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# CONTROLLABILITY FOR MECHANICAL SYSTEMS WITH SYMMETRIES AND CONSTRAINTS

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This paper derives controllability tests for a large class of mechanical systems characterized by nonholonomic constraints and symmetries. Recent research in geometric mechanics has led to a single, simplified framework that describes this class of systems, which includes examples such as wheeled mobile robots; undulatory robotic and biological locomotion systems, such as paramecia, inchworms, and snakes; as well as the reorientation of satellites and underwater vehicles. This geometric framework has also been applied to more unusual examples, such as the snakeboard robot, the wobblestone, and the reorientation of a falling cat. Using results from modern nonlinear control theory, we develop accessibility and controllability tests based on this reduced geometric structure. We also discuss parallels between these tests and the construction of open-loop control algorithms, with analogies to the generation of locomotive gaits for robotic systems.

## 1. Introduction

Mechanical systems provide a fertile area of study for researchers interested in nonlinear control, due to the inherent nonlinearities of these systems, and the Lagrangian structure that they possess. Recently, a great deal of emphasis has been placed on studying systems with nonholonomic (non-integrable) constraints, including mobile wheeled robots and multiple-trailer vehicles, where the wheels provide a no-slip velocity constraint. For the purposes of controls, however, these systems are very often treated as kinematic systems, i.e., the dynamics of these mechanical systems are assumed to be inverted out. Very often this assumption is quite valid, but frequently it is not. There are also a growing number of systems in which this type of assumption is not even approximately valid. We focus in this paper on one such class of systems, namely the class of systems with nonholonomic constraints and symmetries. We will make the definition of symmetries more precise below; however, for mechanical systems symmetries essentially imply an invariance of the system, often with respect to inertial positioning. In the absence of external constraints, these symmetries lead to momentum conservation laws. Motivating the study of this class of systems is a large array of examples of undulatory locomotion systems. These examples will be used throughout this paper to illustrate and motivate the developments contained herein.

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Making use of modern advances in geometric mechanics, researchers have made great progress in analyzing the mechanics of locomotion. This problem asks the fundamental question of how a system uses its control inputs to effect motion from one place to another. By utilizing the inherent mathematical structure found in these types of problems, one can formulate the dynamics of a wide variety of locomotion problems in a very intuitively appealing and insightful manner. Doing so leads to a stronger comprehension of the *mechanics* of locomotion, but leaves open some very basic questions about the *control* of such systems. Preliminary research (Kelly and Murray, 1994; Krishnaprasad and Tsakiris, 1994) suggests that the geometric tools used to formulate the mechanics will also provide a basis for unlocking the answers to many of the questions regarding the control theoretic issues involved.

An important by-product of the mechanics research in locomotion has been the development of a theoretical bridge between systems with two different types of nonholonomic constraints. On one hand, there are systems with *external* (often called kinematic) constraints which include wheeled vehicles (Murray and Sastry, 1993), grasping with point-finger contacts, and some models of snakes (Krishnaprasad and Tsakiris, 1994), paramecia (Shapere and Wilczek, 1989), and even legged locomotion (Goodwine and Burdick, 1996; Kelly and Murray, 1994). Recent work by Kelly and Murray (1994) provides kinematically constrained models for a wide range of systems. However, the models are again restricted to purely *kinematic* systems, which require active input controls to generate movement. A kinematically constrained body in motion will remain in motion only if its control inputs are continually active. Thus it is not possible to build momentum or to "coast"—it is exactly this component of locomotion that has been added in the models considered here.

Characteristic of all of these systems, however, is a second type of nonholonomic constraint, which arises due to Lie group symmetries. For locomotive systems, this is often a "pick-and-place" symmetry, whereby the rigid body dynamics are invariant with respect to inertial positioning. In the absence of external constraints, these invariances imply the existence of *internal* (sometimes called dynamic) constraints on the system, which very often take the form of momentum conservation laws. Examples of systems with internal nonholonomic constraints include satellites in space (Krishnaprasad, 1990; Nakamura and Mukherjee, 1995) and the problem of the "falling cat" (Montgomery, 1990).

Naturally, there exist problems for which both internal and external constraints may exist and interact in a nontrivial manner. Examples of these problems fall generally into two realms: one in which certain of the conservation laws may remain after the addition of constraints, such as in the rolling penny or the constrained particle (Bloch *et al.*, 1996); and one in which the conservation laws are transformed into what Bloch *et al.* term a *generalized momentum equation*, where the momenta are governed by a differential equation. There is strong evidence to suggest that many different modes of locomotion (such as undulatory, legged, etc.) are governed by equations of this form. What is present in the case of mixed constraints (i.e., kinematic *and* dynamic) is the ability to change the momentum of a locomotive body. This is crucial for many types of locomotion, such as running or swimming. Thus, by using internal, *shape* controls it is possible not only to change the position of the system, but to generate velocities and hence truly *locomote* in a dynamic sense.

Recent investigations have led to a unifying geometric framework within which to analyze these types of problems. Along with the problem of mechanics comes a host of associated issues to investigate, including (optimal) control, stabilization, trajectory generation, and path planning. Certainly, extensive work has been done in analyzing these issues for systems with purely kinematic *or* purely dynamic constraints. We will highlight some of the more recent theoretical results on controllability for these types of systems here. While certain initial results do exist for *dynamic* mechanical systems (Bloch *et al.*, 1992), they generally require that the unconstrained dynamics be fully actuated. While this is a stronger result than those derived for kinematic systems, these assumptions still require the continual motion of the inputs to generate motion of the system; that is, there are no "dynamic," or momentum, effects present. In deriving the conditions for controllability presented here, the authors have employed the most advanced tools of which we are presently aware for showing small-time local controllability of nonlinear control systems with drift (Sussman, 1987). For a nice review of the issues involved in local controllability tests, the reader is referred to (Kawski, 1990). Also, since the structure of the equations was largely motivated by developments in locomotion, some mention will be given to the relationship between the controllability tests derived here and trajectory/gait generation. As an example, we will examine the snakeboard model, which has been an important motivating example behind the theoretical progress for this mixed kinematic and dynamic constraint case (Bloch *et al.*, 1996; Ostrowski *et al.*, 1994; 1995).

## 2. Background and Problem Formulation

The use of Lie groups will be important for the analysis performed in this paper. The principal motivation for using Lie groups arises from our studies of robotic locomotion, where displacements occur in some subgroup of $SE(3)$, most often translation and rotation in the plane, $SE(2)$, or rigid body rotation, $SO(3)$. However, the analysis here is valid for general mechanical systems which have some or all of their dynamics evolving on a Lie group. In order to appeal to the general community's intuition of rigid body motion, we will very often make reference to Lie groups as describing position and orientation of a robotic system, but the reader should keep in mind that the results hold much more generally.

Formally, the displacement of a robot's body fixed frame is considered as a *left translation*. That is, if the initial position of a rigid body is denoted by $g$, and it is displaced by an amount $h$, then its final position is $hg$. This displacement can be thought of as a map $L_h : G \to G$ given by $L_h(g) = hg$ for $g \in G$. Hence, we can describe the evolution of the *position* of the robot by referencing it using a Lie group with respect to some inertial frame.

The remaining components of the system are assumed to be controllable, and these configuration variables will be represented by a manifold $M$. Most often for locomotion systems, these variables will describe the internal *shape* of the system. Again, this designation of the controlled subspace with internal shape is convenient for giving an intuitive picture of a locomotive body, but it should be remembered that the manifold $M$ is quite general, and so can describe whatever variables are necessary to complement the Lie group, $G$. Thus, the configuration manifold will be the product manifold given by $Q = G \times M$ and the left translation induces a *left action* of $G$ on $Q$. For those familiar with the mechanics literature, the manifold $Q$ is said to define a *trivial principal fiber bundle* with *fibers*, $G$, over a *base* space, $M$. We will assume the coordinates on $Q$ to be decomposable into fiber and base coordinates, i.e., for $q \in Q$, we can write $q = (g, r) \in G \times M$.

**Definition 1.** A *left action* of a Lie group $G$ on a manifold $Q$ is a smooth map $\Phi : G \times Q \to Q$ such that: (1) $\Phi(e, q) = q$ for all $q \in Q$, where $e$ stands for the identity element of $G$; and (2) $\Phi(h, \Phi(g, q)) = \Phi(hg, q)$ for every $g, h \in G$ and $q \in Q$.

It will be useful to consider the left action as a map from $Q$ into $Q$, with the element $h \in G$ held fixed. Notationally, $\Phi_h : Q \to Q$ is given by $(g, r) \mapsto (\Phi(h, g), r) = (hg, r)$. The *lifted action*, which describes the effect of $\Phi_h$ on velocity vectors in $TQ$, is the linear map, $T_q\Phi_h : T_qQ \to T_{hq}Q$. This is similar to a Jacobian matrix corresponding to the mapping $\Phi_h$, often written as $D_q\Phi_h$ or $(\Phi_h)_*$.

Since we are working with mechanical systems, we will assume the existence of a Lagrangian function, $L(q, \dot{q})$, on $TQ$. In the absence of constraints, the robot's dynamical equations can be derived from Lagrange's equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - \tau_i = 0 \tag{1}$$

where $\tau$ is a forcing function. Our interest, however, is in systems in which nonholonomic constraints are present. These constraints can take many forms, including no-slip wheel conditions and viscous friction. Let us restrict our attention to Pfaffian constraints, which are linear in the velocities. Given $k$ such constraints, we can write them as a vector-valued set of $k$ equations:

$$\omega^i_j(q)\dot{q}^j = 0, \qquad \text{for} \quad i = 1, \ldots, k \tag{2}$$

This class of constraints includes most commonly investigated nonholonomic constraints.

The constraints can be incorporated into the dynamics through the use of Lagrange multipliers. That is, eqn. (1) is modified by adding a force of constraint with an unknown multiplier, $\lambda$, as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} + \lambda_j\omega^j_i - \tau_i = 0 \tag{3}$$

However, once the Lagrange multipliers are solved for, much of the physical intuition of the problem may be lost. This is very often done by choosing generalized coordinates to represent velocities in the unconstrained directions. Doing so is usually an *ad-hoc* procedure that we have found to be greatly improved by using the formalism and structure of Lie groups (Ostrowski, 1995; Ostrowski and Burdick, 1996b).

For systems in which the Lagrangian and the constraints are left-invariant, i.e., for which

$$L(\Phi_h q, T_q\Phi_h\dot{q}) = L(q, \dot{q}) \quad \text{and} \quad \omega^i(q)\dot{q} = \omega^i(h^{-1}q)T_q\Phi_{h^{-1}}\dot{q}, \quad \forall\, h \in G, q \in Q \tag{4}$$

it was shown in (Bloch *et al.*, 1996; Ostrowski, 1995) that the equations of motion can be transformed into the following form:

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} + \mathbb{I}^{-1}(r)p \tag{5}$$

$$\dot{p} = \frac{1}{2}\dot{r}^T\sigma_{\dot{r}\dot{r}}(r)\dot{r} + p^T\sigma_{p\dot{r}}(r)\dot{r} + \frac{1}{2}p^T\sigma_{pp}(r)p \tag{6}$$

$$\ddot{r} = u \tag{7}$$

These equations, of course, deserve a good deal of comment (to gain a much better insight into these equations, the reader is referred to the paper (Bloch *et al.*, 1996)). Equations (5) and (7) are the fiber and base equations, respectively. They will define velocity vectors for the configuration variables (and accelerations for the base variables). Equations (6) is called the *generalized momentum equation*, where $p$ is a momentum vector associated with the momentum along each of the kinematically unconstrained fiber directions. Notice that in (7) we have assumed the base (shape) space to be fully controllable, with acceleration inputs, $u$.

Let us briefly examine how this set of equations reduces in the more familiar limiting cases. First, in the principal kinematic case, the shape space is assumed to be controlled (eqn. (7)), and the invariance of the constraints leads to the kinematic equation

$$g^{-1}\dot{g} = -\mathbb{A}(r)\dot{r} \tag{8}$$

Examples such as the wheeled mobile robot, the $N$-trailer system, and inchworm locomotion can be put in this form.

On the other hand, if there are no constraints, but the Lagrangian is invariant (the first half of (4)), then one can define a momentum for the system that is conserved. In this case, (5) and (6) simplify to

$$p = \frac{\partial L}{\partial \dot{q}}(g, r, \dot{g}, \dot{r}) \quad \Rightarrow \quad g^{-1}\dot{g} = -\mathsf{A}(r)\dot{r} + \mathbb{I}^{-1}(g, r)p \tag{9}$$

$$\dot{p} = 0 \tag{10}$$

Notice that in both cases, the symmetries allow us to pull the group variable $g$ out of the equation. In the controllability analysis of (5)–(7) this fact will be used to greatly simplify the necessary calculations.

Returning to eqns. (5)–(7), notice the central role played by the term $\mathbb{A}(r)$ in (5) ($\mathsf{A}(r)$ in eqn. (9)). In the language of geometric mechanics, $\mathbb{A}$ is said to define a *connection* on the trivial principal fiber bundle $Q$. The connection satisfies certain geometric properties, the most important of which is to specify the relationship between control velocities on $TM$ and spatial (group) velocities on $TG$. In investigating issues of controllability, the connection is extremely useful, because it defines the role of the control inputs in generating spatial motion along the fiber. In testing for controllability, we will see that derivatives of $\mathbb{A}$ have a direct correspondence to Lie brackets of the control and drift vector fields. One of the advantages of dealing with $\mathbb{A}$ directly is that we bypass the need to deal with parameterizations of the manifold, instead of performing calculations on the Lie algebra directly. This can be particularly useful for Lie groups such as $SO(3)$ and $SE(3)$.

The $\mathbb{I}^{-1}p$ term determines the effect of the momentum on the fiber equations. $\mathbb{I}$ is called the *locked inertia tensor*. Its development is beyond the scope of this paper, but essentially $\mathbb{I}$ corresponds to the inertia of the system given a particular (locked) configuration of the shape variables. For the terms, $\sigma_{\dot{r}\dot{r}}$, $\sigma_{p\dot{r}}$, and $\sigma_{pp}$ of the generalized momentum equation, we mention only that they are strictly functions of the base variables, $r$, and so the generalized momentum equation can be written as a function *only* of the base and momentum variables (Ostrowski, 1995; Ostrowski and Burdick, 1996a). We have written (7) in terms of acceleration inputs so that the system of equations can be written in the standard form for a control system with drift:

$$\dot{z} = f(z) + h_i(z)u^i \tag{11}$$

Let $N = G \times \mathbb{R}^p \times M \times T_r M$, where $\dim \mathbb{R}^p$ is the dimension of the unconstrained fiber directions (i.e., the dimension of $p$). Then, using $z = (g, p, r, \dot{r}) \in N$, we see that (5)–(7) can be written in the form of (11) with

$$f(z) = \begin{pmatrix} g\Big( -\mathbb{A}(r)\dot{r} + \mathbb{I}^{-1}(r)p \Big) \\ \frac{1}{2}\dot{r}^T \sigma_{\dot{r}\dot{r}}\dot{r} + p^T\sigma_{p\dot{r}}\dot{r} + \frac{1}{2}p^T\sigma_{pp}p \\ \dot{r} \\ 0 \end{pmatrix}, \qquad h_i(z) = \begin{pmatrix} 0 \\ 0 \\ 0 \\ e_i \end{pmatrix} \tag{12}$$

where $e_i$ is the $m$-vector ($m = \dim M$) with a "1" in the $i$-th row and "0" otherwise.

## 2.1. The Snakeboard Example

Now let us turn to a formulation of the snakeboard problem in terms of the relationships derived above. The *Snakeboard* is a commercial variant of the skateboard, which allows for independent rotation of the wheel trucks. The simplified model of the *Snakeboard* (referred to as the snakeboard model) is shown in Fig. 1, along with a robotic version built in our lab to verify theoretical simulations (shown in Fig. 1). We will briefly recall the description of the snakeboard as developed in (Ostrowski *et al.*, 1994). As a mechanical system the snakeboard has a configuration manifold given by $Q = SE(2) \times \mathbb{S} \times \mathbb{S} \times \mathbb{S}$. Here $SE(2)$ is the group of rigid motions in the plane, and is to be thought of as describing the position of the board with respect to some inertial reference frame. As coordinates for $Q$ we shall use $(x, y, \theta, \psi, \phi_b, \phi_f)$ where $(x, y, \theta)$ describes the position of the board with respect to a reference frame, $\psi$ is the angle of the rotor with respect to the board, and $\phi_b$ and $\phi_f$ are, respectively, the angles of the back and front wheels with respect to the board. Note that the wheels themselves are allowed to spin freely, just as with the traditional skateboard.

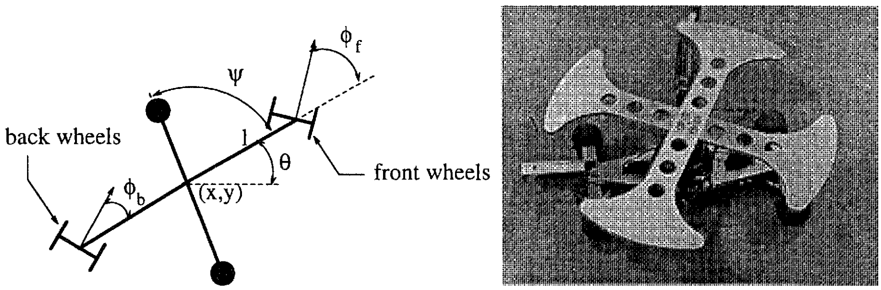


Fig. 1. The simplified model of the *Snakeboard*, along with a demo prototype.

The configuration space easily splits into a trivial fiber bundle structure, with $q = (g, r)$ given by $g = (x, y, \theta) \in G = SE(2)$ and $r = (\psi, \phi_b, \phi_f) \in M = \mathbb{S} \times \mathbb{S} \times \mathbb{S}$. The left action for a group element, $h = (a^1, a^2, \alpha) \in G$, is given by the map:

$$\begin{aligned} &\Phi_h(x, y, \theta, \psi, \phi_b, \phi_f) \\ &\quad = (x \cos\alpha - y \sin\alpha + a^1, x \sin\alpha + y \cos\alpha + a^2, \theta + \alpha, \psi, \phi_b, \phi_f) \end{aligned} \tag{13}$$

and the lifted action takes the form:

$$\begin{aligned} &T_q\Phi_h(\dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}, \dot{\phi}_b, \dot{\phi}_f) \\ &\quad = (\dot{x} \cos\alpha - \dot{y} \sin\alpha, \dot{x} \sin\alpha + \dot{y} \cos\alpha, \dot{\theta}, \dot{\psi}, \dot{\phi}_b, \dot{\phi}_f) \end{aligned} \tag{14}$$

Parameters for the problem are:

| | |
|---|---|
| $m$ | : the mass of the board, |
| $J$ | : the inertia of the board, |
| $J_r$ | : the inertia of the rotor, |
| $J_w$ | : the inertia of the wheels (assumed to be the same), and |
| $l$ | : the length from the board's center of mass to the wheels. |

For the snakeboard, the unconstrained Lagrangian is given simply by kinetic energy terms as

$$L = \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}J\dot{\theta}^2 + \frac{1}{2}J_r(\dot{\psi} + \dot{\theta})^2 + \frac{1}{2}J_w\Big((\dot{\phi}_b + \dot{\theta})^2 + (\dot{\phi}_f + \dot{\theta})^2\Big)$$

The control torques are assumed to be applied to the rotation of the wheels and the rotor. The wheels of the snakeboard are assumed to roll without lateral sliding. At the back wheels, this implies a nonholonomic constraint of the form

$$-\sin(\phi_b + \theta)\dot{x} + \cos(\phi_b + \theta)\dot{y} - l\cos(\phi_b)\dot{\theta} = 0 \tag{15}$$

Similarly at the front wheels the constraint appears as

$$-\sin(\phi_f + \theta)\dot{x} + \cos(\phi_f + \theta)\dot{y} + l\cos(\phi_f)\dot{\theta} = 0 \tag{16}$$

A quick set of calculations shows that both the Lagrangian and the constraint one-forms are invariant with respect to the lifted group action. The momentum is defined along directions which are tangent to the fiber and lie in the constraint distribution (i.e., satisfy the constraints). Thus, we define the *constrained fiber distribution* for this problem to be the one-dimensional subspace,

$$\mathcal{S}_q = \mathrm{sp}\left\{ a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial \theta} \right\}$$

where

$$a = -l\Big[\cos\phi_b \cos(\phi_f + \theta) + \cos\phi_f \cos(\phi_b + \theta)\Big]$$

$$b = -l\Big[\cos\phi_b \sin(\phi_f + \theta) + \cos\phi_f \sin(\phi_b + \theta)\Big]$$

$$c = \sin(\phi_b - \phi_f)$$

All vectors in $\mathcal{S}_q$ are tangent to the group $G$ and satisfy the constraints in (15) and (16). The generalized momentum, $p$, defined by Bloch *et al.* (1996) is then also one-dimensional. Let $\langle\!\langle , \rangle\!\rangle$ denote the inner product defined by the kinetic energy metric for our mechanical system. Then

$$\begin{aligned} p &= \langle\!\langle \dot{q}, X(q) \rangle\!\rangle \\ &= \langle\!\langle (\dot{x}, \dot{y}, \dot{\theta}, \dot{\psi}, \dot{\phi}_b, \dot{\phi}_f), (a, b, c, 0, 0, 0) \rangle\!\rangle \\ &= ma\dot{x} + mb\dot{y} + \hat{J}c\dot{\theta} + J_r c\dot{\psi} + J_w c(\dot{\phi}_b + \dot{\phi}_f) \end{aligned}$$

where $X(q) \in \mathcal{S}_q$ and $\hat{J} = J + J_r + 2J_w$ is the sum of the moments of inertia.

Before continuing with the derivation of the generalized momentum equation, we first make two simplifying assumptions that greatly reduce the complexity of the derivations to follow. First, we assume that the wheels are controlled to move out of phase with each other, in opposite directions. In other words, let $\phi = \phi_b = -\phi_f$. This is motivated by the motions seen by actual riders of the *Snakeboard*. Second, along the lines of Bloch *et al.*, we will assume that $\hat{J} = J + J_r + 2J_w = ml^2$. This will significantly simplify the analysis to follow, without overly constraining the example.

Then, writing the equations as in the form of (12):

$$\begin{aligned} g^{-1}\dot{g} &= -\mathbb{A}(r)\dot{r} + \mathbb{I}^{-1}(r)p \\ \dot{p} &= \frac{1}{2}\dot{r}^T \sigma_{\dot{r}\dot{r}} \dot{r} + p^T \sigma_{p\dot{r}} \dot{r} + \frac{1}{2} p^T \sigma_{pp} p \\ \ddot{r} &= u \end{aligned}$$

we find that for $\dot{r} = (\dot{\psi}, \dot{\phi})$,

$$\mathbb{A} = \begin{pmatrix} -\dfrac{J_r}{2ml}\sin 2\phi & 0 \\ 0 & 0 \\ \dfrac{J_r}{ml^2}\sin^2\phi & 0 \end{pmatrix}, \qquad \mathbb{I}^{-1} = \begin{pmatrix} -\dfrac{1}{2ml} \\ 0 \\ \dfrac{1}{2ml^2}\tan\phi \end{pmatrix}$$

The generalized momentum equation is

$$\dot{p} = 2J_r \cos^2\phi\, \dot{\phi}\dot{\psi} - \tan\phi\, \dot{\phi} p$$

Finally, we can write the shape space dynamics as

$$\left(1 - \frac{J_r}{ml^2}\sin^2\phi\right)\ddot{\psi} = \frac{J_r}{2ml^2}\sin 2\phi\, \dot{\phi}\dot{\psi} - \frac{1}{2ml^2}\dot{\phi}p + \frac{1}{J_r}\tau_\psi \tag{17}$$

$$\ddot{\phi} = \frac{1}{2J_w}\tau_\phi \tag{18}$$

which is easily seen to be controllable. As above, we rewrite this simply as

$$\ddot{\psi} = u_\psi \quad \text{and} \quad \ddot{\phi} = u_\phi \tag{19}$$

Let $z = (x, y, \theta, p, \psi, \phi, \dot{\psi}, \dot{\phi}) \in N$. Then we can write the snakeboard equations in the form of (12):

$$f = \begin{pmatrix} \dfrac{\cos\theta(-p + \dot{\psi} J_r \sin 2\phi)}{2ml} \\ \dfrac{\sin\theta(-p + \dot{\psi} J_r \sin 2\phi)}{2ml} \\ \dfrac{-2\dot{\psi} J_r \sin^2\phi + p\tan\phi}{2ml^2} \\ 2\dot{\phi}\dot{\psi} J_r \cos^2\phi - \dot{\phi} p \tan\phi \\ \dot{\psi} \\ \dot{\phi} \\ 0 \\ 0 \end{pmatrix}, \quad h_\psi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad h_\phi = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

### 2.2. Definitions: Accessibility and Controllability

In order to discuss control theoretic issues regarding a particular system, we must start by precisely defining the types of control goals we seek. In nonlinear control theory, there are two commonly used notions of control—*accessibility* and *controllability*. Putting aside technical definitions for a moment, we would like our control goal to be something like the following: "a system will be said to be controllable if, given any initial point $q_i$ and final point $q_f$, there exists an admissible control law $u$ which drives the system from $q_i$ to $q_f$." For general nonlinear systems, the notion of *small-time local controllability*, in which controllability is shown for local neighborhoods of $q_i$, will be the closest we can come to our goal of controllability. Note that it is still very much a local condition, and, while it is a much stronger condition than accessibility, it is also much more difficult to satisfy. Here we give definitions for these terms and in the next section present an example of how they differ.

Let $\mathcal{R}^V(z_0, T)$ denote the set of reachable points in $N$ from $z_0$ at time $T > 0$, using admissible controls, $u(t)$, and such that the trajectories remain in the neighborhood $V$ of $z_0$ for all $t \le T$. Furthermore, let

$$\mathcal{R}_T^V(z_0) = \bigcup_{t \le T} \mathcal{R}^V(z_0, t)$$

be the set of all reachable points from $z_0$ within time $T$. These two definitions lead us naturally to define the following:

**Definition 2.** (Nijmeijer and Van der Schaft, 1990) The system given by (11) is *locally accessible* if for all $z \in N$, $R_T^V(z)$ contains a non-empty open set of $N$ for all neighborhoods $V$ of $z$ and all $T > 0$.

**Definition 3.** (Sussman, 1987) The system given by (11) is called *small-time locally controllable* (STLC) if for any neighborhood $V$, time $T > 0$ and $z \in N$, $z$ is an interior point of $\mathcal{R}_T^V(z)$ for all $T > 0$.

For driftless systems, local accessibility and local controllability are equivalent. Notice, however, that the general types of systems in which we are interested will *require* the presence of a drift vector field, since this is how the momenta enter into the dynamic equations (notice the $\tilde{I}^{-1}p$ term in (7)). To give a motivating example of how these definitions differ, consider the problem of controlling an airplane in flight. The airplane can in a coarse sense be thought of as a system that is locally accessible, since it can basically reach an open set of points relative to its forward trajectory. It is, however, obviously not STLC, since the open neighborhood that it can reach after flying for some *small* time $T$ does not contain the point at which it started. Notice that here we emphasize that this only holds for small time, or in a local neighborhood. Using these same arguments (which can easily be formalized), one sees that mechanical systems can only be STLC around states with zero initial velocity. If our requirement for a system to be controllable were only that it be able to move between two points, then the airplane would satisfy this condition, since it could perform a circle in order to return to the starting point (or any point in an open neighborhood around the starting point). It is unclear as to what sense of controllability will be most important for the purposes of locomotion. To date, however, there are very few theoretical results concerning questions of global nonlinear controllability, and so we must be satisfied with investigating small-time local controllability.

### 2.3. The Lie Algebra Rank Condition

For general systems of the form:

$$\dot{z} = f(z) + h_i(z)u^i, \qquad z \in N$$

a standard method for determining accessibility is to compute the *accessibility distribution*. To do so, we define a sequence of distributions. Let

$$\Delta_0 = \operatorname{span}\{f, h_1, \dots, h_m\}$$

(the span taken over $C^\infty$ functions on $N$), and iteratively define

$$\Delta_k = \Delta_{k-1} + \operatorname{span}\left\{ [X,Y] \,\middle|\, X, Y \in \Delta_{k-1} \right\}$$

This is a nondecreasing sequence of distributions on $N$, and so terminates at some $k_f$, under certain regularity conditions. We will call $\Delta_{k_f}$ the *accessibility distribution*, and denote it by $C$:

$$C = \Delta_{k_f} = \Delta_\infty$$

A standard result from nonlinear control theory (based on the Frobenius Theorem), known as the Lie algebra rank condition (LARC), equates accessibility with the condition $C = TN$.

**Theorem 1.** *(LARC) If* $\dim C(z) = \dim T_z N$ *for all* $z \in N$*, then the system given by eqn. (11) is locally accessible.*

As a means of illustrating the calculations necessary to compute the accessibility distribution, we include the following example from (Nijmeijer and Van der Schaft, 1990, Example 3.14).

**Example 1.** Let $N = \mathbb{R}^2$ with coordinates $(z_1, z_2)$. Consider the system

$$\begin{pmatrix} \dot{z}_1 \\ \dot{z}_2 \end{pmatrix} = \begin{pmatrix} 0 \\ z_1^2 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix} u$$

We can write $C$ as the span of the vector fields

$$f = z_1^2 \frac{\partial}{\partial z_2}, \qquad h = \frac{\partial}{\partial z_1}$$
$$[h, f] = 2z_1 \frac{\partial}{\partial z_2}, \qquad \big[h, [h, f]\big] = 2\frac{\partial}{\partial z_2}$$

Thus, $\dim C(z) = \dim \mathbb{R}^2 = 2$ for all $(z_1, z_2) \in N$ and so Theorem 1 implies that this system is locally accessible. That is, the controls can always be used to reach an open 2-D subset of $\mathbb{R}^2$. Notice, however, that this system possesses a drift term so that $\dot{z}_2 \geq 0$ everywhere. Given a starting point, $(z_1^0, z_2^0)$, the reachable sets will consist only of points with $z_2 \geq z_2^0$ and hence will not contain $(z_1^0, z_2^0)$ in their interior. Therefore, this system is *not* STLC.

## 3. Review of Previous Work

### 3.1. The Principal Kinematic Case

Kelly and Murray (1995) have derived controllability results for the *principal kinematic* case. The kinematic case implies a driftless system; in this setting, accessibility and controllability are equivalent. The conditions they give for controllability will be useful in the present context for checking accessibility and controllability in systems where momentum terms drive the system. In the kinematic case, however, momenta arising from symmetries are annihilated by the nonholonomic constraints. Therefore, $p \equiv 0$, and the equations of motion reduce to

$$\begin{aligned} g^{-1}\dot{g} &= -\mathbb{A}(r)\dot{r} \\ \dot{r} &= u \end{aligned} \tag{20}$$

Given specified control inputs, the local form of the connection, $\mathbb{A}(r)$, thus determines the motion in the full configuration space.

By using the special structure provided by the Lie group symmetries, Kelly and Murray were able to derive straightforward computational conditions for controllability and suggest methods for generating desired trajectories. In their paper, they establish two important results that will be useful later. First, they observe that by taking the appropriate derivatives, the controllability analysis can be performed on the Lie algebra, i.e., at $g = e$. This is a very important point, as it implies that the controllability analysis can be performed independent of the group variables. The computational burden is thus dramatically reduced, and some of the challenges in dealing with parameterizations of manifolds can be avoided. This fact is used in our controllability tests—the special decoupled structure of the fiber equations makes this possible, as is seen in the explicit calculations below. Furthermore, Kelly and Murray show that the controllability of a kinematic system can be determined solely from the local form of the connection, $\mathbb{A}$, its curvature, and higher covariant derivatives. The reader unfamiliar with exterior derivatives of differential forms is referred to (Abraham *et al.*, 1988).

**Definition 4.** Given a local connection form, $\mathbb{A}$, on $Q$, the *local curvature form* is the 2-form $D\mathbb{A}$ determined by evaluating the exterior derivative of $\mathbb{A}$ on horizontal vectors. In our setting, this implies

$$D\mathbb{A}(X,Y) = \mathrm{d}\mathbb{A}(X,Y) + \left[\mathbb{A}(X), \mathbb{A}(Y)\right] \tag{21}$$

where $X, Y \in \mathfrak{X}(M)$ are base (control) vector fields.

If we rewrite (20) as

$$\dot{q} = X_i^h u^i$$

with

$$X_i^h = \begin{pmatrix} -g\mathbb{A}(e_i) \\ e_i \end{pmatrix}$$

(recall that $e_i$ is the vector in $T_r M$ with a 1 in the $i$-th row), then it is shown in (Kelly and Murray, 1995) that each of the brackets in the accessibility distribution $C$ can be expressed in terms of derivatives of the connection. For example, the first order brackets between control vector fields can be expressed in terms of the curvature:

$$[X_i^h, X_j^h] = \begin{pmatrix} -gD\mathbb{A}(e_i, e_j) \\ 0 \end{pmatrix}$$

and the next higher order bracket in similar fashion:

$$\Big[[X_i^h, X_j^h], X_k^h\Big] = \begin{pmatrix} -g\left(L_{e_k} D\mathbb{A}(e_i, e_j) - \big[\mathbb{A}(e_k), D\mathbb{A}(e_i, e_j)\big]\right) \\ 0 \end{pmatrix}$$

Noting this, they construct a series of subspaces of $\mathfrak{g}$ given by repeatedly taking higher derivatives of the connection:

$$\begin{aligned} \mathfrak{h}_1 &= \operatorname{span}\left\{\mathbb{A}(X) : X \in T_r M\right\} \\ \mathfrak{h}_2 &= \operatorname{span}\left\{D\mathbb{A}(X,Y) : X, Y \in T_r M\right\} \\ \mathfrak{h}_3 &= \operatorname{span}\Big\{L_Z D\mathbb{A}(X,Y) - \big[\mathbb{A}(Z), D\mathbb{A}(X,Y)\big], \\ &\qquad \big[D\mathbb{A}(X,Y), D\mathbb{A}(W,Z)\big] : W, X, Y, Z \in T_r M\Big\} \\ &\ \vdots \\ \mathfrak{h}_k &= \operatorname{span}\left\{L_X\xi - [\mathbb{A}(Z), \xi], [\eta, \xi] : X \in T_r M, \xi \in \mathfrak{h}_{k-1}, \eta \in \mathfrak{h}_2 \oplus \cdots \oplus \mathfrak{h}_{k-1}\right\} \end{aligned} \tag{22}$$

Notice that in the above equations, the connection has been placed in the appropriate mathematical context as a Lie algebra valued one-form on $M$. Thus, derivatives of $\mathbb{A}$ will take their values in $\mathfrak{g}$ when evaluated along the appropriate vector fields on $M$.

Next recall that for driftless systems local controllability and local accessibility are equivalent, so that the results given below in terms of the accessibility distribution apply equally to controllability for systems with purely kinematic constraints. Kelly and Murray define two types of local controllability, adapted for problems of locomotion. *Fiber controllability* implies that we can use control inputs to move to any position in the fiber, but without regards to the intermediate or final conditions of the controlled variables. On the other hand, *total controllability* is a slightly stronger condition, basically equivalent to STLC, which includes the ability to fully specify the motion of the controlled variables.

**Proposition 1.** (Kelly and Murray, 1995) *The system given by eqns. (20) is locally fiber controllable at $q \in Q$ if and only if*

$$\mathfrak{g} = \mathfrak{h}_1 \oplus \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots$$

*and is locally totally controllable if and only if*

$$\mathfrak{g} = \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots$$

The subspaces, $\mathfrak{h}_k \subset \mathfrak{g}$, will be used below to give sufficient conditions for local accessibility (and later controllability) of the general mixed case given by (12).

In order to illustrate the above definitions (and to make clearer the distinction between fiber and total controllability), we include the example of the two-wheeled mobile robot, presented in (Kelly and Murray, 1995).

**Example 2.** *Two-wheeled Mobile Robot* (cont.)

Recall the two-wheeled planar mobile robot described in Section 2. The configuration space is $Q = G \times M = SE(2) \times (\mathbb{S} \times \mathbb{S})$, with coordinates $q = (x, y, \theta, \phi_1, \phi_2)$. The constraints defining the no-slip condition can be written as in (20) so as to highlight their Lie group structure:

$$\begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = - \begin{pmatrix} \frac{\rho}{2} & \frac{\rho}{2} \\ 0 & 0 \\ \frac{\rho}{2w} & -\frac{\rho}{2w} \end{pmatrix} \begin{pmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{pmatrix}$$

From this, it is clear that the local form of the connection is given by

$$\mathbb{A}(r) = \begin{pmatrix} \frac{\rho}{2} & \frac{\rho}{2} \\ 0 & 0 \\ \frac{\rho}{2w} & -\frac{\rho}{2w} \end{pmatrix} \tag{23}$$

Also, we note that it is easy to show that the base directions are controllable, in a manner similar to that done with the snakeboard in (17) and (18) above.

The connection is used to define $\mathfrak{h}_1$ for the controllability calculations, and so

$$\mathfrak{h}_1 = \text{span}\left\{ \left(1, 0, \frac{1}{w}\right)^T, \left(1, 0, -\frac{1}{w}\right)^T \right\}$$

In order to compute the curvature, $D\mathbb{A}$, we use the formula $D\mathbb{A} = \mathrm{d}\mathbb{A} + [\mathbb{A}, \mathbb{A}]$, for which we will need the structure constants of the Lie algebra. A straightforward calculation shows that for $\xi, \eta \in \mathfrak{g}$,

$$[\xi, \eta] = \begin{pmatrix} \xi^2\eta^3 - \xi^3\eta^2 \\ \xi^1\eta^3 - \xi^3\eta^1 \\ 0 \end{pmatrix}$$

using the standard basis for $se(2)$. If we write $\mathbb{A}$ using differential forms as

$$\mathbb{A} = \begin{pmatrix} \frac{\rho}{2}\,\mathrm{d}\phi_1 + \frac{\rho}{2}\,\mathrm{d}\phi_2 \\ 0 \\ \frac{\rho}{2w}\,\mathrm{d}\phi_1 - \frac{\rho}{2w}\,\mathrm{d}\phi_2 \end{pmatrix}$$

then it is easy to see that $\mathrm{d}\mathbb{A} = 0$. Calculating the bracket, we get

$$D\mathbb{A} = [\mathbb{A}, \mathbb{A}] = - \begin{pmatrix} 0 \\ \frac{\rho^2}{2w}\,\mathrm{d}\phi_1 \wedge \mathrm{d}\phi_2 \\ 0 \end{pmatrix}$$

Clearly, the Lie algebra element $(0,1,0)^T \in \mathfrak{h}_2$ is in the span of $D\mathbb{A}$, when applied to the appropriate tangent vectors. Thus, the two-wheeled mobile robot is fiber controllable, since $\mathfrak{h}_1 + \mathfrak{h}_2 = \mathfrak{g}$. However, Kelly and Murray (1995) show that the higher-order derivatives of $\mathbb{A}(r)$ will never lead to terms with nonzero elements in the third slot (i.e., terms like $(*,*,1)$), and so the mobile robot in this example is *not* totally controllable (since $\mathfrak{h}_2 + \mathfrak{h}_3 + \ldots \neq \mathfrak{g}$). This surprising result is related to the geometric relationship between the two wheels, and the paths they must follow.

### 3.2. Unconstrained Systems with Symmetries

In the same manner as for the principal kinematic case above, Montgomery (1993) showed that similar tests can be used to show controllability for an unconstrained dynamical system with Lie group symmetries. His result applies to the case where the spatial momentum $\mu$ is zero (and hence the body momentum $p = \mathrm{Ad}_g^*\mu = 0$), so that all motion is horizontal. For this situation, we see that, since the momentum is zero and constant (recall Noether's theorem for unconstrained systems), the equations reduce to those of the principal kinematic case (eqn. (20)),

$$g^{-1}\dot{g} = -\mathsf{A}(r)\dot{r}$$

$$\dot{r} = u$$

where $\mathsf{A}$ is again the mechanical connection. Using the same construction above, his result states that if

$$\mathfrak{g} = \mathfrak{h}_2 \oplus \mathfrak{h}_3 \oplus \cdots$$

then any two configurations $q_0$ and $q_1$ can be connected by a horizontal path, i.e., one which satisfies the $\mu = 0$ constraint. In other words, even though we have a fully dynamical system, it is possible to give simple controllability conditions based on the connection. Notice, however, that for $\mu \not\equiv 0$ this presents a drift term which implies that controllability and accessibility are no longer equivalent, and so Chow's theorem (LARC) implies only accessibility. One of our goals in the following sections is to derive tests for general systems with symmetries and constraints in order to establish basic controllability results.

## 4. Local Accessibility

Having reviewed the previous results on each of the limiting cases—the principal kinematic and unconstrained cases—we now turn towards deriving new results for the more general case of mixed constraints and symmetries. In doing so, however, we will attempt to build upon the structure used in these previous works. Let us begin by examining a few of the lower order brackets in the accessibility distribution, $C$, which play an important role in the accessibility and controllability analyses to follow. Notice that we have chosen the control vector fields in such a manner that they are mutually orthogonal, and such that

$$[h_i, h_j] = 0, \quad \forall\, i,j \in \{1,\ldots,m\}$$

The remaining first order brackets (those in $\Delta_1$) will be of the form of a control vector field bracketed with the drift vector field. A quick calculation shows that

$$\alpha_i := [h_i, f] = \begin{pmatrix} -\mathbb{A}_i(r) \\ (\sigma_{\dot{r}\dot{r}})_{ij}\dot{r}^j + (\sigma_{p\dot{r}})_i^j p_j \\ e_i \\ 0 \end{pmatrix}$$

At this point we direct the reader's attention to the similarity between this set of vector fields and those for the kinematic case. If one disregards the variables that are eliminated in the kinematic case, i.e., the momentum and acceleration variables, then the two sets of equations are identical. A loose mathematical interpretation of this similarity is that the bracket operation pairing the drift and torque controls (given by $\alpha_i = [h_i, f]$) yields a vector field that is "equivalent" to having integrated the input control torques, converting them to something approximating velocity controls. Hence they take on a form reminiscent of the kinematic case, where the control inputs are velocities. This, of course, is just a naive way of describing the similarities between the brackets $\alpha_i$ and the inputs in the principal kinematic case.

Moving to the second order brackets, an interesting thing happens when we bracket $h_i$ with $\alpha_j$:

$$\beta_{ij} := [h_i, \alpha_j] = \Big[h_i, [h_j, f]\Big] = \begin{pmatrix} 0 \\ (\sigma_{\dot{r}\dot{r}})_{ij} \\ 0 \\ 0 \end{pmatrix}$$

Thus, the $\sigma_{\dot{r}\dot{r}}$ term, which is a cross-coupling term for the base variables, directly affects the momentum variables via the $\beta_{ij}$ brackets. Viewing this coupling as a map, $\sigma_{\dot{r}\dot{r}} : TM \times TM \to \mathbb{R}^p$, then $\sigma_{\dot{r}\dot{r}}$ being surjective implies that all of the momentum directions can be generated via this second-order bracket. This mapping will be quite useful for a variety of reasons, as detailed below.

**Proposition 2.** *Assume that $\sigma_{\dot{r}\dot{r}}$ is onto and that*

$$\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{h}_3 + \cdots$$

*where the $\mathfrak{h}_k$'s are defined as above (eqns. (22)) using the local form of the connection given in (12). Then the system given by (12) is locally accessible.*

*Proof.* To show accessibility, we need to show that the distribution $\Delta_\infty$ spans $TN$ at each point $z$. The assumption on $\sigma_{\dot{r}\dot{r}}$ implies that the bracket given by $[h_i, [h_j, f]]$ will span the momentum directions, so it remains only to show that $\Delta_\infty$ contains the fiber and base directions. To do this, we begin with $\Delta_1$. It will contain vectors of the form

$$h_i = \begin{pmatrix} 0 \\ 0 \\ 0 \\ \delta_i \end{pmatrix}, \qquad \alpha_i = \begin{pmatrix} -\mathbb{A}_i(r) \\ (\sigma_{\dot r \dot r})_{ij}\dot r^j + (\sigma_{p\dot r})_i^j p_j \\ e_i \\ 0 \end{pmatrix}$$

Thus, the base directions (velocity and acceleration vectors on $M$) will be contained in $\Delta_1$. Next, we examine $\Delta_2$. It will contain the vectors, $\alpha_i$, and also vectors of the form

$$\beta_{ij} = \begin{pmatrix} 0 \\ (\sigma_{\dot r \dot r})_{ij} \\ 0 \\ 0 \end{pmatrix}$$

Thus, for each $z \in N$ we can cancel off the terms in $\alpha_i$ which act in the momentum direction, and so define a new set of vector fields to operate on

$$\tilde\alpha_i = \begin{pmatrix} -\mathbb{A}_i(r) \\ 0 \\ e_i \\ 0 \end{pmatrix}$$

Using these vector fields, we define

$$\tilde\Delta_2 := \operatorname{span}\{f, h_i, \beta_{ij}, \tilde\alpha_i\}$$

and the subsequent distributions, $\tilde\Delta_k$, similar to before. Then $\tilde\Delta_\infty \subset \Delta_\infty$. As in the kinematic case, higher-order bracketing of $\tilde\alpha_i$ and $\tilde\alpha_j$ will lead to higher order derivatives of the connection, $\mathbb{A}(r)$. By the assumption that $\mathfrak{h} = \mathfrak{g}_2 + \mathfrak{g}_3 + \ldots$, we have it that $\tilde\Delta_\infty = T_z N$, for each $z \in N$. The result follows since $\Delta_\infty \supset \tilde\Delta_\infty = T_z N$. ■

The criterion given in Proposition 2 will be used in the following sections as a basis for checking local controllability and for demonstrating accessibility and controllability properties for the snakeboard example.

## 5. Local Controllability

Unfortunately, for nonlinear systems with drift we have seen above that local accessibility may be quite different from local controllability. In order to provide a result for controllability, we will need to show that certain of the higher-order brackets either vanish or can be written as a linear combination of lower order brackets. This result is due to Sussman (1987), and is the strongest statement of local controllability for nonlinear control systems with drift of which we are currently aware. For further details on this construction, please refer to (Bloch *et al.*, 1992; Sussman, 1987). In order to use these results, we first need to develop a notion of degree of a Lie bracket. This development can be done much more formally using the formalism of free Lie algebras (Ostrowski, 1995; Serre, 1992), but instead we rely on common intuition to provide the necessary understanding of what is meant by the two definitions of degree developed here.

Let $\boldsymbol{X} = (X_0, \ldots, X_m)$ be a finite set of vector fields (in Serre's notation, this would be a finite sequence of indeterminates). Then, denote by $\mathrm{Br}(\boldsymbol{X})$ the set of all possible iterated Lie brackets involving $X_0, \ldots, X_m$.

It should be clear now that we can use this free Lie algebra to define a notion of degree for a Lie bracket. Let the set $\boldsymbol{X} = (X_0, \ldots, X_m)$ be a finite sequence of indeterminates.

**Definition 5.** Let the *degree of* $B \in \mathrm{Br}(\boldsymbol{X})$ *relative to* $X_a$, denoted by $\delta^a(B)$, be the integer number of times that $X_a$ appears in the bracket $B$. The *degree* of $B \in \mathrm{Br}(\boldsymbol{X})$ is then given by

$$\delta(B) = \sum_{a=0}^{m} \delta^a(B) \tag{24}$$

To illustrate this, suppose that $m = 2$. Then the degrees for

$$Y = \Big[X_0, [X_1, X_2], [X_0, X_1]\Big] \quad \text{and} \quad Z = [X_1, X_2]$$

are $\delta^0(Y) = 2$, $\delta^1(Y) = 2$, $\delta^2(Y) = 1$ and $\delta^0(Z) = 0$, $\delta^1(Z) = 1$, $\delta^2(Z) = 1$, respectively.

The degree, $\delta$, of a vector field should (hopefully) correspond to one's intuitive notion of the degree of a Lie bracket. There are several results on generating a complete set (basis) of iterated Lie brackets, e.g., a Philip Hall basis (Murray and Sastry, 1993; Serre, 1992). We use a result given in (Lewis, 1995):

**Proposition 3.** *Every element of the free Lie algebra* $L_{\boldsymbol{X}}$ *can be written as a linear combination of repeated brackets of the form*

$$\Big[X_k, \Big[X_{k-1}, [\ldots, [X_2, X_1] \ldots]\Big]\Big]$$

*where* $X_i \in \boldsymbol{X}$, $i = 1, \ldots, k$.

Let $h_0 := f$ so that $\Delta_0 = \mathrm{span}\,\{h_0 = f, h_1, \ldots, h_m\}$. We represent this set of vector fields as $\boldsymbol{X} = (h_0, h_1, \ldots, h_m)$ with the set of all possible Lie brackets given by $\mathrm{Br}(\boldsymbol{X})$. Then, we have the following theorem due to Sussman (the version we have written here is greatly simplified, but retains essentially all of the content of the original theorem—more details can be found in (Sussman, 1987; Ostrowski, 1995)):

**Theorem 2.** (Sussman, 1987) *Given the system of (11), with* $h_0(z_0) = f(z_0) = 0$ *at an equilibrium point* $z_0 \in N$*, assume that* $\boldsymbol{X} = (h_0, \ldots, h_m)$ *satisfies the LARC at* $z_0$*. Further, assume that whenever* $X \in \mathrm{Br}(\boldsymbol{X})$ *is a bracket for which* $\delta^0(X)$ *is odd and* $\delta^1(X), \ldots, \delta^m(X)$ *are all even, then there exist brackets* $Y_1, \ldots, Y_k$ *such that*

$$X = \alpha^i Y_i$$

*for some* $\alpha^1, \ldots, \alpha^k \in \mathbb{R}$*, and*

$$\delta(Y_i) < \delta(X), \quad i = 1, \ldots, m$$

*Then the system defined by (11) is STLC from* $z_0$*.*

With this theorem in mind, we define a "bad" bracket to be those brackets for which the *drift term* appears an *odd* number of times and for which the *control vector fields* **each** appear an *even* number of times (including zero times). The sufficient conditions for small-time local controllability, then, can be simply restated as requiring that all "bad" brackets be expressible in terms of brackets of lower degree.

**Proposition 4.** *Assume that the system defined by (12) is locally accessible, that the map* $\sigma_{\dot{r}\dot{r}}$ *is onto, and that* $(\sigma_{\dot{r}\dot{r}})_{ii} \equiv 0$ *for* $i = 1, \ldots, m$ *(no summation over* $i$*). Then this system is small-time locally controllable (STLC) from all equilibrium points,* $z_0 \in N$*.*

*Proof.* In order to show controllability, we begin by demonstrating that all "bad" brackets as defined by Sussman will either be zero or be expressible in terms of lower order "good" brackets (in fact, of order 3). This, along with the assumption that the LARC is satisfied (using the results from the kinematic case), will give the result via Theorem 2.

First, we restrict our attention to the point $z_0 = (0, 0, 0, 0) \in Gr \times \mathbb{R}^p \times M \times T_r M$. It is easy to show that the result will hold for all equilibrium points, $z \in N$ (of the form $z = (g, 0, r, 0)$), by translating (12) appropriately. Also notice that $f(z_0) = 0$, satisfying the first requirement of Theorem 2.

Next, recall the definition of the degree of a bracket and notice two important facts that must be true of any bad bracket $X$: 1) $\delta(X)$ must be odd, and 2) $\delta^0(X) \neq \sum_{i=1}^m \delta^i(X)$. These are both made true by virtue of there being exactly one odd term in the summation of eqn. (24). The first condition implies that all even order brackets are necessarily "good" brackets, while the second condition implies that for bad brackets the quantity

$$\gamma(X) := \delta^0(X) - \sum_{i=1}^m \delta^i(X)$$

is always odd, and thus never zero.

More specific to the system of eqns. (12), let $\mathcal{O}(k)$ denote a function in $(z, \dot{z})$ which is a homogeneous polynomial of order $k$ in $(\dot{r}, p)$. Thus, $f(z) = (\mathcal{O}(1), \mathcal{O}(2), \mathcal{O}(1), 0)$. A straightforward set of calculations shows that for any bracket involving the drift vector field, $f$, bracketing by $f$ will increase the order of these functions by 1, and that bracketing by any of the $h_i$'s will decrease the order of the bracket by 1. Thus, we will find that for any bad bracket, $X$, (which by definition must contain at least one $X_0$ in the bracket), it will evaluate to a vector field with the form $X = (\mathcal{O}(\gamma(X)), \mathcal{O}(\gamma(X)+1), \mathcal{O}(\gamma(X)), 0)^1$, or will be identically zero, e.g., any bracket involving $[h_1, h_2] \equiv 0$. Viewed this way, it is easy to see that all bad brackets for which $\gamma(X) \neq -1$ must have $X = 0$ when evaluated at the equilibrium points.

Thus, the only bad brackets that we need worry about are those with $\gamma(X) = -1$, for which $X = (0, \mathcal{O}(0), 0, 0)$. These are brackets which lie in the momentum direction. But we have already assumed that the map $\sigma_{\dot{r}\dot{r}}$ is onto, which means that these directions are captured by a bracket of degree 3:

$$\beta_{ij} = \Big[X_i, [X_j, X_0]\Big] = \begin{pmatrix} 0 \\ -(\sigma_{\dot{r}\dot{r}})_{ij} \\ 0 \\ 0 \end{pmatrix}$$

Unfortunately, brackets of the form $[X_i, [X_i, X_0]]$ $(i = j)$ are also bad brackets, which explains the necessity of assuming that $(\sigma_{\dot{r}\dot{r}})_{ii} \equiv 0$. Given this, however, we see that any bad bracket which is not zero at $z_0$ can be rewritten in terms of brackets of the form $[h_i, [h_j, f]]$, where $i \neq j$. ■

A few remarks are in order. First, the reader should note that while the condition that $(\sigma_{\dot{r}\dot{r}})_{ii} \equiv 0$ in Proposition 4 may appear slightly artificial, it is *required* in order to satisfy Sussman's criterion for controllability. In fact, research by Lewis and Murray (1995) suggest that similar conditions may be needed for general mechanical systems. They study accessibility and controllability for unconstrained mechanical systems, and report similar conditions on these third-order brackets of the type $[h_i, [f, h_i]]$. In their case, these brackets are allowed to be nonzero if they are contained in the control input vector field; however, it is not difficult to show that for our purposes these brackets must be identically zero.

Also, one of the main tenets driving this research is that the process of locomotion can be described as a coupling of internal shape changes creating net external motion and that this process can be modeled using a mathematical connection on a principal fiber bundle. Naturally, there arises the question of what role the connection and its derivatives really do play in describing the actual motion of the system. In particular, what is the relationship between the connection and its derivatives and locomotive gaits? We address this issue in the following section.

## 6. Locomotive Gaits

Let us briefly consider an important aspect of locomotion that is intricately related to the study of control for these types of systems. A very common observation of

[1] We have allowed $\gamma(X)$ to be negative, and so define $\mathcal{O}(k) = 0$ for all $k < 0$.

locomotion is that it is most often generated by *cyclical* shape changes (Collins and Stewart, 1992; Hildebrand, 1965). The motion takes on a characteristic form, called a *gait.*

**Definition 6.** A locomotive *gait* is a specified cyclic pattern of internal shape changes (inputs) which couple to produce a net motion.

One very interesting phenomenon that arises in the study of locomotion is the presence of a very limited set of basic motion patterns. For each species, there usually exist at most a handful of gaits, often tailored for specific needs or environments. For instance, a human will walk or run, depending on the desired speed, but may also hop or skip (though these two gaits do not seem to serve any evolutionary function). On the other hand, snakes will generally move in a serpentine fashion, but can adapt to other environments. For instance, on a slippery surface, a snake may push off the walls of its environment and move in a concertina (inch-worm) gait. Alternatively, snakes in the desert are known to use a sidewinding gait in order to minimize the amount of time that body surfaces spend in contact with the hot sand, and maximize the time that surfaces are off the ground and hence cooled by the air. What is interesting about all of this is that there is a small set of gaits that are used, and almost universally these gaits are based on a single frequency of oscillation. In studying locomotion, and in particular when examining related control issues, it will be important to ask the question of how our models and control laws reflect these naturally occurring patterns of motion.

We provide here a brief discussion of the gaits that have been found for the example of the snakeboard. Obviously, the analysis of gaits is intricately related to issues of controllability for locomotion systems.

**The Snakeboard.** We return to the snakeboard example to investigate controllability and gait patterns. Obviously, the bracket of the control inputs, $[h_\psi, h_\phi]$, is identically zero. The only other first order brackets are those mixing the drift vector field with the control inputs:

$$\alpha_\psi = [h_\psi, f] = \Big(\frac{J_r \sin 2\phi}{2ml}\cos\theta,\ \frac{J_r \sin 2\phi}{2ml}\sin\theta,\ \frac{-J_r \sin^2\phi}{ml^2},\\ 2\dot{\phi} J_r \cos^2\phi,\ 1,\ 0,\ 0,\ 0\Big)^T$$

and

$$\alpha_\phi = [h_\phi, f] = \Big(0,\ 0,\ 0,\ 2\dot{\psi} J_r \cos^2\phi - p\tan\phi,\ 0,\ 1,\ 0,\ 0\Big)^T$$

Notice that these vector fields have "1's" in the appropriate velocity directions. As mentioned above, this loosely corresponds to *integrating* the control torques to velocity controls. Notice that this will also encode the information given by the local form of the connection, $\mathbb{A}(r)$, since the connection relates input velocities to fiber velocities.

The vector fields above imply control of the base (assumed to be controllable). In order to show accessibility and controllability (STLC), the first criteria to be satisfied are the conditions on $\sigma_{\dot{r}\dot{r}}$, given by the following third-order brackets. First, we need the diagonal elements of $\sigma_{\dot{r}\dot{r}}$ to be zero. This is seen to be true via a direct calculation:

$$\beta_{\phi\phi} = \beta_{\psi\psi} = 0$$

Then we look at off diagonal terms to show that $\sigma_{\dot{r}\dot{r}}$ is onto (and hence that the momentum direction is contained in the accessibility distribution). To see this, we simply write down the necessary bracket:

$$\beta_{\phi\psi} = [h_\psi, \alpha_\phi] = \left(0,\ 0,\ 0,\ 2J_r \cos^2\phi,\ 0,\ 0,\ 0,\ 0\right)^T$$

which is nonzero for all $\phi \neq \pi/2$.

Finally, to demonstrate that the snakeboard is controllable, we need to show that $\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{h}_3 + \ldots$, using the connection, $\mathbb{A}(r)$. We begin by computing $[\alpha_\phi, \alpha_\psi]$, which gives us the curvature of the connection, $D\mathbb{A}$. This yields terms of the form:

$$\left(\frac{J_r}{ml}\cos 2\phi,\ 0,\ -\frac{J_r}{ml^2}\sin 2\phi\right)^T \in \mathfrak{h}_2$$

Then, $[\alpha_\phi, [\alpha_\phi, \alpha_\psi]]$ yields

$$\left(-\frac{2J_r}{ml}\sin 2\phi,\ 0,\ -\frac{2J_r}{ml^2}\cos 2\phi\right)^T \in \mathfrak{h}_3$$

and $[\alpha_\phi, [\alpha_\psi, [\alpha_\psi, [\alpha_\psi, \alpha_\phi]]]]$ gives

$$\left(0,\ \frac{2{J_r}^2}{m^2 l^3}\cos 2\phi,\ 0\right)^T \in \mathfrak{h}_5$$

Thus, $\mathfrak{g} = \mathfrak{h}_2 + \mathfrak{h}_3 + \mathfrak{h}_5$, and the conditions for Proposition 4 are satisfied.

As an aside, we comment that the roller racer example in (Krishnaprasad and Tsakiris, 1995) fails to satisfy these conditions ($\sigma_{ii} \neq 0$), but also has been shown not to be STLC (a general result on control systems with single inputs (Lewis, 1997)). Finally, having shown that the snakeboard is controllable, we return to the question of how these calculations relate to the gait patterns demonstrated by the snakeboard. A major part of this issue, then, is asking the question, "what role do the connection and its derivatives *really* play in describing the actual motion of the system?" In particular, "what is the relationship between the connection and its derivatives and locomotive gaits?" Although the results at present are only qualitative, they certainly suggest that we are on the right track. Along with this, they provide some hints as to what directions to follow in future research.

Extensive simulations of the snakeboard gaits can be found in (Ostrowski *et al.*, 1994), some of which are included here to provide a new perspective on how these results fit into the present context. To date, there have been three basic gait patterns studied for the snakeboard: the "drive" (or "serpentine") gait, the "rotate" gait, and the "parallel parking" gait. In each of these, we assume complete control of the base variables, $\phi$ and $\psi$, and specify their trajectories as sinusoidal inputs of the form:

$$\phi = a_\phi \sin(\omega_\phi t), \qquad \psi = a_\psi \sin(\omega_\psi t)$$

A gait will be referenced by an integer ratio of the form $\omega_\phi : \omega_\psi$, corresponding to the ratio between $\omega_\phi$ and $\omega_\psi$. For instance, a 3:2 gait (the parallel parking gait) would correspond to $\omega_\phi = 3$ and $\omega_\psi = 2$. For the simulations, the following parameters were used:

| | |
|---|---|
| $m$ | : 6 kg |
| $J$ | : 0.06 kg·m$^2$ |
| $J_r$ | : 0.167 kg·m$^2$ |
| $J_w$ | : 0.00167 kg·m$^2$ |
| $l$ | : 0.3 m |

These values roughly reflect the physical parameters used to build a working prototype snakeboard (shown in Fig. 1).

*The "drive" gait*

The drive gait is characterized by a 1:1 frequency ratio, and demonstrates a forward, serpentine motion resembling that of a snake. A simulation of this gait is shown in Fig. 2, using the parameters: $a_\phi = 0.7\,\text{rad}$, $a_\psi = -1\,\text{rad}$, and $\omega_\phi = \omega_\psi = 1\,\text{rad/sec}$. We remark that the scaling of the axes given in this figure and those to follow is chosen so as to maximize the visibility and spread of the data presented in these figures, and so this must be taken into account when interpreting the results in terms of physical quantities. Notice that in Fig. 2 the amplitude of the motion in the transverse or $y$-direction steadily increases. This is due to the fact that momentum is continually being built up by this gait. Human riders use feedback to control this effect, and are visibly seen modifying their input patterns once a desired speed is reached.

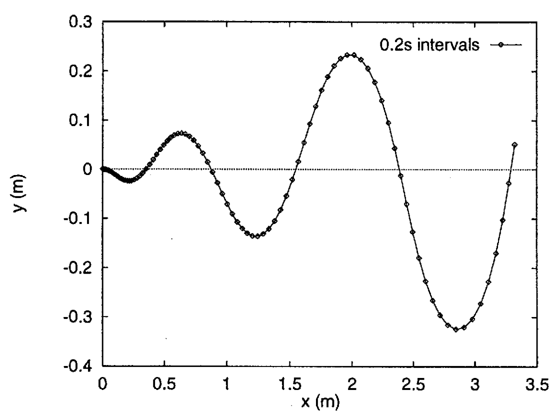


Fig. 2. Position of the center of mass for the 1:1 (drive) gait.

In relationship to the Lie bracket calculations, we notice that the 1:1 frequency ratio has a direct correspondence to the 1:1 bracket, $[\alpha_\phi, \alpha_\psi]$. In fact, evaluated at $\phi = 0$ (the center of the wheels' rotation), the bracket gives a Lie algebra element of $(J_r/ml,\ 0,\ 0)^T$. This is written in the body frame of the board, and so corresponds to forward motion, along the length of the board.

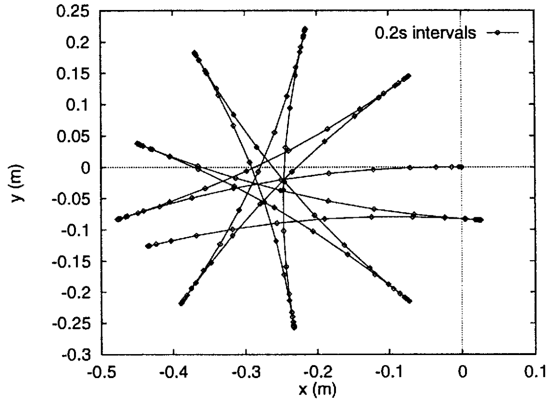


Fig. 3. Position of the center of mass for the 2:1 (rotate) gait.

*The "rotate" gait*

The rotate gait uses a 2:1 frequency ratio, and generates a rotational motion (in $\theta$) that leaves the $(x, y)$ position relatively unchanged in the mean. The input parameters for the simulation shown in Fig. 3 were $a_\phi = 0.7\,\text{rad}$, $a_\psi = 1\,\text{rad}$, $\omega_\phi = 2\,\text{rad/sec}$, and $\omega_\psi = 1\,\text{rad/sec}$. The snakeboard moves steadily around a central point, while undergoing large rotations—moving $\pi$ radians, or one half rotation, in approximately four cycles.

Again, we return to examine the correspondence of this motion with the Lie bracket. We see that the necessary bracket direction, the $\theta$-direction, is given by a 2:1 Lie bracket. Namely, $[\alpha_\phi, [\alpha_\phi, \alpha_\psi]]|_{\phi=0}$ produces the element $(0,\ 0,\ -2J_r/ml^2)^T$.

*The "parking" gait*

The final gait studied is the parallel parking gait, so called because its motion resembles that of a car performing a parallel parking maneuver (see Fig. 4). It is based on a 3:2 frequency ratio and generates a net lateral motion, transverse to the length of the board. The parameters used in the simulation were $a_\phi = 0.7\,\text{rad}$, $a_\psi = 1\,\text{rad}$, $\omega_\phi = 3\,\text{rad/sec}$, and $\omega_\psi = 2\,\text{rad/sec}$.

The 3:2 bracket, $[\alpha_\phi, [\alpha_\psi, [\alpha_\psi, [\alpha_\psi, \alpha_\phi]]]]$, in which $\alpha_\psi$ appears 3 times and $\alpha_\phi$ appears twice, gives $(0,\ 2{J_r}^2/m^2l^3,\ 0)^T$. Other permutations of the fifth order, 3:2 bracket give Lie algebra elements that are either in the same direction or are identically zero. The nonzero entry in the second position of the Lie algebra element above corresponds directly to the direction transverse to the board, namely the $y$-direction when the board is at $\theta = 0$.

## 7. Conclusion

This paper establishes easily computable controllability results for systems with Lie group symmetries and external nonholonomic constraints. These types of systems

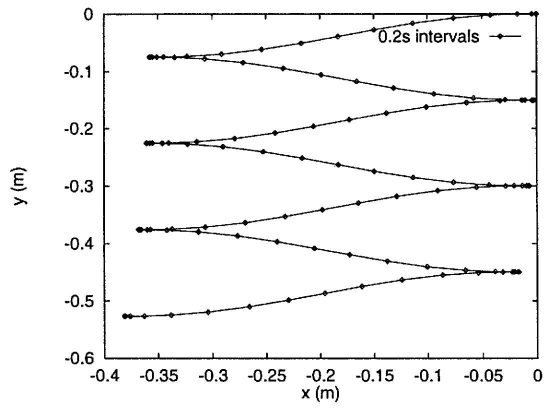


Fig. 4. Position of the center of mass for the 3:2 (parking) gait.

are characterized by the existence of a connection, which relates the motion of the system to motion in its control inputs. The connection has been discussed in its direct implications for accessibility and controllability. Further, these types of systems will very often include drift vector fields, in the form of momentum terms. This added complexity makes the analysis slightly more difficult, but gives the additional possible benefit of increased control over the system dynamics. Research has shown that many, if not most, problems of locomotion can be formulated in terms of this dynamical structure, and so motivate further analysis of the controllability of these types of systems. We have presented an initial survey of the relationship between gaits (as input patterns) and the Lie bracket directions generated in the controllability tests. Future work will be concerned with further exploiting the geometric structure of the problem, and in developing means for establishing results governing trajectory generation and optimal control of locomotive gaits.

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