Appl. Math. and Comp. Sci., 1997, Vol.7, No.4, 869–879

A NEW PERSPECTIVE ON CONTROLLABILITY PROPERTIES FOR DYNAMICAL SYSTEMS

PAULA ROCHA*, JEFFREY WOOD**

In this paper, we study the properties of weak and strong controllability as newly defined in (Rocha, 1995) for delay-differential systems in a behavioural setting, now for the multidimensional case. Further, we give an overview of the relationships between these properties and the original behavioural controllability concept introduced in (Willems, 1988).

1. Introduction

Whereas the classical notion of controllability is a property of state space representations, within the behavioural approach to dynamical systems controllability is defined at the level of the external (joint input-output) variables. According to the definition given in (Willems, 1988) for the 1D case, a system is controllable if it is possible to concatenate any of its past and of its future trajectories so as to still obtain an admissible (global) trajectory. The generalization of this property to multidimensional systems has been studied in (Rocha and Willems, 1991) for the 2D case, and, more recently, in (Wood *et al.*, 1997; Zerz, 1996) for the general ND case. The case of delay-differential systems is treated in (Glüssing-Lüerssen, 1995; Rocha and Willems, 1997), where behavioural controllability is characterized and compared with the other existing controllability notions, namely within the frameworks of the functional analytical approach—spectral and approximate controllability—(Manitius, 1981), and of the algebraic approach—weak and strong (or strict) controllability—(Lévy, 1981; Morse, 1976). The properties of weak and strong controllability have been introduced in (Lévy, 1981; Morse, 1976) for delay-differential systems with pseudo-state representation $(A(\Delta), B(\Delta), C(\Delta), D(\Delta))$, where Δ is the unit delay operator and $A(z), B(z), C(z), D(z)$ are polynomial matrices in z, as rank conditions on the matrix $[B(z) \mid A(z)B(z) \mid \ldots \mid A(z)^{n-1}B(z)]$. This goes against the spirit of the behavioural approach, according to which the structural properties of a system should be defined as attributes of its behaviour (i.e. of the set of its admissible trajectories). The need for a behavioural, system theoretic interpretation of the aforementioned rank conditions has led, in (Rocha, 1995), to the redefinition of weak and strong controllability for general delay-differential systems. Since the new definitions are given in terms of the characteristics of the behaviour and are not based on the delay-differential nature

* Department of Mathematics, University of Aveiro, 3810 Aveiro, Portugal, e-mail: procha@mat.ua.pt.

** Department of Electronics and Computer Sience, University of Southampton, Southampton SO17 1BJ, United Kingdom, e-mail: jjw@ecs.soton.ac.ukl.

of its description, the corresponding controllability properties are also meaningful for 1D systems other than delay-differential ones, as well as for multidimensional systems.

Our paper has two main goals: the first one is to complete the existing results on weak and strong controllability by making a detailed study for the multidimensional case; the other one is to present an overview of all the controllability-type properties defined up till now within the behavioural framework and of the way they relate to one another.

The definitions of the different controllability properties are given in Section 2. In Section 3 we consider the delay-differential case and develop the preliminary results obtained in (Rocha, 1995). The discrete ND case is studied in Section 4. Finally, Section 5 contains our concluding remarks.

2. Controllability Properties

In this paper, we consider dynamical systems over a (one- or multidimensional) domain T, whose laws can be described by means of an equation of the form:

$$R(\underline{\Omega})w = 0 \tag{1}$$

where the system trajectories w are elements of $(\mathbb{K}^q)^T$, $(\mathbb{K} = \mathbb{R}, \mathbb{C})$, for some $q \in \mathbb{N}$, $R(\underline{z})$ is a polynomial matrix in $\underline{z} := (z_1, \dots, z_L), L \geq 1$, and $\underline{\Omega} := (\Omega_1, \dots, \Omega_L)$ consists of commuting linear operators Ω_i. In particular, we will be concerned with the cases $T = \mathbb{R}$ and $\underline{\Omega} = \mathrm{d}/\mathrm{d}t$ or $\underline{\Omega} = (\mathrm{d}/\mathrm{d}t, \Delta)$, (where Δ is the unit delay $\Delta w(t) = w(t-1)$), which respectively correspond to the cases of differential systems and of delay-differential systems, and the case $T = \mathbb{Z}^N$ and $\underline{\Omega} = (\underline{\sigma}, \underline{\sigma}^{-1}) := (\sigma_1, \dots, \sigma_N, \sigma_1^{-1}, \dots, \sigma_N^{-1})$, $(N \geq 1)$, with $\sigma_i w(\underline{t}) = w(\underline{t} + e_i)$ (where e_i is the i-th canonical element of $\mathbb{Z}^N$), which corresponds to ND-systems over the discrete domain $\mathbb{Z}^N$.

Since the set $\mathcal{B}$ of admissible trajectories (the system *behaviour*) is given as the kernel of the operator $R(\underline{\Omega})$, we will say that (1)—or, equivalently, $\mathcal{B} = \ker R(\underline{\Omega})$—is a *kernel representation* of $\mathcal{B}$.

The original (behavioural) controllability notion introduced in (Willems, 1988) for the one-dimensional case can be extended to (possibly) higher dimensions in the following way (cf. Rocha and Willems, 1991; Wood *et al.*, 1997).

Definition 1. A behaviour $\mathcal{B}$ defined over the domain T is said to be *controllable* if there exists $d > 0$ such that for all $T_1, T_2 \subset T$ satisfying the condition $d(T_1, T_2) > d$:

$$\mathcal{B}|_{T_1 \cup T_2} = \mathcal{B}|_{T_1} \times \mathcal{B}|_{T_2}$$

In other words, $\mathcal{B}$ is controllable if any two partial trajectories $w_1 \in \mathcal{B}|_{T_1}$ and $w_2 \in \mathcal{B}|_{T_2}$ can be concatenated so as to obtain a global trajectory $w \in \mathcal{B}$, provided that the distance between T_1 and T_2 is sufficiently large.

The definition of weak controllability is based on the concept of hermetic sub-behaviour (which has been introduced in (Rocha, 1995) for the 1D case under the name of "absorbing sub-behaviour").

If $\mathcal{B}$ is a behaviour with kernel representation, we will say that $\mathcal{B}'$ is a *sub-behaviour* of $\mathcal{B}$ if $\mathcal{B}' \subset \mathcal{B}$ and $\mathcal{B}'$ has also a kernel representation. Further, if $\mathcal{B} = \ker R$ and $\mathcal{B}' = \ker R'$, we will say that $\mathcal{B}'$ is a *regular* sub-behaviour if there exists a polynomial matrix $F(\underline{z})$ such that $R(\underline{z}) = F(\underline{z})R'(\underline{z})$. It should be noticed that, while in all the other considered cases every sub-behaviour is regular, the same does not hold true in the delay-differential case; for instance, $\ker(\mathrm{d}/\mathrm{d}t)$ is a non-regular sub-behaviour of $\ker(\Delta - 1)$.

Given an N-dimensional domain T, a subset $T' \subset T$ is called *q-complementary* if $T \backslash T'$ is a shift of an ND quadrant.

Definition 2. Let $\mathcal{B}$ be a behaviour with kernel representation defined over a domain T. A sub-behaviour $\mathcal{B}'$ of $\mathcal{B}$ is said to be *hermetic* if for every q-complementary subset T' of T:

$$\{w' \in \mathcal{B}, w'|_{T'} \in \mathcal{B}'|_{T'}\} \Rightarrow \{w' \in \mathcal{B}'\}$$

This means that, when $\mathcal{B}'$ is hermetic, if $w' \in \mathcal{B}$ is a partial trajectory of $\mathcal{B}'$ on certain sufficiently large subsets T' of T, then it can only be extended as a trajectory of $\mathcal{B}'$, i.e. it cannot leave $\mathcal{B}'$. For example in the 1D case, if a trajectory $w' \in \mathcal{B}$ "starts off" in $\mathcal{B}'$, then it must remain in $\mathcal{B}'$, and if it "ends" in $\mathcal{B}'$, then it must have always been there. Hence the term "hermetic".

Definition 3. A behaviour $\mathcal{B}$ with kernel representation is said to be *weakly controllable* if it does not contain any proper hermetic sub-behaviour.

Example 1. Let $\mathcal{B}$ be the behaviour of a delay-differential system with kernel representation $\mathcal{B} := \ker R(\mathrm{d}/\mathrm{d}t, \Delta)$, where $R(z_1, z_2) = [z_1^2 - z_2^2 \quad z_1 - z_2]$, and define $\mathcal{B}' \subset \mathcal{B}$ as $\mathcal{B}' := \ker R'(\mathrm{d}/\mathrm{d}t, \Delta)$, with $R'(z_1, z_2) = [z_1 + z_2 \quad 1]$. Suppose that $w' \in \mathcal{B}$ is such that $w'|_{T'} \in \mathcal{B}'|_{T'}$, for an arbitrary q-complementary subset T' of $T = \mathbb{R}$, (i.e. for an arbitrary unbounded interval of $\mathbb{R}$). The fact that $w' \in \mathcal{B}$ implies that $v := [\mathrm{d}/\mathrm{d}t + \Delta \quad 1]w'$ is an element of $\ker(\mathrm{d}/\mathrm{d}t - \Delta)$; moreover, since $w'|_{T'} \in \mathcal{B}'|_{T'}$, $v|_{T'} \equiv 0$. But the only trajectory in $\ker(\mathrm{d}/\mathrm{d}t - \Delta)$ which is null in T' is the zero trajectory, and therefore we conclude that $v \equiv 0$, which is equivalent to saying that $w' \in \mathcal{B}'$. This shows that $\mathcal{B}'$ is a (proper) hermetic sub-behaviour and hence $\mathcal{B}$ is not weakly controllable. ♦

In order to define strong controllability, we first introduce the notion of rectifiability. A behaviour $\mathcal{B} \subset (\mathbb{K}^q)^T$ is said to be *rectified* if $\mathcal{B} = \{w = \mathrm{col}(w_0, w_f) : T \to \mathbb{K}^q \mid w_0(t) = 0 \ \forall t \in T\}$. This means that the first components w_0 of the trajectories of $\mathcal{B}$ must be zero and that the remaining ones, w_f, are free. We will say that a behaviour with kernel representation $\mathcal{B} = \ker R(\underline{\Omega})$ is *rectifiable* if there exists an invertible polynomial operator $U(\underline{\Omega})$ such that $U(\underline{\Omega})(\mathcal{B})$ is rectified.

Definition 4. A behaviour $\mathcal{B}$ with kernel representation is said to be *strongly controllable* if it is rectifiable.

The next result is a consequence of Definitions 1, 3 and 4.

Proposition 1. *Let $\mathcal{B} \subset (\mathbb{K}^q)^T$ be a behaviour with kernel representation. Then the following implications hold:*

$$\mathcal{B} \text{ is strongly controllable} \Rightarrow \mathcal{B} \text{ is controllable} \Rightarrow \mathcal{B} \text{ is weakly controllable}$$

Proof. The first implication is obvious. In order to prove the second one, suppose that $\mathcal{B}$ is not weakly controllable. Let $\mathcal{B}'$ be a proper hermetic sub-behaviour of $\mathcal{B}$, and consider a trajectory $w_1 \in \mathcal{B} \backslash \mathcal{B}'$. Then there exists a subset T_1 contained in an ND quadrant of T such that $w_1|_{T_1} \notin \mathcal{B}'|_{T_1}$. Given $d > 0$, take $T_2 \subset T$ as a q-complementary subset such that $d(T_1, T_2) > d$. Let further $w_2 \in \mathcal{B}'$. Since $\mathcal{B}'$ is hermetic, every trajectory $w \in \mathcal{B}$ which coincides with w_2 in T_2 must be an element of $\mathcal{B}'$, and hence differs from w_1 in T_1. Consequently, $w_1|_{T_1}$ and $w_2|_{T_2}$ are not concatenable so as to obtain a (global) system trajectory, showing that $\mathcal{B}$ is not controllable. ■

Note that all the controllability properties we have defined correspond to a smaller or greater extent to a lack of "determinism", in the sense that the knowledge about the restrictions of a system trajectory to certain subsets of the domain does not bring any further information about the whole trajectory. An extreme opposite of this situation is when the coincidence of two partial trajectories implies the coincidence of the corresponding global trajectories—in this case we say that the behaviour is *autonomous*. For systems with kernel representation, due to linearity, we can take the following definition of autonomy.

Definition 5. A behaviour $\mathcal{B}$ with kernel representation defined over a domain T is said to be *autonomous* if for any q-complementary subset T' of T $\{w \in \mathcal{B}, w|_{T'} \equiv 0\} \Rightarrow \{w \equiv 0\}$.

The next theorem gives a characterization of autonomy.

Theorem 1. *Let $\mathcal{B} = \ker R(\Omega)$ be a behaviour with kernel representation. Then the following statements are equivalent:*

1. *$\mathcal{B}$ is autonomous.*
2. *$R(\underline{z})$ is a full column rank polynomial matrix.*
3. *$\mathcal{B}$ has no free variables.*

Remark 1. We say that a component w_i of $w \in \mathcal{B}$ is a *free variable* if for all $v \in \mathbb{K}^T$ there is a trajectory $w \in \mathcal{B}$ such that the corresponding component w_i is equal to v.

Proof. The equivalence between 2 and 3 has been proven in (Fornasini *et al.*, 1993; Glüssing-Lüerssen, 1995; Willems, 1988; Wood *et al.*, 1997; Zerz, 1996), and the implication $1 \Rightarrow 3$ can be easily derived. In order to complete the proof we will see that $2 \Rightarrow 1$. Suppose that $R(\underline{z})$ has full column rank. Then there exists a polynomial matrix $S(\underline{z})$ such that $SR = d(\underline{z})I$. This implies that $\mathcal{B}$ is a sub-behaviour of $\ker d(\underline{\Omega})I =: \mathcal{D}$. Since every component w_i of the trajectories w in $\mathcal{D}$ must satisfy the restriction $d(\underline{\Omega})w_i = 0$, it turns out that $w_i|_{T'} = 0 \Rightarrow w_i = 0$ for any q-complementary subset T' of T. This means that $\mathcal{D}$ is autonomous, and hence the same happens with $\mathcal{B}$. ■

3. The Delay-Differential Case

We begin our study of the controllability properties of Section 2 by considering the case of general delay-differential systems over $T = \mathbb{R}$ with kernel representation $\mathcal{B} = \ker R(\mathrm{d}/\mathrm{d}t, \Delta)$, where $R(z_1, z_2)$ is a 2D polynomial matrix of rank r and $R(\mathrm{d}/\mathrm{d}t, \Delta)$ is regarded as an operator acting on C^∞ trajectories. This includes the pure differential case—where $R(\mathrm{d}/\mathrm{d}t, \Delta) = R(\mathrm{d}/\mathrm{d}t)$—as well as the pure delay case—where $R(\mathrm{d}/\mathrm{d}t, \Delta) = R(\Delta)$.

As regards the property of (behavioural) controllability (cf. Definition 1), the following result has been obtained in (Rocha and Willems, 1997).

Theorem 2. *With the previous notation, the following statements are equivalent:*

1. $\mathcal{B} = \ker R(\mathrm{d}/\mathrm{d}t, \Delta)$ *is controllable.*
2. $\mathcal{B} = \mathrm{im}\, M(\mathrm{d}/\mathrm{d}t, \Delta)$ *for some 2D polynomial matrix* $M(z_1, z_2)$.
3. $\mathrm{rank}\, R(\lambda, e^{-\lambda}) = r, \ \forall \lambda \in \mathbb{C}$.

Whenever Condition 2 is satisfied, we will say that $\mathcal{B}$ has an *image representation.* In particular, if $R(\mathrm{d}/\mathrm{d}t, \Delta) = R(\mathrm{d}/\mathrm{d}t)$, this result reduces to the characterization of controllability given in (Willems, 1988) for the pure differential case. Moreover, it follows from this theorem that, for systems with a pseudo-state space description:

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = A(\Delta)x + Bu \\ y = Cx + Du \end{cases}$$

the behaviour $\mathcal{B} = \ker \mathrm{col}([\mathrm{d}/\mathrm{d}t - A(\Delta) \mid -B) \mid 0], [C \mid D \mid -I])$ of the joint (x, u, y)-variable is controllable if and only if $\mathrm{rank}[\lambda I - A(e^{-\lambda}) \mid B]$ is full for all $\lambda \in \mathbb{C}$. This corresponds to the condition of spectral controllability introduced in (Manitius, 1981).

As for weak and strong controllability (cf. Definitions 3 and 4, respectively), the conclusions of the investigation carried out in (Rocha, 1995) can be reformulated as follows. A 2D polynomial matrix $R(z_1, z_2)$ of rank s can be factored as $R = FR_1$, where F and R_1 are 2D polynomial matrices with full column rank and full row rank s, respectively; we will call such a decomposition an FCR/FRR factorization. $R(z_1, z_2)$ is said to be *DD factor left-prime* (DDFLP) if every FCR/FRR factorization $R = FR_1$ is such that $\mathrm{rank}\, F(\lambda, e^{-\lambda}) = s$ for all $\lambda \in \mathbb{C}$. (This means that $\ker F(\mathrm{d}/\mathrm{d}t, \Delta) = \{0\}$ and hence $\ker R(\mathrm{d}/\mathrm{d}t, \Delta) = \ker R_1(\mathrm{d}/\mathrm{d}t, \Delta)$.) For instance,

$$R(z_1, z_2) = \begin{bmatrix} z_1 - 1 \\ z_2 - 1 \end{bmatrix} \begin{bmatrix} z_1 + 3 & 1 \end{bmatrix}$$

is DDFLP, since its FRR/FCR factorizations are unique up to constants and $[\lambda - 1 \; e^{-\lambda} - 1]^T$ has rank 1 for all $\lambda \in \mathbb{C}$. For matrices with full row rank, DD factor left-primeness reduces to factor left-primeness (FLP), i.e. to the absence of nonunimodular left factors.

Theorem 3. *A delay-differential system with behaviour* $\mathcal{B} = \ker R(\mathrm{d}/\mathrm{d}t, \Delta)$ *is weakly controllable if and only if* $R(z_1, z_2)$ *is DDFLP.*

Proof. Since $R(z_1, z_2)$ is a 2D polynomial matrix, if it is not DDFLP, there exist 2D polynomial matrices $F(z_1, z_2)$ with full column rank and $R_1(z_1, z_2)$ with full row rank such that:

$$R(z_1, z_2) = F(z_1, z_2) R_1(z_1, z_2)$$

but $F(\lambda, e^{-\lambda})$ has a rank drop for some $\lambda \in \mathbb{C}$. This implies that $\ker R_1(z_1, z_2)$ is a proper sub-behaviour of $\mathcal{B}$. Now, since $F(z_1, z_2)$ has full column rank, $\ker F(\mathrm{d}/\mathrm{d}t, \Delta)$ is autonomous, and hence:

$$\Big(v \in \ker F(\mathrm{d}/\mathrm{d}t, \Delta), v|_{T'} = 0\Big) \Rightarrow (v \equiv 0)$$

for every q-complementary subset T' of $T = \mathbb{R}$. Therefore, if w is a trajectory in $\mathcal{B}$ and $(R_1(\mathrm{d}/\mathrm{d}t, \Delta)w)|_{T'} = 0$, then $R_1(\mathrm{d}/\mathrm{d}t, \Delta)w \equiv 0$. In other words, if $w \in \mathcal{B}$ and $w|_{T'} \in \ker R_1|_{T'}$, then $w \in \ker R_1$. So, $\ker R_1$ is a proper hermetic sub-behaviour of $\mathcal{B}$, meaning that $\mathcal{B}$ is not weakly controllable. Thus, the fact that $R(z_1, z_2)$ is DDLFP is a necessary condition for the weak controllability of $\mathcal{B}$.

In order to show the converse implication, suppose that $\mathcal{B}$ is not weakly controllable, admitting therefore a proper hermetic sub-behaviour. It is possible to show that in this setting $\mathcal{B}$ also has a regular proper hermetic sub-behaviour $\mathcal{B}' = \ker R'(\mathrm{d}/\mathrm{d}t, \Delta)$ such that $R'(z_1, z_2)$ has full row rank. Let $F(z_1, z_2)$ be such that $R = FR'$. Taking into account that $\mathcal{B}'$ is hermetic, it follows that $F(z_1, z_2)$ must have full column rank. This implies that $\operatorname{rank} R' = \operatorname{rank} R$. However, since $\mathcal{B}'$ is a proper sub-behaviour, $\ker F(\mathrm{d}/\mathrm{d}t, \Delta) \neq \{0\}$. Consequently, R is not DDFLP, proving that the DD factor left-primeness of $R(z_1, z_2)$ is also sufficient condition for the weak controllability of $\mathcal{B}$. ■

Theorem 4. *A delay-differential system with behaviour* $\mathcal{B} = \ker R(\mathrm{d}/\mathrm{d}t, \Delta)$, *with* $\operatorname{rank} R(z_1, z_2) = r$, *is strongly controllable if and only if* $\operatorname{rank} R(\lambda_1, \lambda_2) = r$ $\forall (\lambda_1, \lambda_2) \in \mathbb{C} \times \mathbb{C}$.

If the condition of the theorem is satisfied we will say that $R(z_1, z_2)$ is *zero prime in the generalized sense* (GZP). For matrices with full row (resp. column) rank, generalized zero primeness corresponds to zero left-primeness (ZLP) (resp., zero right-primeness (ZRP)), i.e. to the fact that the row (resp. column) rank is full for all $(\lambda_1, \lambda_2) \in \mathbb{C} \times \mathbb{C}$.

Proof. If $R(z_1, z_2)$ is GZP, there exist unimodular matrices $U(z_1, z_2)$ and $W(z_1, z_2)$ such that

$$WRU = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}$$

Therefore

$$Rw = 0 \Leftrightarrow [I \;\; 0]Vw = 0$$

where $V(z_1, z_2)$ is the inverse of $U(z_1, z_2)$. This implies that $V(D, \Delta)(\mathcal{B}) = \ker[I \;\; 0]$, and so $\mathcal{B}$ is rectifiable, i.e. it is strongly controllable.
Suppose now that $\mathcal{B}$ is strongly controllable. Then there exists an invertible operator $V(\mathrm{d}/\mathrm{d}t, \Delta)$ such that $V(\mathrm{d}/\mathrm{d}t, \Delta)(\ker R) = \ker[I \;\; 0]$ and, consequently, $\ker R = \ker[I \;\; 0]V$. Since $V(\mathrm{d}/\mathrm{d}t, \Delta)$ is invertible, $V(z_1, z_2)$ is unimodular, and therefore $[I \;\; 0]V(z_1, z_2)$ is zero left-prime. This implies that $R(z_1, z_2)$ is GZP. ■

The preceding results imply that, for systems with a pseudo-state space description,

$$\begin{cases} \mathrm{d}x/\mathrm{d}t = A(\Delta)x + B(\Delta)u \\ y = C(\Delta)x + D(\Delta)u \end{cases}$$

the behaviour $\mathcal{B} = \ker \mathrm{col}([\mathrm{d}/\mathrm{d}t - A(\Delta) \mid -B(\Delta) \mid 0], [C(\Delta) \mid D(\Delta) \mid -I])$ of the joint (x, u, y)-variable is weakly controllable if and only if $[z_1 I - A(z_2) \mid B(z_2)]$ is factor left-prime and strongly controllable if and only if $[z_1 I - A(z_2) \mid B(z_2)]$ is zero left-prime. These characterizations coincide with the definitions given in (Lévy, 1981), showing that our redefinitions of weak and strong controllability are adequate.

As we have seen in Proposition 1, strong controllability implies controllability, which in turn implies weak controllability. The examples below show that the reciprocal implications are not true.

Example 2. The delay-differential system

$$[\Delta - 1 \;\; \mathrm{d}/\mathrm{d}t]w = 0$$

has an associated 2D polynomial matrix $R(z_1, z_2) = [z_2 - 1 \;\; z_1]$, which is (DD)FLP, and hence weakly controllable. However, $R(\lambda, e^{-\lambda}) = [e^{-\lambda} - 1 \;\; \lambda]$ drops in rank for $\lambda = 0$, and therefore the system is not controllable. ♦

Example 3. The system $R(\mathrm{d}/\mathrm{d}t, \Delta)w = 0$ with $R(z_1, z_2) = [z_2 + 1 \;\; z_1]$ is controllable since $\mathrm{rank}[e^{-\lambda} + 1 \;\; \lambda] = 1 \; \forall \lambda \in \mathbb{C}$. But, because $R(\lambda_1, \lambda_2)$ drops in rank for $(\lambda_1, \lambda_2) = (1, -1)$, the system is not strongly controllable. ♦

4. The Discrete ND Case

Let us now consider systems over $T = \mathbb{Z}^N$ $(N \geq 1)$ with kernel representation $\mathcal{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1})$, where $(\underline{\sigma}, \underline{\sigma}^{-1}) := (\sigma_1, \dots, \sigma_N, \sigma_1^{-1}, \dots, \sigma_N^{-1})$, σ_i denotes the i-th ND shift (cf. Section 2), and $R(\underline{z}, \underline{z}^{-1}) := R(z_1, \dots, z_N, z_1^{-1}, \dots, z_N^{-1})$ is an ND Laurent polynomial matrix. For this class of systems, (behavioural) controllability has been studied in (Rocha and Willems, 1991; Willems, 1988), respectively for the 1D and the 2D cases, and, more recently, in (Wood *et al.*, 1997; Zerz, 1996) for the general ND case.

In the characterization of this property, the following notion of primeness plays an important role. An ND (Laurent) polynomial matrix R is said to be *factor left-prime in the generalized sense* (GFLP) if the existence of a factorization $R = FR_1$ with $\operatorname{rank} R_1 = \operatorname{rank} R$ implies the existence of an (ND (Laurent) polynomial) matrix E such that $R_1 = ER$ (Zerz, 1996). As shown in (Zerz, 1996), for matrices with full row rank generalized factor left primeness implies factor left primeness, but the converse is not true. However, it is not difficult to show that these two properties are equivalent for $N \leq 2$.

Theorem 5. (Rocha and Willems, 1991; Willems, 1988; Wood *et al.*, 1997; Zerz, 1996) *Let $\mathcal{B}$ be a behaviour over $T = \mathbb{Z}^N$ with kernel representation $\mathcal{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1})$. Then the following statements are equivalent:*

1. *$\mathcal{B}$ is controllable.*
2. *$\mathcal{B} = \operatorname{im} M(\underline{\sigma}, \underline{\sigma}^{-1})$ for some ND Laurent polynomial matrix $M(\underline{z}, \underline{z}^{-1})$.*
3. *$R(\underline{z}, \underline{z}^{-1})$ is GFLP.*

If $N \leq 2$, controllability is still equivalent to the existence of a kernel representation associated with a full row rank FLP Laurent polynomial matrix $\bar{R}$ (Rocha and Willems, 1991; Willems, 1988). In the 1D case, this means that $\operatorname{rank} \bar{R}(\lambda, \lambda^{-1})$ is full for all $\lambda \in \mathbb{C} \setminus \{0\}$, or, equivalently, that all the kernel representations of $\mathcal{B}$ are associated with Laurent polynomial matrices which have constant rank when evaluated for $\lambda \in \mathbb{C} \setminus \{0\}$ (Willems, 1988).

Similarly to the delay-differential case, strong controllability can be characterized by means of the following rank constancy condition.

Theorem 6. *The ND behaviour $\mathcal{B} = \ker R(\underline{\sigma}, \underline{\sigma}^{-1})$ is strongly controllable if and only if $\operatorname{rank} R(\underline{\lambda}, \underline{\lambda}^{-1}) = s \ \ \forall \underline{\lambda} \in (\mathbb{C} \setminus \{0\})^N$, where s denotes the rank of the ND Laurent polynomial matrix $R(\underline{z}, \underline{z}^{-1})$.*

The proof of this result is analogous to that of Theorem 4. As before, if the condition of the theorem is satisfied, we will say that $R(\underline{z}, \underline{z}^{-1})$ is zero prime in the generalized sense.

In order to study weak controllability, we will first investigate the properties of hermetic sub-behaviours. Let $\mathcal{B}$ be a behaviour over the domain T with trajectories $w = \operatorname{col}(w_1, \ \ldots, w_q)$ taking values in $\mathbb{K}^q$. We will say that a choice $w' = \operatorname{col}(w_{i_1}, \ldots, w_{i_m})$ of components of w is a *set of free variables* if for all $v \in (\mathbb{K}^m)^T$ there exists a trajectory $w \in \mathcal{B}$ such that the corresponding choice of variables w' is equal to v. The *number of free variables* of a behaviour $\mathcal{B}$ is defined as the length of its largest set(s) of free variables.

Theorem 7. *Let $\mathcal{B}$ be an ND behaviour (over $T = \mathbb{Z}^N$) with kernel representation, and $\mathcal{B}'$ a sub-behaviour of $\mathcal{B}$. The following statements are equivalent:*

1. *$\mathcal{B}'$ is a hermetic sub-behaviour of $\mathcal{B}$.*

2. *$\mathcal{B}'$ and $\mathcal{B}$ have the same number of free variables.*

3. *$\mathcal{B}/\mathcal{B}'$ is autonomous.*

4. *$\mathcal{B}'$ has a sub-behaviour of the form $r(\underline{\sigma},\underline{\sigma}^{-1})(\mathcal{B})$, for some nonzero ND Laurent polynomial $r(\underline{z},\underline{z}^{-1})$.*

Proof. $1 \Rightarrow 2$: Suppose that $\mathcal{B}$ and $\mathcal{B}'$ do not have the same number of free variables. Fix a maximal set w' of free variables in $\mathcal{B}'$. Then w' is also a set of free variables in $\mathcal{B}$, so there must be an additional free variable w_j in $\mathcal{B}$. Let T' be an arbitrary q-complementary set in T. It is possible to construct $w \in \mathcal{B}$ such that $w|_{T'} \in \mathcal{B}'|_{T'}$ but with the component w_j chosen to ensure that $w \notin \mathcal{B}'$, otherwise w_j would be free in $\mathcal{B}'$. Hence $\mathcal{B}'$ is not a hermetic sub- behaviour of $\mathcal{B}$.

$2 \Rightarrow 3$: Suppose that $\mathcal{B}'$ and $\mathcal{B}$ have the same number of free variables. Then it follows from the results in (Wood *et al.*, 1997) that $\mathcal{B}/\mathcal{B}'$ is a behaviour with kernel representation which has no free variables, and is therefore autonomous.

$3 \Rightarrow 4$: If $\mathcal{B}/\mathcal{B}'$ is autonomous, then there exists a nonzero ND Laurent polynomial $r(\underline{z},\underline{z}^{-1})$ such that $r(\underline{\sigma},\underline{\sigma}^{-1})(w+\mathcal{B}') = \mathcal{B}'$ for all $w \in \mathcal{B}$. Equivalently, $rw \in \mathcal{B}'$ for all $w \in \mathcal{B}$, so $r(\mathcal{B})$ is contained in $\mathcal{B}'$. Since $r(\mathcal{B})$ is the image of $\mathcal{B}$ under a polynomial ND shift operator, it has a kernel representation and is therefore a sub-behaviour of $\mathcal{B}'$.

$4 \Rightarrow 1$: Suppose that $r(\underline{\sigma},\underline{\sigma}^{-1})(\mathcal{B})$ is a sub-behaviour of $\mathcal{B}'$, for some nonzero ND Laurent polynomial r. Let $\mathcal{B}' = \ker R'$ be a kernel representation of $\mathcal{B}'$. Then $\mathcal{B} \subset \ker rR' =: \mathcal{B}^{\text{ext}}$. We will show that $\mathcal{B}'$ is a hermetic sub-behaviour of $\mathcal{B}^{\text{ext}}$. Let T' be an arbitrary q-complementary subset of T, and suppose that $w \in \mathcal{B}^{\text{ext}}$ is such that $w|_{T'} \in \mathcal{B}'|_{T'}$. Then there exists $w' \in \mathcal{B}'$ such that $(w-w')|_{T'} = 0$, and so $(R'(w-w'))|_{T''} = 0$ for some q-complementary subset $T'' \subset T'$. Since $(w-w') \in \mathcal{B}^{\text{ext}}$, $R'(w-w') \in \ker r$, and, taking into account that $\ker r$ is autonomous, the fact that $(R'(w-w'))|_{T''} = 0$ implies that $R'(w-w') = 0$. So, $w - w' \in \mathcal{B}'$, and $w \in \mathcal{B}'$, showing that $\mathcal{B}'$ is a hermetic sub-behaviour of $\mathcal{B}^{\text{ext}}$. Finally, since $\mathcal{B} \in \mathcal{B}^{\text{ext}}$, this implies that $\mathcal{B}'$ is also a hermetic sub-behaviour of $\mathcal{B}$. ■

We call $\mathcal{B}$ a *divisible* behaviour if $r(\underline{\sigma},\underline{\sigma}^{-1})(\mathcal{B}) = \mathcal{B}$ for every nonzero ND Laurent polynomial r. Our next result follows immediately from Theorem 7 (namely from the equivalence between 1 and 4) and provides a characterization for weak controllability.

Corollary 1. *Let $\mathcal{B}$ be a behaviour with kernel representation over $T = \mathbb{Z}^N$. Then $\mathcal{B}$ is weakly controllable if and only if it is divisible.*

It is shown in (Wood *et al.*, 1997) that, for the ND case, controllability is equivalent to divisibility. The following result is a straightforward consequence of this fact.

Corollary 2. *Let $\mathcal{B}$ be a behaviour with kernel representation over $T = \mathbb{Z}^N$. Then $\mathcal{B}$ is weakly controllable if and only if it is controllable.*

Taking Theorem 5 into account, this yelds further characterizations of weak controllability in terms of the existence of an image representation or, equivalently, of the GFL primeness of the kernel representations.

5. Concluding Remarks

Based on the results of the previous sections we conclude that, for all the considered cases, controllabilty is equivalent to the existence of an image representation, whereas strong controllability is equivalent to a rank constancy condition (GZP) on the ND (Laurent) polynomial matrices associated with the corresponding kernel representations. These two properties coincide (only) in the 1D case. Weak controllability is equivalent to controllability in the general ND case, but not in the delay-differential case.

The characterization of the different controllability properties of a behaviour $\mathcal{B} = \ker R(\underline{\Omega})$ in terms of the (Laurent) polynomial matrix R can be summarized as follows. In the delay-differential case, weak controllability is equivalent to the DDFL primeness of $R(z_1, z_2)$, controllability is equivalent to the constancy of the rank of $R(\lambda, e^{-\lambda})$, and strong controllability amounts to the constancy of the rank of $R(\lambda_1, \lambda_2)$. As we previously mentioned, these characterizations show that strong controllability implies controllability, which in turn implies weak controllability, but the reciprocal implications are not true.

In the 1D case, the three controllability properties coincide and are equivalent to the constancy of the rank of $R(\lambda, \lambda^{-1})$.

Weak controllability and controllability coincide in the 2D case, and are equivalent to the GFL primeness of R, as well as to the existence of a kernel representation such that the associated Laurent polynomial matrix is FLP. Strong controllability is strictly stronger than these properties and is equivalent to the rank constancy of $R(\lambda_1, \lambda_2, \lambda_1^{-1}, \lambda_2^{-1})$.

The situation is almost analogous for the ND case ($N \geq 3$), with the difference that now (weakly) controllable behaviours do not necessarily admit FLP representations.

References

Fornasini E., Rocha P. and Zampieri S. (1993): *State space realization of 2-D finite-dimensional behaviours.* — SIAM J. Contr. Optim., Vol.31, No.6, pp.1502–1517.

Glüssing-Lüerssen H. (1995): *A behavioral approach to delay-differential systems.* — Preprint.

Lévy B.C. (1981): *2-D polynomial and rational matrices, and their applications for the modelling of 2-D dynamical systems.* — Ph.D. Thesis, Stanford University.

Manitius A. (1981): *Necessary and sufficient conditions of approximate controllability for general linear retarded systems.* — SIAM J. Contr. Optim., Vol.19, No.4, pp.516–532.

Morse A.S. (1976): *Ring models for delay-differential systems.* — Automatica, Vol.12, pp.529–531.

Rocha P. (1995): *Behavioural interpretation of weak and strong controllability for delay-differential systems.* — Prep. IFAC Conf. *System Structure and Control*, Nantes, France, 4–7 July 95, pp.506–510.

Rocha P. and Willems J.C. (1991): *Controllability of 2-D systems.* — IEEE Trans. Automat. Contr., Vol.36, No.4, pp.413–423.

Rocha P. and Willems J.C. (1997): *Behavioural controllability of delay-differential systems.* — SIAM J. Contr. Optim., Vol.35, No.1, pp.254–264.

Willems J.C. (1988): *Models for dynamics.* — Dynamics Reported, Vol.2, pp.171–269.

Wood J., Rogers E. and Owens D.H. (1997): *Controllable and autonomous nD linear systems.* — Preprint.

Zerz E. (1996): *Primeness of multivariate polynomial matrices.* — Syst. Contr. Lett., Vol.29, No.3, pp.139–146.