REALIZATION PROBLEM FOR DISCRETE-TIME POSITIVE LINEAR SYSTEMS

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Sufficient conditions for the existence of a realization of a given proper transfer matrix are established. A procedure for calculation of a realization of a transfer matrix is given and illustrated by an example.

1. Introduction

A positive linear system is a dynamical system in which the input, state, and output space are spaces over the positive real numbers. Positive linear systems have been used in biomathematics, economics, chemometrics and other research areas (Anderson *et al.*, 1996; Berman and Plemmous, 1979; Minc, 1988; Ohta *et al.*, 1984; Van den Hof, 1995; 1996). A special class of positive linear systems is that of compartmental systems (Maeda *et al.*, 1977; Van den Hof, 1996). The reachability, observability and realizability of continuous-time positive systems have been discussed in (Ohta *et al.*, 1984). The realization problem of a given positive impulse response function has been considered in (Anderson *et al.*, 1996; Van den Hof, 1995) and the realization of the n-th-order linear difference equation has been treated in (Maeda and Kodama, 1981).

In this paper, the realization problem of a given proper transfer matrix is discussed. Sufficient conditions for the existence of a realization of a given transfer matrix are established and a procedure for calculation of the realization is presented.

2. Preliminaries and Problem Formulation

Let $\mathbb{R}_+$ be the set of nonnegative real numbers and let $\mathbb{Z}_+$ be the set of nonnegative integers. Denote by $\mathbb{R}_+^n$ the set of n-tuples of nonnegative real numbers and by $\mathbb{R}_+^{n\times m}$ the set of nonnegative real $n\times m$ matrices. The set of rational $p\times m$ matrices in variable z with real coefficients will be denoted by $\mathbb{R}^{p\times m}(z)$.

Consider a linear discrete-time system described by the equations

$$\begin{aligned} x_{i+1} &= Ax_i + Bu_i \\ y_i &= Cx_i + Du_i \end{aligned} \qquad i \in \mathbb{Z}_+ \tag{1}$$

where $x_i \in \mathbb{R}^n$ is the state vector, $u_i \in \mathbb{R}^m$ is the input vector, $y_i \in \mathbb{R}^p$ is the output vector, and A, B, C and D are real matrices.

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Definition 1. The system (1) is called a discrete-time positive linear system **if** for every $x_0 \in \mathbb{R}_+^n$ and for all $u_i \in \mathbb{R}_+^m$, $i \in \mathbb{Z}_+$ we have $x_i \in \mathbb{R}_+^n$ and $y_i \in \mathbb{R}_+^p$ for $i \in \mathbb{Z}_+$.

It is easy to show that the system (1) is a positive linear system if and only if

$$A \in \mathbb{R}_+^{n\times n}, \quad B \in \mathbb{R}_+^{n\times m}, \quad C \in \mathbb{R}_+^{p\times n}, \quad D \in \mathbb{R}_+^{p\times m} \tag{2}$$

The transfer matrix of (1) is given by

$$T(z) = C(I_n z - A)^{-1}B + D \in \mathbb{R}^{p\times m}(z) \tag{3}$$

where I_n is the $n\times n$ identity matrix.

Definition 2. (Kaczorek, 1992) Matrices A, B, C, D are called a realization of a given $T(z)$ if they satisfy (3). A realization is called minimal if the matrix A has minimal dimension among all realizations of $T(z)$.

The problem under consideration can be stated as follows: Given a proper transfer matrix $T(z) \in \mathbb{R}^{p\times m}(z)$ we shall:

(1) establish conditions on $T(z)$ under which there exists a realization (2) of $T(z)$,

(2) give a procedure for calculation of a realization (A, B, C, D) of the linear positive system (1) of $T(z)$.

3. Existence of a Realization of a Given Transfer Matrix

Using (3) it is easy to show (Kaczorek, 1992) that

$$T_\infty := \lim_{z\to\infty} T(z) = D \tag{4}$$

The strictly proper transfer matrix

$$T_{sp}(z) = T(z) - D \tag{5}$$

may be written as

$$T_{sp}(z) = \frac{N(z)}{d(z)} \tag{6}$$

where $N(z)$ is the $p\times m$ polynomial matrix in z and

$$d(z) = z^q + a_{q-1}z^{q-1} + \cdots + a_1 z + a_0 \tag{7}$$

To simplify the considerations, let us assume first that $d(z) = 0$ has only distinct roots $z_1, z_2, \ldots, z_q$, i.e. $d(z) = (z - z_1)(z - z_2)\cdots(z - z_q)$. In this case, (5) may be written in the form

$$T_{sp}(z) = \sum_{k=1}^{q} \frac{T_k}{z - z_k} \tag{8}$$

where

$$T_k := \lim_{z \to z_k} (z - z_k) T_{sp}(z) = \frac{N(z_k)}{\prod\limits_{\substack{i=1 \\ i \neq k}}^{q} (z_k - z_i)} \tag{9}$$

Theorem 1. *There exists a realization (2) of a given transfer matrix $T(z)$ if:*

(i) $z_k \in \mathbb{R}_+$ *for* $k = 1, \dots, q$

(ii) $T_\infty \in \mathbb{R}_+^{p \times m}$ *and* $T_k \in \mathbb{R}_+^{p \times m}$ *for* $k = 1, \dots, q$

Moreover, a realization of $T(z)$ is given by

$$A = \operatorname{diag}\Big[I_{r_1} z_1, I_{r_2} z_2, \dots, I_{r_q} z_q\Big] \in \mathbb{R}_+^{n \times n}, \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix} \in \mathbb{R}_+^{n \times m} \tag{10}$$

$$C = [C_1 \, C_2 \dots C_q] \in \mathbb{R}^{p \times n}, \quad D = T_\infty$$

where

$$r_k = \operatorname{rank}_+ T_k, \quad k = 1, \dots, q, \quad n = r_1 + r_2 + \cdots + r_q \tag{11}$$

(function of the $\operatorname{rank}_+ T_k$ *is defined in Appendix),*

$$T_k = C_k B_k, \quad C_k \in \mathbb{R}_+^{p \times r_k}, \quad B_k \in \mathbb{R}_+^{r_k \times m}, \quad k = 1, \dots, q \tag{12}$$

Proof. Note that $z_k \in \mathbb{R}_+$, $k = 1, \dots, q$ implies $A \in \mathbb{R}_+^{n \times n}$. By Lemma A1 (given in Appendix), if $T_k \in \mathbb{R}_+^{p \times m}$, then there exist matrices $B_k \in \mathbb{R}_+^{r_k \times m}$, $C_k \in \mathbb{R}_+^{p \times r_k}$ such that $T_k = C_k B_k$ for $k = 1, \dots, q$ and $B \in \mathbb{R}_+^{n \times m}$, $C \in \mathbb{R}_+^{p \times n}$. From (4) and the assumption $T_\infty \in \mathbb{R}_+^{p \times m}$ it follows that $D \in \mathbb{R}_+^{p \times m}$.

To show that (10) is a realization of $T(z)$ using (8) and (5) we calculate

$$C(I_n z - A)^{-1} B + D = [C_1, C_2, \dots, C_q]$$

$$\times \operatorname{diag}\Big[I_{r_1}(z - z_1)^{-1}, I_{r_2}(z - z_2)^{-1}, \dots, I_{r_q}(z - z_q)^{-1}\Big] \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}$$

$$+ T_\infty = \sum_{k=1}^{q} \frac{C_k B_k}{z - z_k} + T_\infty = \sum_{k=1}^{q} \frac{T_k}{z - z_k} + T_\infty = T(z)$$

Thus, the matrices (10) are a realization of $T(z)$. ■

4. Procedure for Calculation of a Realization

From Theorem 1, we have the following procedure for calculating a realization (2) of a given $T(z)$ satisfying the assumptions (i) and (ii).

Procedure:

Step 1. Using (4) find $D = T_\infty$.

Step 2. Using (5), (6), (7) and (9) find $T_{sp}(z)$, roots $z_1, z_2, \dots, z_q$ of $d(z) = 0$ and T_k for $k = 1, \dots, q$.

Step 3. Using Lemma A1 find $C_k \in \mathbb{R}_+^{p \times r_k}$ and $B_k \in \mathbb{R}_+^{r_k \times m}$ satysfying (12), where r_k is defined by (11).

Step 4. Find the desired realization (10).

The procedure is illustrated by the following example.

5. Example

Find a realization (2) of the transfer matrix

$$T(z) = \begin{bmatrix} \dfrac{2z-4}{(z-1)(z-3)} & 0 & \dfrac{3z-7}{(z-2)(z-3)} \\ \dfrac{3}{z-3} & \dfrac{2z-3}{(z-1)(z-2)} & \dfrac{2}{z-3} \end{bmatrix} \tag{13}$$

In this case, $p = 2$, $m = 3$, $T_\infty = 0$ and $T_{sp}(z) = T(z)$.

The equation

$$d(z) = (z-1)(z-2)(z-3) = 0$$

has the distinct roots $z_1 = 1$, $z_2 = 2$, $z_3 = 3$. Using (9), we obtain

$$T_1 = \lim_{z \to 1}(z-1)T(z) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T_2 = \lim_{z \to 2}(z-2)T(z) = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$T_3 = \lim_{z \to 3}(z-3)T(z) = \begin{bmatrix} 1 & 0 & 2 \\ 1 & 0 & 2 \end{bmatrix}$$

Hence, $\operatorname{rank} T_1 = \operatorname{rank} T_2 = 2$, $\operatorname{rank} T_3 = 1$, $r_1 = r_2 = 2$, $r_3 = 1$, $n = r_1 + r_2 + r_3 = 5$ and we may choose

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_2 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B_3 = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix}, \quad C_3 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Using (10), we obtain

$$A = \operatorname{diag}\Big[I_{r1}z_1,\ I_{r2}z_2,\ I_{r3}z_3\Big] = \operatorname{diag}\Big[1, 1, 2, 2, 3\Big]$$

$$B = \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 2 \end{bmatrix} \tag{14}$$

$$C = \Big[C_1 \quad C_2 \quad C_3\Big] = \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \end{bmatrix}, \quad D = 0$$

It is easy to check that the matrices (14) are a realization (2) of (13).

If the equation $d(s) = 0$ has the roots $z_1, z_2, \ldots, z_q$ with multiplicities $m_1, m_2, \ldots, m_q$, respectively, then the strictly proper transfer matrix (6) may be written in the form

$$T_{sp}(z) = \sum_{k=1}^{q} \sum_{i=1}^{m_k} \frac{T_{ki}}{(z - z_k)^{m_k - i + 1}} \tag{15}$$

where

$$T_{ki} = \frac{1}{(i-1)!} \frac{\mathrm{d}^{i-1}}{\mathrm{d}z^{i-1}} \Big[(z - z_k)^{m_k} T(z)\Big]\Big|_{z=z_k} \tag{16}$$

In this case

$$T_{ki} = \sum_{j=1}^{i} C_{kj} B_{k, m_k + j - i} \quad \text{for} \quad k = 1, \ldots, q; \quad i = 1, \ldots, m_k \tag{17}$$

and a realization of $T_{ps}(z)$ has the form

$$A = \operatorname{diag}\Big[J_1, J_2, \ldots, J_q\Big], \quad B = \begin{bmatrix} B_1 \\ B_2 \\ \vdots \\ B_q \end{bmatrix}, \quad C = \Big[C_1, C_2, \ldots, C_q\Big] \tag{18}$$

where

$$J_k = \begin{bmatrix} z_k & 1 & 0 & \cdots & 0 & 0 \\ 0 & z_k & 1 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & z_k & 1 \\ 0 & 0 & 0 & \cdots & 0 & z_k \end{bmatrix} \in \mathbb{R}_+^{m_k \times m_k} \qquad (k = 1, \dots, q) \tag{19}$$

$$B_k = \begin{bmatrix} B_{k_1} \\ B_{k_2} \\ \vdots \\ B_{km_k} \end{bmatrix}, \qquad C_k = [C_{k_1}, C_{k_2}, \dots, C_{km_k}]$$

Taking into account that

$$(I_{m_k} z - J_k)^{-1} = \begin{bmatrix} \frac{1}{z - z_k} & \frac{1}{(z - z_k)^2} & \cdots & \frac{1}{(z - z_k)^{m_k - 1}} \\ 0 & \frac{1}{z - z_k} & \cdots & \frac{1}{(z - z_k)^{m_k - 1}} \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & \frac{1}{z - z_k} \end{bmatrix}$$

we obtain

$$C_k(I_{m_k} z - J_k)^{-1} B_k = \frac{1}{z - z_k} \sum_{i=1}^{m_k} C_{ki} B_{ki} + \frac{1}{(z - z_k)^2} \sum_{i=1}^{m_k - 1} C_{kiB_k,\, i+1} + \cdots$$

$$+ \frac{1}{(z - z_k)^{m_k}} C_{k1} B_{km_k} = \sum_{j=1}^{m_k} \frac{T_{kj}}{(z - z_k)^{m_k - j + 1}}$$

Hence from (18), (19), (17) and (15) we have

$$C(Iz - A)^{-1} B = \sum_{k=1}^{q} C_k (I_{m_k} z - J_k)^{-1} B_k = \sum_{k=1}^{q} \sum_{j=1}^{m_k} \frac{T_{kj}}{(z - z_k)^{m_k - j + 1}} = T_{sp}(z)$$

Thus, the matrices (18), (19) are a realization of $T_{sp}(z)$.

In the general case, the condition $T_k \in \mathbb{R}_+^{p \times m}$ should be replaced by $T_{ki} \in \mathbb{R}_+^{p \times m}$ for $k = 1, \dots, q$ and $i = 1, \dots, m_k$.

6. Concluding Remarks

Sufficient conditions for the existence of a realization (2) of a given proper transfer matrix $T(z)$ have been established. A procedure for calculating the realization (2) of $T(z)$ has been presented. The problem of the relationship between the minimality of a realization obtained by the procedure and its controllability and observability will be considered in a successive paper.

Appendix

Let $A \in \mathbb{R}_+^{m\times m}$ and r be its positive column rank defined as follows (Cohen, 1993). The positive column rank of A (denoted by $\operatorname{rank}_+ A$) is the smallest nonnegative integer r for which there exist linearly independent vectors $v_i \in \mathbb{R}_+^m$, $i = 1, \dots, r$ such that each column of A is a linear combination of vectors with nonnegative coefficients.

It can be shown (Cohen, 1993) that

$$\operatorname{rank} \le \operatorname{rank}_+ A \le \min(m, n)$$

and, if for $A \in \mathbb{R}_+^{m\times n}$, $m < 4$, and $n < 4$, then $\operatorname{rank}_+ A = \operatorname{rank} A$.

Lemma A1. *Let* $A \in \mathbb{R}_+^{m\times n}$ *and* $r = \operatorname{rank}_+ A$. *Then there exists a pair of matrices* $B \in \mathbb{R}_+^{r\times m}$, $C \in \mathbb{R}_+^{p\times r}$ *such that* $A = CB$.

Proof. Let $r < \min(m,n)$ and $C := [v_1, v_2, \dots, v_r] \in \mathbb{R}_+^{m\times r}$. By definition of the positive column rank, the j-th column A_j of A is a linear combination of the vectors $v_1, v_2, \dots, v_r$ with nonnegative coefficients b_{ij}, i.e. $A_j = \sum_{i=1}^{r} b_{ij} v_i$ for $j = 1, \dots, n$ or $A = CB$ where $B = \Big[b_{ij}\Big]\Big|_{\substack{i=1,\dots,r \\ j=1,\dots,n}} \in \mathbb{R}_+^{r\times n}$.

If $r = m$, then we may choose $C = I_m$ and $B = A$. Similarly, if $r = n$, then we may choose $B = I_n$ and $C = A$. ■

Remark. If $\operatorname{rank}_+ A = \operatorname{rank} A = r < \min(m,n)$, then we may choose r linearly independent columns of A as the columns of $C \in \mathbb{R}_+^{m\times r}$. Using elementary row operations, we perform the reduction

$$\left[C \ \vdots \ A \right] \to \left[\begin{array}{c:c} I_r & V \\ \hdashline 0 & 0 \end{array} \right]$$

It is easy to show that $B = V$ and $B \in \mathbb{R}_+^{r\times m}$.

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