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AN INFINITESIMAL OBSERVABILITY TEST FOR INVARIANT SYSTEMS ON LIE GROUPS

NIKOLAOS APOSTOLOU*, DEMOSTHENES KAZAKOS*

This paper deals with infinitesimal observability for invariant control systems on Lie groups having an invariant output function. For this class of systems an infinitesimal observability criterion is stated and proved. It is also shown how this criterion can be applied in order to conclude more general observability results for such systems.

1. Introduction

Observability is one of the essential topics of control theory and in the past years much attention has been paid to its various aspects (see e.g. Celle *et al.*, 1989; Cheng *et al.*, 1990; Ciccarella *et al.*, 1993; Gauthier and Bornard, 1981). In a very interesting recent paper (Gauthier and Kupka, 1994) the authors characterized general nonlinear systems that are observable independently of the inputs by introducing a new concept of observability which they termed infinitesimal observability and which is different from the standard observability. In the same paper they proved some of the properties of the new concept and showed how infinitesimal observability can lead to the characterization of systems that are observable independently of the inputs. The importance of this new notion motivated the present work. This paper deals with infinitesimal observability for invariant control systems evolving on Lie groups and having a suitable output function. For such systems an infinitesimal observability criterion will be proved.

This work is organized as follows. In the present section, some fundamental definitions and facts concerning infinitesimal observability for general systems are given. In the next section, invariant control systems on Lie groups are considered and basic assumptions are made. Finally, in Section 3 the main result is stated and proved.

Consider now the following general system Σ:

$$\dot{x} = \psi(x, u)$$

$$y = f(x)$$

where $x \in M$, an analytic connected manifold, $y \in \mathbb{R}$ and u belongs to the input space U, an analytic connected manifold. It is assumed that ψ is analytic in both

* Control Systems Laboratory, Department of Electrical and Computer Engineering, University of Patras, 26500 Patras, Greece, e-mail: kazakos@ee.upatras.gr.

x and u, and that f is analytic in x. The admissible inputs are measurable and bounded functions $u_T : [0,T] \to U$. Assume that for every initial state x_0 and for any control law u_T the Cauchy problem:

$$\dot{x} = \psi(x, u_T), \quad x(0) = x_0$$

has a solution defined on the interval $[0,T]$. For $t \in [0,T]$ we denote by $x(t, u_T)$ the points on the orbit generated by u_T and starting at $x \in M$. Now the following classical definition can be stated:

Definition 1. Σ is observable if for every x_1, $x_2 \in M$ with $x_1 \neq x_2$ there exists a control law u_T such that:

$$f\Big(x_1(t, u_T)\Big) \neq f\Big(x_2(t, u_T)\Big)$$

for every t in a subset of $[0,T]$ with positive measure, that is if Σ distinguishes between any two distinct initial states.

Let us consider next infinitesimal observability of Σ. In order to define infinitesimal observability, the notion of "lifting" $T\Sigma$ of Σ to the tangent bundle TM of M is necessary. The map $\psi : M \times U \to TM$ gives a parametrized vector field on M. Thus the tangent map (with respect to x) $T\psi : TM \times U \to T(TM)$ ($T(TM)$ denotes the tangent bundle of TM) determines a parametrized vector field on TM. Furthermore, the differential of the function f is the map $\mathrm{d}f : TM \to \mathbb{R}$. The definition of the lifting $T\Sigma$ of Σ is the following (see also Gauthier and Kupka, 1994):

Definition 2. The lifting $T\Sigma$ of Σ is the system:

$$\dot{\xi} = T\psi(\xi, u)$$

$$z = \mathrm{d}f(\xi)$$

where $\xi \in TM$.

The orbits of Σ and $T\Sigma$ are related in the following way. Let $\pi : TM \to M$ be the canonical projection. Also let, for a control law u_T and for $t \in [0,T]$, $\sigma(t, u_T) : M \to M$ be a map such that $\sigma(t, u_T)(x) = x(t, u_T)$. If $\xi(t, u_T)$ is an orbit of $T\Sigma$ for some initial $\xi \in TM$, then $\pi(\xi(t, u_T))$ is the orbit of Σ starting at $\pi(\xi)$ and generated by the same input function u_T, i.e.

$$\pi(\xi)(t, u_T) = \pi\Big(\xi(t, u_T)\Big) \tag{1}$$

Conversely, if $x(t, u_T)$ is an orbit of Σ for some $x \in M$, then $T\sigma(t, u_T)(\xi)$ is the orbit of $T\Sigma$ starting at $\xi \in T_xM$ and generated by the same input function u_T, i.e.

$$\xi(t, u_T) = T\sigma(t, u_T)(\xi) \tag{2}$$

Now the following definition of infinitesimal observability can be stated:

Definition 3. Let $x \in M$ and u_T be a control law. Σ is called infinitesimally observable at (x, u_T) if for every $\xi_1, \xi_2 \in T_x M$ with $\xi_1 \neq \xi_2$:

$$\mathrm{d}f\Big(\xi_1(t, u_T)\Big) \neq \mathrm{d}f\Big(\xi_2(t, u_T)\Big)$$

for every t in a subset of $[0, T]$ with positive measure, i.e. if $T\Sigma$ with u_T distinguishes between any two distinct initial states $\xi_1, \xi_2 \in T_x M$.

For further details, as well as for the relation between infinitesimal observability and the characterization of systems that are observable independently of the inputs, the reader is referred to (Gauthier and Kupka, 1994). Here only the following property is mentioned: If Σ is infinitesimally observable at (x, u_T), then there exists an open neighborhood W of x such that Σ with u_T distinguishes between any two distinct initial states $x_1, x_2 \in W$. In the next sections, an infinitesimal observability criterion for a particular class of systems, namely for invariant control systems on Lie groups, is stated and proved. In view of the foregoing property of infinitesimal observability, this criterion can also be regarded as a local observability result for such systems.

2. Invariant Systems on Lie Groups

In this section invariant control systems on Lie groups are considered. Here basic assumptions are made and some mathematical preliminaries are given. These can also be found in (Helgason, 1978; Varadarajan, 1984). Let G be a real, analytic, and connected Lie group with identity element e. G will be the state space of every system Σ occuring in the sequel. Also let $L = \mathrm{Lie}\,(G)$ be the Lie algebra of left invariant vector fields on G and L^* be the dual of L, i.e. the space of all left invariant one forms on G. We write $\exp tX$, $t \in \mathbb{R}$ for the integral curve of $X \in L$ passing through e at $t = 0$. The integral curve through $g \in G$ is then $g \cdot \exp tX$. We denote by $L(\cdot)$, $R(\cdot)$, $TL(\cdot)$, $TR(\cdot)$ the left and right translation and the corresponding tangent maps, respectively. Let $I(g) : G \to G$ where $g \in G$ be a map such that $I(g)(r) = g \cdot r \cdot g^{-1}$, i.e. $I(g) = L(g) \circ R(g^{-1}) = R(g^{-1}) \circ L(g)$. $I(\cdot)$ is obviously a diffeomorphism. The tangent map at e $TI(g)_e : L \to L$ is also denoted by $Ad(g)$ and is an automorphism of L. The map $Ad : G \to Aut(G)$ is the well-known adjoint representation of G. If V_1 and V_2 are linear spaces, V_1^* and V_2^* are respectively their duals and $\Lambda : V_1 \to V_2$ is any linear map, then a linear map $\Lambda^* : V_2^* \to V_1^*$ is induced in the following way: $\Lambda^*(\omega)v = \omega(\Lambda v)$ for every $v \in V_1$, $\omega \in V_2^*$. In particular, $TL(\cdot)$, $TR(\cdot)$, $TI(\cdot)$ induce such maps denoted by $T^*L(\cdot)$, $T^*R(\cdot)$, $T^*I(\cdot)$, respectively.

Consider now a control system on G of the following form:

$$\dot{g} = A_g + \sum_{i=1}^{\nu} u^i B_g^i$$

where $g \in G$, A, $B^i \in L$, $u^i \in \mathbb{R}$ for $i = 1, \ldots, \nu$. Throughout this paper, such a system will be called invariant. An invariant system can be characterized by the

Since $Ad(g)$ is an automorphism of L for every $g \in G$, it is immediate that $\ker(Ad(g)) = 0$. Thus eqn. (3) shows that Σ is infinitesimally observable at (g, u_T) if and only if $\bigcap\{\ker(\phi(g(t)) : t \in [0,T]\} = 0$ and this is equivalent to

$$\operatorname{span}\left\{\phi\Big(g(t)\Big) : t \in [0,T]\right\} = L^*$$

■

Remark 1. An analog of Theorem 1 can be stated in the case of several invariant output functions $f_1, \ldots, f_m$. In this case, a linear map $F(g) : L \to \mathbb{R}^m$ with $F(g) = (\phi_1(g), \ldots, \phi_m(g))$ is induced for every $g \in G$. Then infinitesimal observability at (g, u_T) is equivalent to

$$\bigcap\left\{\ker F\Big(g(t,u_T)\Big) : t \in [0,T]\right\} = 0$$

Next the case of piecewise constant inputs is treated. Such an input u_T can be identified with a family of pairs $\{(X_1,t_1),\ldots,(X_k,t_k)\}$ with $X_i \in L$, $t_i > 0$ for $i = 1,\ldots,k$ and $\Sigma t_i = T$. This means that from time 0 to t_1 the vector field X_1 is applied, from time t_1 to $t_1 + t_2$ the vector field X_2 is applied and so forth.

In the following proposition it is proved that if the vector fields incorporated by a piecewise constant control law u_T are fixed points of $Ad(g)$, then infinitesimal observability at (g, u_T) can be possibly achieved only when g belongs to the centre of G. If, in addition, u_T incorporates a vector field which commutes with every other vector field incorporated by u_T, then infinitesimal observability at (g, u_T) can be possibly achieved only when this specific vector field belongs to the centre of L.

Proposition 1. *Let Σ be an invariant system on G with an invariant output function f, $u_T = \{(X_1,t_1),\ldots,(X_k,t_k)\}$ be a piecewise constant control law for Σ and $g \in G$. Assume that $Ad(g)X_i = X_i$ for $i = 1,\ldots,k$. If Σ is infinitesimally observable at (g,u_T), then $g \in \operatorname{centre}(G)$. If in addition, there exists some $1 \le m \le k$ such that $[X_i, X_m] = 0$ for every $i = 1,\ldots,k$, then $X_m \in \operatorname{centre}(L)$.*

Proof. As in the proof of Theorem 1, setting $g(t,u_T) = g(t)$ and $e(t,u_T) = e(t)$, we have $g(t) = g \cdot e(t)$. By definition of u_T it follows that $e(t) = \exp(t_1X_1)\cdots\exp(\lambda_q X_q)$ for some $1 \le q \le k$, $\lambda_q \in (0,t_q]$ with $t_1 + \cdots + \lambda_q = t$. Taking into account the formula (cf. Varadarajan, 1984, p.104)

$$r \cdot \exp(tZ) \cdot r^{-1} = \exp\Big(tAd(r)Z\Big) \quad \text{for every } r \in G,\ Z \in L,\ t \in \mathbb{R}$$

it is clear that $g \cdot e(t) = g \cdot \exp(t_1X_1)\cdots\exp(\lambda_qX_q) = \exp(t_1X_1)\cdots\exp(\lambda_qX_q)\cdot g = e(t)\cdot g$ for every $t \in [0,T]$, since $Ad(g)X_i = X_i$ for every $i = 1,\ldots,k$. By virtue of Lemma 1 we have

$$Ad\Big(e(t)\Big)^*\phi\Big(g(t)\Big) = \phi\Big(e(t)^{-1}\cdot g\cdot e(t)\cdot e(t)\Big) = \phi\Big(g\cdot e(t)\Big) = \phi\Big(g(t)\Big)$$

for every $t \in [0,T]$. Furthermore, it is obvious that $Ad(g(t))^*\phi(g(t)) = \phi(g(t))$ for every $t \in [0,T]$. Hence

$$\phi\Big(g(t)\Big)\Big(Ad\Big(g(t)\Big)Z - Ad\Big(e(t)\Big)Z\Big) = 0 \quad \text{for every } t \in [0,T],\ Z \in L$$

Since infinitesimal observability is equivalent to $\text{span}\{\phi(g(t)) : t \in [0,T]\} = L^*$, it follows that

$$\bigcap \Big\{ \text{range}\Big(Ad\big(g(t)\big) - Ad\big(e(t)\big) \Big) : \ t \in [0,T] \Big\} = 0 \tag{4}$$

But for every $t \in [0,T]$ we have

$$\begin{aligned} Ad\big(g(t)\big) - Ad\big(e(t)\big) &= Ad(g)Ad\big(e(t)\big) - Ad\big(e(t)\big) \\ &= \big(Ad(g) - I_L\big)Ad\big(e(t)\big) \end{aligned}$$

where I_L is the identity map from L onto itself. From the relation above it is concluded that $\text{range}(Ad(g(t)) - Ad(e(t))) = \text{range}(Ad(g) - I_L)$ because $Ad(e(t))$ is an automorphism of L, which means that $\text{range}(Ad(e(t))) = L$. Thus, from eqn. (4) it follows that $Ad(g) = I_L$, which implies the first assertion (cf. Varadarajan, 1984).

We now turn to the second assertion. Since $[X_i, X_m] = 0$, one has $\exp(X_i)\exp(X_m) = \exp(X_i + X_m) = \exp(X_m)\exp(X_i)$ for every $i = 1, \ldots, k$. Then one can write for $\tau \in \mathbb{R}$, $t \in [0,T]$

$$\begin{aligned} \phi\big(g(t)\big) &= \phi\big(g \cdot \exp(t_1 X_1) \cdots \exp(\lambda_q X_q)\big) \\ &= \phi\big(g \cdot \exp(t_1 X_1) \cdots \exp(\lambda_q X_q) \cdot \exp(-\tau X_m) \cdot \exp(\tau X_m)\big) \\ &= \phi\big(\exp(-\tau X_m) \cdot g \cdot \exp(t_1 X_1) \cdots \exp(\lambda_q X_q) \cdot \exp(\tau X_m)\big) \\ &= Ad(\exp \tau X_m)^* \phi\big(g(t)\big) \end{aligned}$$

Since $\text{span}\{\phi(g(t)) : t \in [0,T]\} = L^*$, it is immediate that $Ad(\exp \tau X_m) = I_L$ for every $\tau \in \mathbb{R}$. This implies that $X_m \in \text{centre}(L)$ and the proof is complete. ∎

Corollary 1. *Let G be nonabelian and Σ, f, u_T, g be as in Proposition 1. Assume that $Ad(g)X_i = X_i$, $[X_i, X_j] = 0$ for $i, j = 1, \ldots, k$. Then Σ cannot be infinitesimally observable at g, u_T).*

Proof. Suppose that Σ is infinitesimally observable at (g, u_T). The subset of G consisting of the points r such that Σ is infinitesimally observable at (r, u_T) is open (cf. Gauthier and Kupka, 1994, Theorem 2.0). Thus there exists an open neighborhood W of g such that Σ is infinitesimally observable at (g_1, u_T) for every $g_1 \in W$. There also exist an open neighborhood V of g (assume without loss of generality that $V \subset W$) and an open neighborhood B of 0 in L such that the map $Z \to g \cdot \exp Z$ is a diffeomorphism of B onto V (cf. Varadarajan, 1984, p. 86). This means that every point $g_1 \in V$ has a unique expression of the form $g_1 = g \cdot \exp Z$ for some $Z \in B$ Hence:

$$Ad(g_1) = Ad(g)Ad(\exp Z)$$

Let $adZ(X) = [Z, X]$ for every $X \in L$. According to Proposition 1 $X_i \in \text{centre}(L)$ for every $i = 1, \ldots, k$. Then for $i = 1, \ldots, k$ it follows that

$$Ad(\exp Z)X_i = e^{adZ}(X_i) = X_i$$

This implies that $Ad(g_1)X_i = Ad(g)X_i = X_i$. Applying again Proposition 1 it is immediate that $g_1 \in \text{centre}(G)$ for every $g_1 \in V$ which contradicts the hypothesis of nonabelian G. ■

Corollary 2. *Let G be nonabelian and Σ, f, u_T be as in Proposition 1. If $X_1, \ldots, X_k \in \text{centre}(L)$, then Σ with u_T cannot distinguish between any two distinct initial states.*

Proof. If Σ with u_T distinguishes between any two distinct initial states, then the set of points g such that Σ is infinitesimally observable at (g, u_T) is dense in G (cf. Gauthier and Kupka, 1994, Theorem 2.0). As in the proof of Corollary 1, write every $g \in V$ with V an open neighborhood of e, as $g = \exp Z$ with $Z \in B$, where B is an open neighborhood of 0 in L. Then for $g \in V$, $i = 1, \ldots, k$,

$$Ad(g)X_i = Ad(\exp Z)X_i = e^{adZ}(X_i) = X_i$$

Thus, from Corollary 1 it follows that Σ cannot be infinitesimally observable at (g, u_T) for every $g \in V$ which finishes the proof. ■

Example. Let G be the connected component of $GL(2, \mathbb{R})$, the group of 2×2 invertible real matrices, which contains the identity element. Then $L = \text{Lie}(G)$ consists of all 2×2 real matrices. Let also Σ be an invariant system on G and $X_1, X_2, X_3 \in L$ with

$$X_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad X_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad X_3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

Assume that $u_3 = \{(X_1, 1),\ (X_2, 1),\ (X_3, 1)\}$ is a control law for Σ. Consider also the invariant output function $f : G \to \mathbb{R}$ with $f(g) = \text{trace}(g)$. G is equipped with the natural topology of $\mathbb{R}^4$ and has a global coordinate system. Write $g = \begin{bmatrix} x & y \\ z & w \end{bmatrix}$. Then $\mathrm{d}f_g = \left(\frac{\partial f}{\partial x}\Big|_g \ \frac{\partial f}{\partial y}\Big|_g \ \frac{\partial f}{\partial z}\Big|_g \ \frac{\partial f}{\partial w}\Big|_g \right)$. After a simple calculation it can easily be shown that $\phi(g) = (x \ z \ y \ w)$. Furthermore,

$$\phi\Big(e(0, u_3)\Big) = \Big(1 \quad 0 \quad 0 \quad 1\Big)$$

$$\phi\Big(e(1, u_3)\Big) = \Big(1 \quad 0 \quad 0 \quad \exp(1)\Big)$$

$$\phi\Big(e(2, u_3)\Big) = \Big(1 \quad \exp(1) \quad 0 \quad \exp(1)\Big)$$

$$\phi\Big(e(3, u_3)\Big) = \Big(\cos(1) \quad \exp(1)\Big(\cos(1) - \sin(1)\Big) \quad \sin(1) \quad \exp(1)\Big(\cos(1) + \sin(1)\Big)\Big)$$

Thus Σ is infinitesimally observable at (e, u_3), since $\text{span}\{\phi(e(t, u_3)) : t \in [0, 3]\} = L^*$. Replace now X_1, X_2 with $X_{12} = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$. Applying Corollary 1 it is immediate that Σ is not infinitesimally observable at (e, u_3) since $[X_{12}, X_3] = 0$.

4. Conclusion

In this paper infinitesimal observability of systems on Lie groups was studied. The structure of these systems was incorporated in order to obtain an infinitesimal observability criterion. In the special case of piecewise constant inputs, the aforementioned criterion leads to simpler tests which also relate infinitesimal observability to local observability. Finally, an example was given to illustrate the results.

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