left[\nabla R^t(x) \nabla R(x)\right] d = -\nabla R^t(x) R(x)$$

Step 2. Line search.

Compute α, the first number of the sequence $\{1, \nu, \nu^2, \nu^3, \ldots\}$ satisfying

$$F(x + \alpha d) \leq f(x) + \alpha \eta \nabla F^t(x) d$$

Step 3. Updates.

(i) Set

$$x := x + \alpha d$$

(ii) Go back to Step 1.

4. Applications

The present model is applied to four illustrative test cases. The designs obtained are compared with the alternative solutions from the literature.

Hole in a Biaxial Stress Field. The determination of the optimal shape of a hole in a biaxial 2D plane stress field is studied. Figure 3 shows the initial design, loading and biaxial symmetric boundary conditions, as well as the finite-element mesh, the initial control points and the directions of movement. The thickness is $h = 5\,\mathrm{mm}$, Young's modulus $E = 210\,\mathrm{GPa}$, Poisson's ratio $\nu = 0.3$, the load per unit length $q = 2.5\,\mathrm{N/mm}$. The target stress is taken as $\bar{\sigma}_i = 16\,\mathrm{MPa}$.

Figure 4 shows the Von Mises contours of the initial design. The optimal design and stress contours are presented in Fig. 5. The iteration history for the objective function and the Von Mises stress is shown in Fig. 6.

The final shape obtained after 7 iterations looks like the ellipse obtained analytically under the assumption that the plate is infinite (Timoshenko and Goodier, 1970). The present solution is also in good agreement with the optimal numerical design of Rasmussen *et al.* (1992) whose objective was to find the shape that minimizes the largest stress concentration on the boundary of the hole. This solution was obtained after 12 iterations.

Fillet Problem. The initial design, boundary conditions, and two loadings given by q_1 and q_2, as well as the finite-element mesh, the initial control points and the directions of movement are shown in Fig. 7. The data used are $h = 1\,\mathrm{mm}$, $E = 226\,\mathrm{GPa}$, $\nu = .3$, $q_1 = 1.425\,\mathrm{N/mm}$, $q_2 = 2\,\mathrm{N/mm}$ and $\bar{\sigma}_i = 100\,\mathrm{MPa}$. Figure 8 shows the contours of the Von Mises stresses for the initial design due to load q_1. The optimal design, stress distribution and iteration history are shown in Figs. 9 to 11.

It can be observed that, even if in the initial shape only one of the loadings is critical, both of them become active in the final design, with fully stressed points on the boundary. If only q_1 is considered, an optimal fillet geometry is obtained that is very similar to the designs presented by Rasmussen *et al.* (1992) and Mota Soares *et al.* (1984), among others.

Portal Frame. Figure 12a shows the initial shape of a portal frame, boundary conditions, loading, control points and moving directions. The finite-element mesh is shown in Fig. 12b. The following data are used: $h = 8\,\mathrm{mm}$, $E = 210\,\mathrm{GPa}$, $\nu = .3$, $q = 2\,\mathrm{N/mm}$, and $\bar{\sigma}_i = 300\,\mathrm{MPa}$. The Von Mises stress contours are shown in Fig. 13, for the initial design. Figures 14a and 14b illustrate the final design and the corresponding stress distribution. The iteration history is shown in Fig. 15.

The optimal shape design is achieved after 8 iterations and is very similar to the one obtained by Rasmussen *et al.* (1992) for volume minimization with the bound on the maximum Von Mises stress of 300 MPa. The latter design was obtained after 23 iterations, using the simplex algorithm. The near optimum was reached after 14 iterations.

Open-End Spanner. The objective is to find the outer and the inner shape of an open-end spanner. The initial shape, loading, boundary conditions and finite-element discretization are shown in Fig. 16. Figure 17 shows the control points, moving contour and directions of movement. The following data are used: $h = 5\,\mathrm{mm}$, $E = 210\,\mathrm{GPa}$, $\nu = .3$, $q = 5\,\mathrm{N/mm}$, and $\bar{\sigma}_i = 250\,\mathrm{MPa}$. The initial stress distribution is shown in Fig. 18. Figures 19a and 19b illustrate the obtained shape and the corresponding mesh and stress distribution. The iterative process, represented in Fig. 20, stops after 10 iterations, but a very satisfactory solution is obtained after 7 iterations.

The present design is similar to the one obtained by Rasmussen *et al.* (1992) and other authors for a spanner, with different data. Rasmussen *et al.* minimized the volume subject to an upper bound on the Von Mises stresses and on the compliance. The optimum was obtained after 20 iterations, but a very good solution was reached after 10 iterations.

5. Conclusions

The present model is very simple regarding the mathematical formulation and implementation in a computer code. This fact suggests that it should be possible to employ this technique in large problems and to extend it to 3-D structures. The computational effort required by one iteration is lower than with traditional techniques, since only an unconstrained minimization is performed. The number of iterations for the cases studied is also reduced. This seems to be due to the fact that the Gauss-Newton algorithm has a rate of convergence quadratic or close to quadratic.

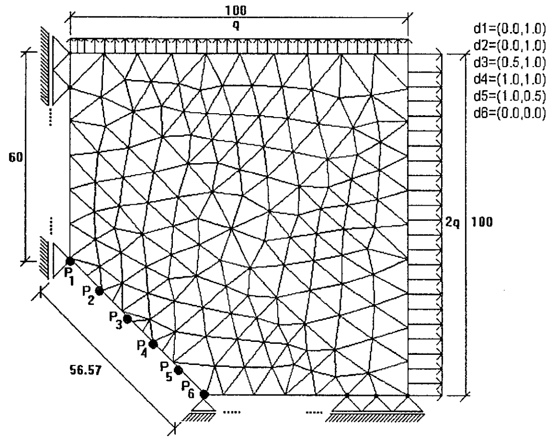


Fig. 3. Hole in biaxial stress field. A quarter-symmetry discrete model.

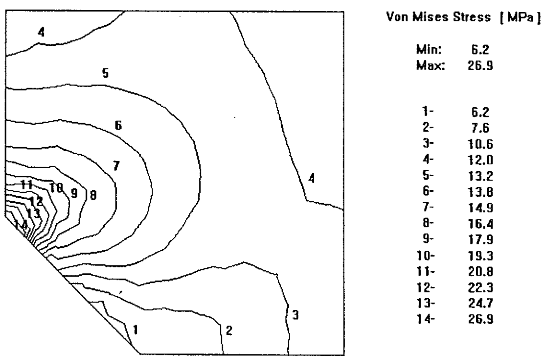


Fig. 4. Hole in biaxial stress field. Contours of the Von Mises stresses for the initial design.

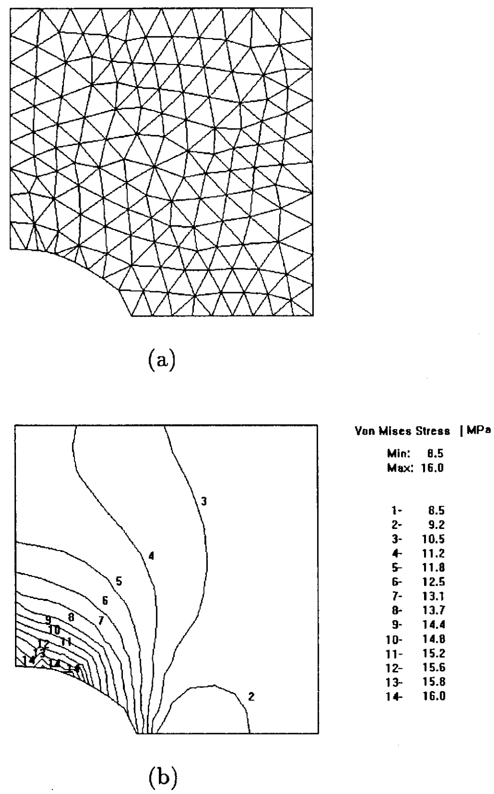


Fig. 5. Hole in biaxial stress field (a) optimal design, (b) contours of the Von Mises stresses for the optimal design.

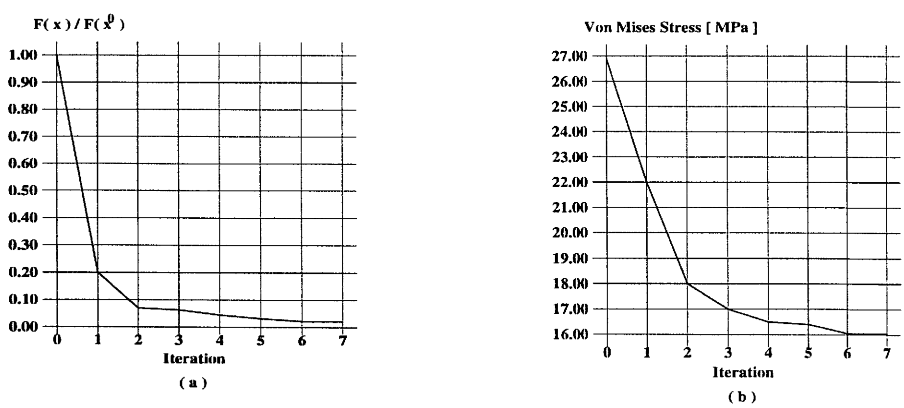


Fig. 6. Hole in biaxial stress field. Iteration history.

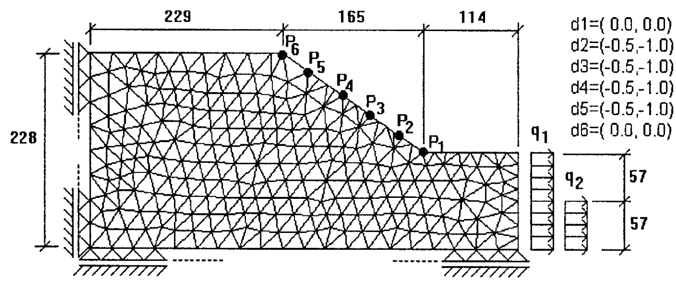


Fig. 7. Fillet problem. A unixial symmetric discrete model.

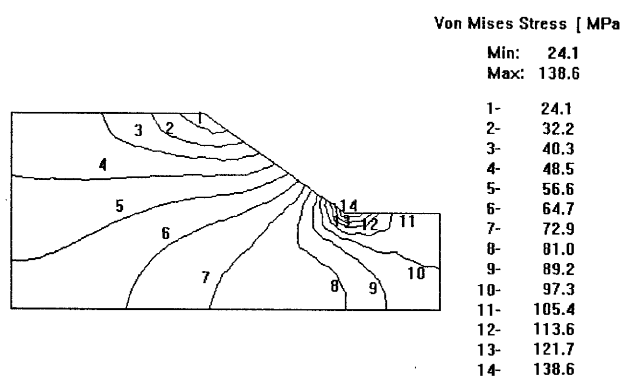


Fig. 8. Fillet problem. Contours of the Von Mises stresses for the initial design, with load q_1.

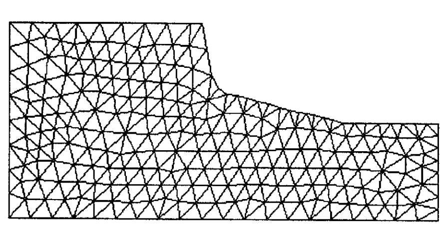


Fig. 9. Fillet problem. Optimal design.

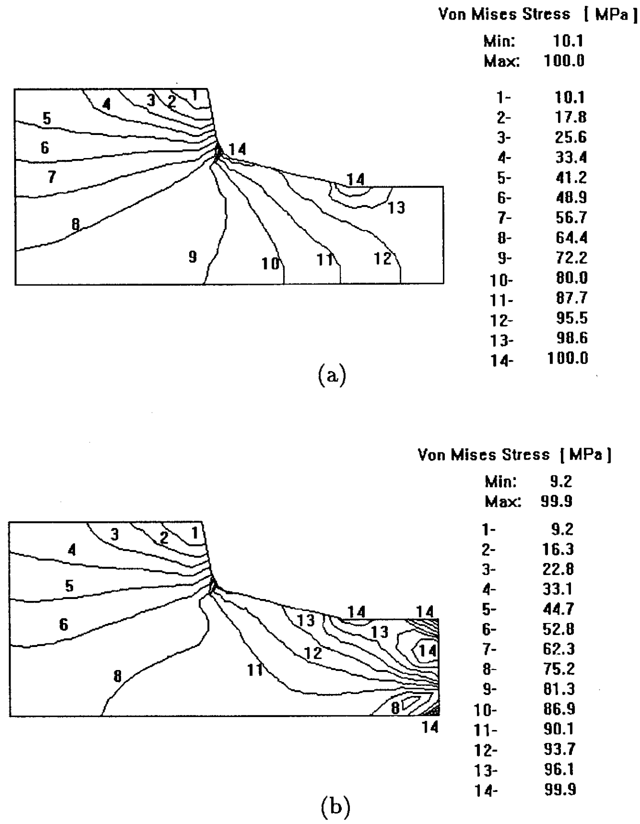


Fig. 10. Fillet problem. Contours of the Von Mises stresses for the optimal design (a) load q_1, (b) load q_2.

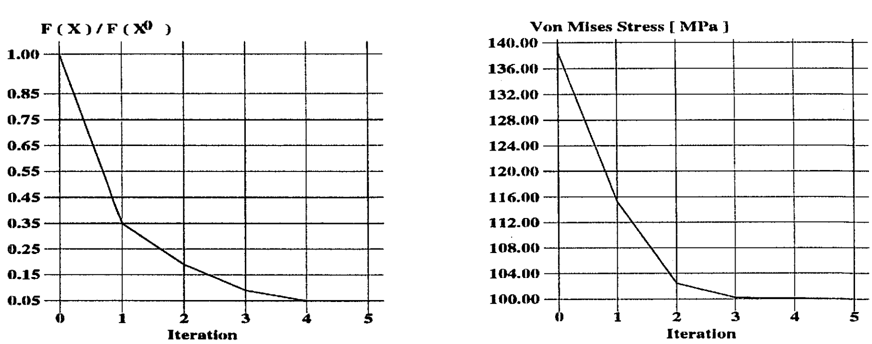
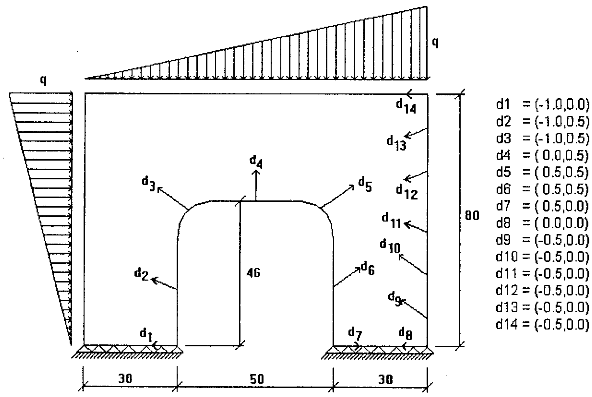
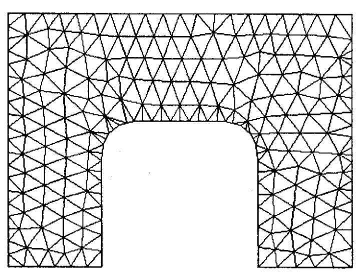


Fig. 11. Fillet problem. Iteration history.



(a)



(b)

Fig. 12. (a) Portal frame, (b) initial discrete finite-element model.

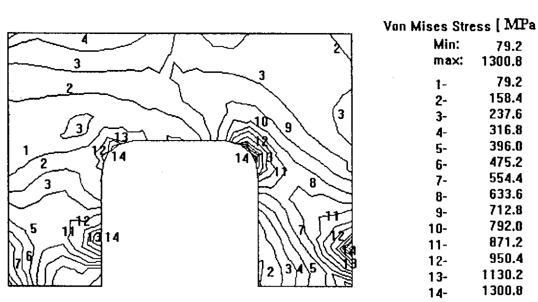
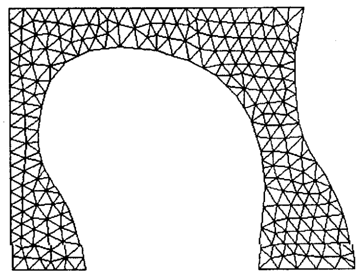
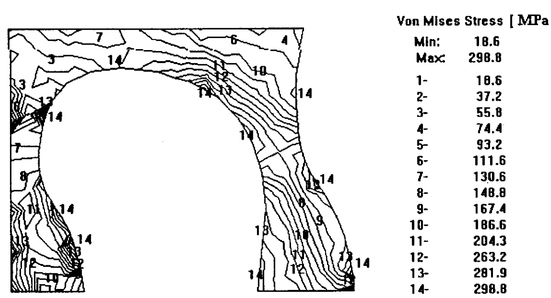


Fig. 13. Portal frame. Contours of the Von Mises stresses for the initial design.

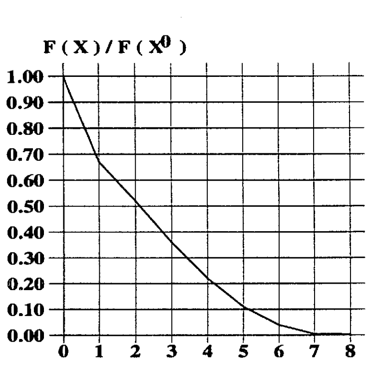


(a)

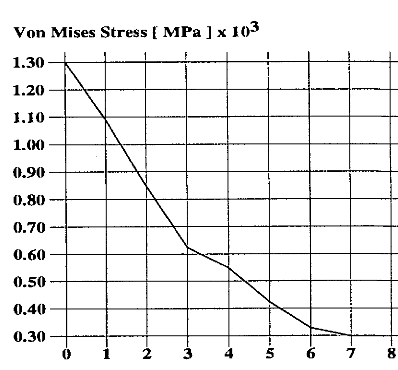


(b)

Fig. 14. Portal frame (a) optimal design, (b) contours of the Von Mises stresses for the optimal design.



(a)



(b)

Fig. 15. Portal frame. Iteration history.

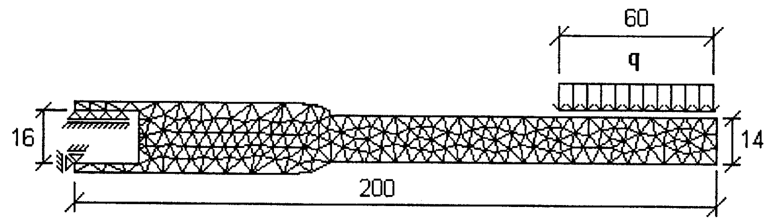


Fig. 16. Open-end spanner. Initial discrete finite-element model.

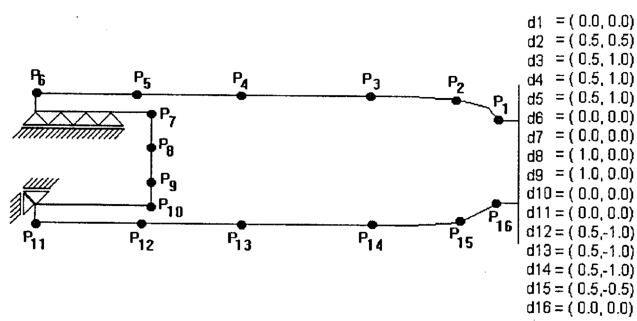


Fig. 17. Open-end spanner. Control points and directions of movement.

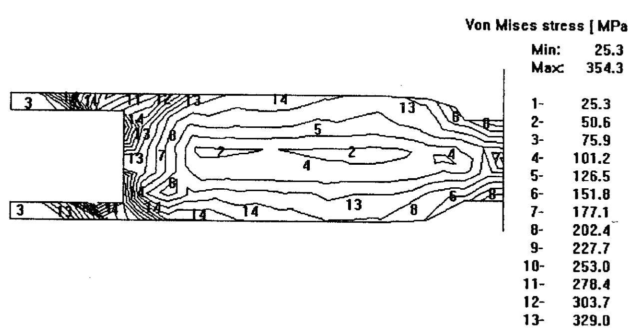
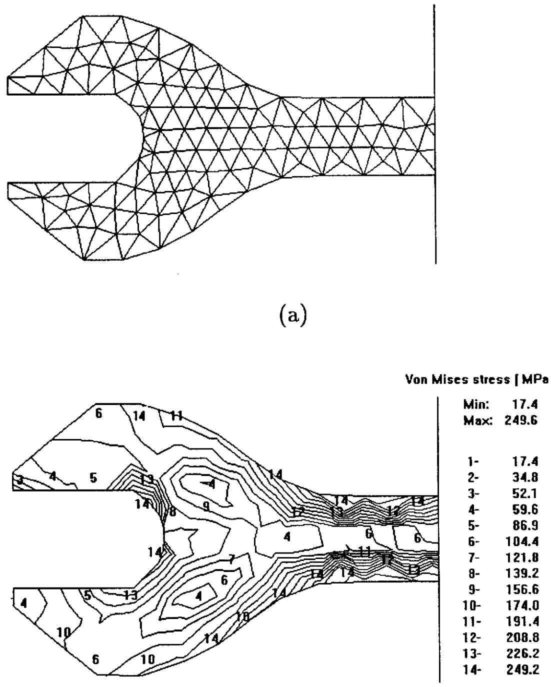


Fig. 18. Open-end spanner. Contours of the Von Mises stresses for the initial design.



(b)

Fig. 19. Open-end spanner (a) optimal design, (b) contours of the Von Mises stresses for the optimal design.

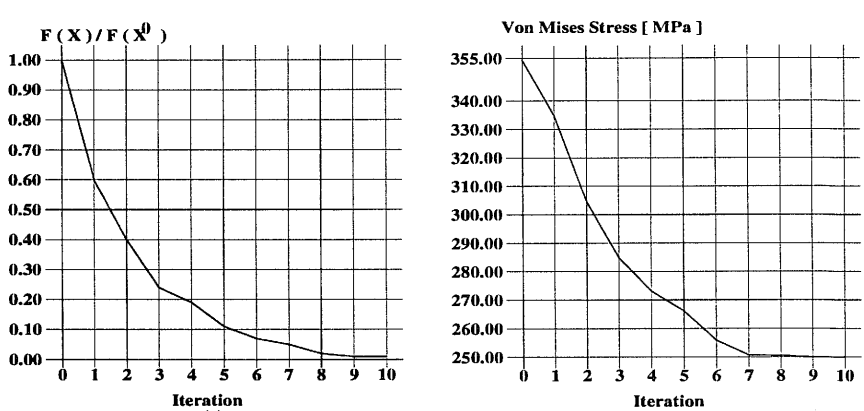


Fig. 20. Open-end spanner. Iteration history.

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