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STABILITY AND OPTIMAL CONTROL OF NONLINEAR DESCRIPTOR SYSTEMS: A SURVEY

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In recent years the analysis and synthesis of control systems in descriptor form have been established. This general description of dynamical systems is important for many applications in mechanics and mechatronics, in electrical and electronic engineering, and in chemical engineering as well. This contribution presents a survey of results on the stability analysis and on the optimal control design of such systems. Lyapunov's stability theory is generalized for descriptor systems based on the stability theory with respect to a part of variables. Similarly, the calculus of variations and Pontryagin's maximum principle are checked for a possible application in descriptor systems. Here, the notion of causality plays an important role in whether or not Pontryagin's maximum principle can be applied.

1. Introduction

In the last decade significant progress in dynamic systems described by differential-algebraic equations (DAE) has been observed. In control theory descriptor (or singular) control systems have been investigated (Dai, 1989) and effective codes have been developed in numerical mathematics to simulate such systems (Brenan *et al.*, 1989; Führer, 1988; Griepentrog and März, 1986; Hairer *et al.*, 1989; Simeon, 1994). But still there are many unsolved problems related to the analysis, design and simulation of descriptor systems. Recent progress in parameter identification of linear mechanical descriptor systems was outlined in (Schmidt, 1994). New methods for the analysis and control design of linear mechanical descriptor systems can be found in (Schüpphaus, 1995). A thorough investigation of observers for general linear descriptor systems was conducted in (Hou, 1995). Also the stability behaviour of such systems has been discussed (Bajić, 1992; Müller, 1993) and first attempts to deal with the stability problems of nonlinear mechanical descriptor systems can be found (Müller, 1994). In (Müller, 1993; 1994) more references on related investigations are given. Following (Müller, 1996b), in this contribution some new results on the stability of nonlinear descriptor systems are presented.

Similarly, the calculus of variations and Pontryagin's maximum principle are checked for a possible application in descriptor systems. The main problem of the optimal control design consists in the causality or non-causality of the descriptor system

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(Müller, 1997b). Causality plays an important role in whether or not Pontryagin's maximum principle can be applied. Causality and non-causality distinguish between the cases where the descriptor system is exclusively governed by the control input or by its time-derivatives additionally. In the unusual case of non-causal systems a quite different problem of optimal control design has to be considered (Müller, 1997a; 1996a).

2. Problem Statement

Controlled time-invariant finite-dimensional descriptor systems can be described in semi-explicit form by

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) \tag{1}$$

$$\boldsymbol{0} = \boldsymbol{f}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) \tag{2}$$

where $\boldsymbol{x}_i$, $i = 1,2$ are n_i-dimensional vectors and $n_1 + n_2 = n$. For the following discussion of stability and of optimal control design we assume that the vector functions are sufficiently smooth. Additionally, the uncontrolled system may have an equilibrium point

$$\boldsymbol{x}_1 = \boldsymbol{0}, \qquad \boldsymbol{x}_2 = \boldsymbol{0}, \qquad \boldsymbol{f}_1(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{0}, \qquad \boldsymbol{f}_2(\boldsymbol{0},\boldsymbol{0},\boldsymbol{0}) = \boldsymbol{0} \tag{3}$$

A special class of differential-algebraic equations (1), (2) consists of mechanical descriptor systems with holonomic constraints described by Lagrange's equation of first kind:

$$\boldsymbol{M}(\boldsymbol{z})\ddot{\boldsymbol{z}} + \boldsymbol{k}(\boldsymbol{z}, \dot{\boldsymbol{z}}) = \boldsymbol{F}^T(\boldsymbol{z})\boldsymbol{\lambda} + \boldsymbol{T}(\mathbf{z})\boldsymbol{u} \tag{4}$$

$$\mathbf{f}(\boldsymbol{z}) + \boldsymbol{R}\boldsymbol{u} = \boldsymbol{0} \tag{5}$$

We confine ourselves to this case, but nonholonomic constraints could be considered too. The f-dimensional vector of displacements is denoted by $\boldsymbol{z}$ and $\boldsymbol{u}$ is the r-dimensional vector of control inputs. The p-dimensional vector $\boldsymbol{\lambda}$ represents constraint forces due to p holonomic constraints (5) (in the calculus of variations they are Lagrange multipiers). The matrix of inertia $\boldsymbol{M}(\boldsymbol{z})$ is assumed to be symmetric, bounded and (uniformely) positive definite. The vector function $\boldsymbol{k}(\boldsymbol{z}, \dot{\boldsymbol{z}})$ includes Coriolis and centrifugal forces and uncontrolled applied forces as well. $\boldsymbol{T}$ is the input matrix of a suitable dimension. The $p \times f$-matrix $\boldsymbol{F}(\boldsymbol{z})$ is the Jacobian matrix of the constraint (5):

$$\boldsymbol{F}(\boldsymbol{z}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{z}^T} \tag{6}$$

It is assumed that the constraints are independent. In the case of linear constraints this assumption is guaranteed by the condition

$$\operatorname{rank} \boldsymbol{F} = p \tag{7}$$

The matrix $\boldsymbol{R}$ represents a possible constant control input matrix in the constraints. The form (1), (2) of the mechanical descriptor system (4), (5) is obtained by defining

$$\boldsymbol{x}_1 = \begin{bmatrix} \boldsymbol{z} \\ \dot{\boldsymbol{z}} \end{bmatrix}, \quad \boldsymbol{x}_2 = \boldsymbol{\lambda} \tag{8}$$

$$\boldsymbol{f}_1 = \begin{bmatrix} \dot{\boldsymbol{z}} \\ -\boldsymbol{M}^{-1}(\boldsymbol{k} - \boldsymbol{F}^T\boldsymbol{\lambda} - \boldsymbol{T}\boldsymbol{u}) \end{bmatrix}, \quad \boldsymbol{f}_2 = \boldsymbol{f} + \boldsymbol{R}\boldsymbol{u} \tag{9}$$

$$n_1 = 2f, \quad n_2 = p \tag{10}$$

The problems of stability and optimal control of nonlinear descriptor systems consist on the one hand in the discussion of the stability of the equilibrium point (3) of the system (1), (2) and, on the other hand, in the control design for (1), (2) with respect to the performance criterion

$$J = \int_0^T f_0(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u})\, dt \longrightarrow \underset{\boldsymbol{u}\in U}{\text{minimum}} \tag{11}$$

where U is a set of bounded or unbounded control functions $\boldsymbol{u}(t)$. Again it is assumed that the function f_0 is sufficiently smooth with respect to its arguments.

3. Representations of the System

To study descriptor systems carefully it is advantageous to have in mind different forms of representation of the dynamical system (1), (2) which are related either to the generation of differential equations for the $\boldsymbol{x}_2$-vector differentiating the algebraic equations (2) as long as necessary or to the elimination of redundant coordinates generating the state space differential equations. In both the cases the key for that is the index k of the system (1), (2) which is roughly speaking the number of differentiations of the algebraic equations (2) to get the underlying set of ordinary differential equations (ODE). For simplicity, we assume that all algebraic equations have a uniform index, i.e. that they all have the same index. Then we have in the case of index k:

$$\frac{\mathrm{d}^j \boldsymbol{f}_2}{\mathrm{d}t^j} \equiv L^j(\boldsymbol{f}_2) = \boldsymbol{0}, \qquad j = 0, \ldots, k-1 \tag{12}$$

$$\dot{\boldsymbol{x}}_2 = -\left(\frac{\partial}{\partial \boldsymbol{x}_2^T} L_{f_1}^{k-1}(\boldsymbol{f}_2)\right)^{-1} L^k(\boldsymbol{f}_2) = \bar{\boldsymbol{f}}_2(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}, \ldots, \boldsymbol{u}^{(k)}) \tag{13}$$

where

$$L(\cdot) = L_{f_1}(\cdot) + L_{\dot{u}}(\cdot) + \frac{\Delta}{\Delta t}(\cdot) \tag{14}$$

$$L_{f_1}(\cdot) = \frac{\partial(\cdot)}{\partial \boldsymbol{x}_1^T}\boldsymbol{f}_1, \quad L_{u^{(j)}}(\cdot) = \frac{\partial(\cdot)}{\partial \boldsymbol{u}^T}\boldsymbol{u}^{(j)}, \quad \frac{\Delta}{\Delta t}L_{u^{(j)}}(\cdot) = L_{u^{(j+1)}}(\cdot) \tag{15}$$

The operator (14) is defined by the operators (15) where the first two are some Lie derivatives and the last one is only applied to $L_{u^{(j)}}$ by differentiating the control input function with respect to time. The vector functions $L^j(\boldsymbol{f}_2)$, $j = 0, \ldots, k-2$ depend on $\boldsymbol{x}_1$ but not on $\boldsymbol{x}_2$. The function $L^{k-1}(\boldsymbol{f}_2)$ depends on $\boldsymbol{x}_2$ such that the related Jacobian matrix is regular and the differential equation (13) can be derived. $L^{k-1}(\boldsymbol{f}_2) = \boldsymbol{0}$ is a first integral of (13). Additionally, the functions $L^j(\boldsymbol{f}_2)$ depend generally on the time derivatives of the control input $\boldsymbol{u}$: $\boldsymbol{u}, \dot{\boldsymbol{u}}, \ldots, \boldsymbol{u}^{(j)}$, $j = 0, \ldots, k$.

Therefore the descriptor system (1), (2) can also be represented by the differential equations (1) and (13) on a manifold described by the invariants (12).

Additionally, in principle it is possible to use the invariants (12) to eliminate kn_2 redundant variables and to end up with a set of $n_1 - (k-1)n_2$ ordinary differential equations in the state space. If

$$\operatorname{rank}\left\{\frac{\partial}{\partial \mathbf{x}_1^T}\begin{bmatrix} L^0(\boldsymbol{f}_2) \\ \vdots \\ L^{k-2}(\boldsymbol{f}_2)\end{bmatrix}\right\} = (k-1)n_2 \tag{16}$$

which we may assume, the invariants $L^j(\boldsymbol{f}_2) = \boldsymbol{0}$, $j = 0, \ldots, k-2$ can be solved with respect to some redundant variables $\boldsymbol{x}_{1r}$ depending on some essential variables $\boldsymbol{x}_{1e}$:

$$\boldsymbol{x}_{1r} = \boldsymbol{g}_{1r}\left(\boldsymbol{x}_{1e}, \boldsymbol{u}, \ldots, \boldsymbol{u}^{(k-2)}\right) \tag{17}$$

with the splitting

$$\begin{bmatrix} \boldsymbol{x}_{1e} \\ \boldsymbol{x}_{1r} \end{bmatrix} = \boldsymbol{P}\boldsymbol{x}_1 \tag{18}$$

where $\boldsymbol{P}$ is a regular permutation matrix. As regards the dimensions, we have

$$\dim \boldsymbol{x}_{1r} = (k-1)n_2, \quad \dim \boldsymbol{x}_{1e} = n_1 - (k-1)n_2 \tag{19}$$

The last invariant $L^{k-1}(\boldsymbol{f}_2) = \boldsymbol{0}$ results in

$$\boldsymbol{x}_2 = \bar{\boldsymbol{g}}_{2r}\left(\boldsymbol{x}_1, \boldsymbol{u}, \ldots, \boldsymbol{u}^{(k-1)}\right) = \boldsymbol{g}_{2r}\left(\boldsymbol{x}_{1e}, \boldsymbol{u}, \ldots, \boldsymbol{u}^{(k-1)}\right) \tag{20}$$

where (17) was used in (20). By these eliminations we end with

$$\begin{bmatrix} \dot{\boldsymbol{x}}_{1e} \\ \dot{\boldsymbol{x}}_{1r} \end{bmatrix} = \boldsymbol{P}\boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) = \begin{bmatrix} \boldsymbol{f}_{1e}(\boldsymbol{x}_{1e}, \boldsymbol{u}, \ldots, \boldsymbol{u}^{(k-1)}) \\ \boldsymbol{f}_{1r}(\boldsymbol{x}_{1e}, \boldsymbol{u}, \ldots, \boldsymbol{u}^{(k-1)}) \end{bmatrix} \tag{21}$$

The related state space ODE of the DAE system (1), (2) is given by the essential part of (21):

$$\dot{\boldsymbol{x}}_{1e} = \boldsymbol{f}_{1e}\left(\boldsymbol{x}_{1e}, \boldsymbol{u}, \ldots, \boldsymbol{u}^{(k-1)}\right) \tag{22}$$

In principle, the control system under consideration can be represented in three descriptions, either in the form of a descriptor system by DAE's (1), (2) or in the form of ODE's (1), (13) on the manifold (12), or in the form of the state space ODE's (22). The description (1), (2) is the most natural one since it applies usual modelling procedures. The description (22) is the most favourable one, but in many applications with nonlinear dynamics it is almost impossible to obtain it. Finally, the description (1), (13), (12) is a connecting link helpfully used in analytical and numerical investigations.

The three types of descriptions have to be supplemented by consistent initial conditions satisfying the invariants (12),

$$L^j(\boldsymbol{f}_2)\Big|_{t=0} = \mathbf{0}, \quad j = 0, \dots, k-1 \tag{23}$$

or the relations (17), (20):

$$\begin{cases} \boldsymbol{x}_{1r0} &= \boldsymbol{g}_{1r}\left(\boldsymbol{x}_{1e0}, \boldsymbol{u}_0, \dots, \boldsymbol{u}_0^{(k-2)}\right) \\ \boldsymbol{x}_{20} &= \boldsymbol{g}_{2r}\left(\boldsymbol{x}_{1e0}, \boldsymbol{u}_0, \dots, \boldsymbol{u}_0^{(k-1)}\right), \quad \boldsymbol{x}_{1e0} \text{ arbitrary} \end{cases} \tag{24}$$

4. Causality

The different representations of the control system in the preceding section show that the system behaviour may depend not only on the control input $\boldsymbol{u}$ but also on its time-derivatives $\dot{\boldsymbol{u}}, \ddot{\boldsymbol{u}}, \dots, \boldsymbol{u}^{(k)}$. Although the DAE description (1), (2) shows explicitly only the input $\boldsymbol{u}$, there may be hidden effects related to time-derivatives $\dot{\boldsymbol{u}}, \dots, \boldsymbol{u}^{(k)}$ as it is shown by the representations (1), (12), (13) or (22). This situation is very different from the common state space discussions. Any control design method has to take care of this unconvenient problem. Therefore it is necessary to clarify this unusual situation. Similarly to a definition by Dai (1989) for discrete-time systems, the notion of "causality" is introduced.

Definition 1. The descriptor system (1), (2) is called *causal* if the solution $[\boldsymbol{x}_1(t), \boldsymbol{x}_2(t)]$ does not depend on $\dot{\boldsymbol{u}}(t), \dots, \boldsymbol{u}^{(k-1)}(t)$ but only on $\boldsymbol{u}(t)$.

This definition does not have to consider $\boldsymbol{u}^{(k)}$ as it may be expected by the notation of (13), because $L^{(k-1)}(\boldsymbol{f}_2) = \mathbf{0}$ is a first integral of (13) depending only on $\dot{\boldsymbol{u}}(t), \dots, \boldsymbol{u}^{(k-1)}(t)$. It should be mentioned that descriptor systems of index $k = 1$ are always causal. Non-causal systems may only appear for $k \geq 2$.

Now the question arises how causality can be checked. For linear descriptor systems characterized by regular matrix pencils a necessary and sufficient condition is available if the system is represented in its Weierstrass-Kronecker canonical form (Dai, 1989). But this solution does not help us in the case of nonlinear descriptor systems (1), (2). Therefore, in (Müller, 1997a; 1996a; 1997b) a new criterion was introduced by assuming sufficient smoothness of the vector functions. Based on the description (1), (12), (13) the following theorem can be proved.

Theorem 1. (Müller, 1997b) *Given a uniform index k, the descriptor system (1), (2) or (1), (12), (13) is causal if*

$$\frac{\partial}{\partial \boldsymbol{u}^T}\left(L^j_{f_1}(\boldsymbol{f}_2)\right) = \boldsymbol{0}, \quad j = 0, \dots, k-2 \tag{25}$$

In this case the invariants (12) are of the form

$$L^j(\boldsymbol{f}_2) \equiv L^j_{f_1}(\boldsymbol{f}_2) \equiv \boldsymbol{f}_{2j}(\boldsymbol{x}_1, \boldsymbol{u}) = \boldsymbol{0}, \quad j = 0, \dots, k-2 \tag{26}$$

$$L^{(k-1)}(\boldsymbol{f}_2) \equiv L^{(k-1)}_{f_1}(\boldsymbol{f}_2) \equiv \boldsymbol{f}_{2,k-1}(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{u}) = \boldsymbol{0} \tag{27}$$

They do not depend on the time-derivatives of the control input.

Accordingly, all the functions in (20)–(24) do not depend on the time-derivatives of the control input, but only on $\boldsymbol{u}(t)$.

To illustrate this result, a linear descriptor system in semi-explicit form is considered:

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{A}_{11}\boldsymbol{x}_1 + \boldsymbol{A}_{12}\boldsymbol{x}_2 + \boldsymbol{B}_1\boldsymbol{u} \tag{28}$$

$$\boldsymbol{0} = \boldsymbol{A}_{21}\boldsymbol{x}_1 + \boldsymbol{A}_{22}\boldsymbol{x}_2 + \boldsymbol{B}_2\boldsymbol{u} \tag{29}$$

It is assumed that the index k holds uniformly in all algebraic equations (29), i.e.

$$k = 1 \;:\; \boldsymbol{A}_{22} \text{ regular} \tag{30}$$

$$k = 2 \;:\; \boldsymbol{A}_{22} = \boldsymbol{0}, \quad \boldsymbol{A}_{21}\boldsymbol{A}_{12} \text{ regular} \tag{31}$$

$$k \geq 3 \;:\; \boldsymbol{A}_{22} = \boldsymbol{0}, \quad \boldsymbol{A}_{21}\boldsymbol{A}_{11}^j\boldsymbol{A}_{12} = \boldsymbol{0}, \quad j = 0, \dots, k-3, \quad \boldsymbol{A}_{21}\boldsymbol{A}_{11}^{k-2}\boldsymbol{A}_{12} \text{ regular} \tag{32}$$

Then the descriptor system (28), (29) is causal if and only if

$$k = 1 \;:\; \text{always} \tag{33}$$

$$k = 2 \;:\; \boldsymbol{B}_2 = \boldsymbol{0} \tag{34}$$

$$k \geq 3 \;:\; \boldsymbol{B}_2 = \boldsymbol{0}, \quad \boldsymbol{A}_{21}\boldsymbol{A}_{11}^j\boldsymbol{B}_1 = \boldsymbol{0}, \quad j = 0, \dots, k-3 \tag{35}$$

Causality will be later an essential point to settle whether Pontryagin's maximum principle can be applied as usual or not.

5. Stability

5.1. General Theory

For standard state-space systems the notion of "stability" is well-defined (even if there is not only one definition of stability but a great number of definitions related to different requirements of applications) and a related stability theory is well-established (Hahn, 1967). For example, let us recall only the notions of stability in the sense of Lyapunov, asymptotic stability, absolute stability, and the well-known stability theory of Lyapunov.

When compared with the large amount of research related to stability problems of state-space systems almost nothing has been investigated regarding the stability behaviour of descriptor systems. First general attempts were presented by Bajić (Bajić, 1986; 1987; 1988a; 1988b; 1992; Bajić and Milić, 1987; Bajić *et al.*, 1989) and by a few other authors (Dolezal, 1987; Griepentrog and März, 1986; Hill and Mareels, 1990). In the case of linear time-invariant descriptor systems stability was defined by the eigenvalues of a related matrix pencil (Dai, 1989). Some recent results for nonlinear descriptor systems were reported in (Müller, 1993; 1994; 1996b). One essential difficulty of a suitable stability definition is the problem in which (sub-)space stability has to be defined. To avoid impulse solutions of (1), (2), the initial conditions $\boldsymbol{x}_1(t_0) = \boldsymbol{x}_{10}$, $\boldsymbol{x}_2(t_0) = \boldsymbol{x}_{20}$ must be consistent with the invariants (23). By these algebraic constraints consistent solutions belong to a consistent manifold which is a subspace of the generalized state space $(\boldsymbol{x}_1, \boldsymbol{x}_2)$. By restricting the stability problem to the consistent manifold, the stability of an equilibrium point (3) can be defined in the sense of Lyapunov confining the perturbed motion to the consistent manifold. Similarly, asymptotic stability is defined with respect to this manifold.

In this restricted sense, stability results for linear time-invariant descriptor systems were reported in (Müller, 1993) based on the discussion of a generalized Lyapunov matrix equation generalizing the results of (Owens and Debeljković, 1985). For linear mechanical descriptor systems the well-known theorem of Thomson, Tait and Chetaev (Müller, 1977) was generalized. Additionally, in (Müller, 1994) the results of (Müller, 1993) were brought forward to nonlinear mechanical descriptor systems using the Hamiltonian as a Lyapunov function. In (Müller, 1996b) an extension for general nonlinear descriptor systems was considered. Summarizing these results, we can derive the Lyapunov-like stability theorem. In the following, we discuss the control-free stability problem for $\boldsymbol{u} = \boldsymbol{0}$ using the notation of (26). Introducing

$$\overline{\boldsymbol{x}}_2 = \boldsymbol{x}_2 + \sum_{i=0}^{k-2} \alpha_i \boldsymbol{f}_{2j}(\boldsymbol{x}_1, \boldsymbol{0}) \tag{36}$$

such that $\dot{\boldsymbol{x}}_1 = \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{0}) = \overline{\boldsymbol{f}}_1(\boldsymbol{x}_1, \overline{\boldsymbol{x}}_2, \boldsymbol{0})$ and assuming a Lyapunov function

$$v(\boldsymbol{x}_1) = v_0(\boldsymbol{x}_1) + \sum_{j=0}^{k-3} \beta_j \boldsymbol{f}_{2j}^T(\boldsymbol{x}_1, \boldsymbol{0}) \boldsymbol{f}_{2j}(\boldsymbol{x}_1, \boldsymbol{0}) \tag{37}$$

with the property

$$\frac{\partial v_0(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1^T} \overline{\boldsymbol{f}}_1(\boldsymbol{x}_1, \overline{\boldsymbol{x}}_2, \boldsymbol{0}) = \frac{\partial v_0(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1^T} \overline{\boldsymbol{f}}_1(\boldsymbol{x}_1, \boldsymbol{0}, \boldsymbol{0}) + \sum_{j=0}^{k-2} \boldsymbol{f}_{2j}^T(\boldsymbol{x}_1, \boldsymbol{0}) \boldsymbol{h}_j(\overline{\boldsymbol{x}}_2) \tag{38}$$

for suitable functions $\boldsymbol{h}_j(\overline{\boldsymbol{x}}_2)$, we have the following result.

Theorem 2. (Müller, 1996b) *If there exist constants* α_i, $i = 0, \dots, k-2$, $\beta_i \geq 0$, $i = 0, \dots, k-3$, *and a function* $v_0(\boldsymbol{x}_1)$ *such that*

$$v(\boldsymbol{x}_1) > 0 \quad \textit{for all} \quad \boldsymbol{x}_1 \neq \boldsymbol{0}, \qquad v(\boldsymbol{0}) = 0 \tag{39}$$

condition (38) is satisfied, and

$$\dot{v}(\boldsymbol{x}_1) = \frac{\partial v_0(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1^T} \overline{\boldsymbol{f}}_1(\boldsymbol{x}_1, \boldsymbol{0}, \boldsymbol{0}) + 2 \sum_{j=0}^{k-3} \beta_j \boldsymbol{f}_{2j}^T(\boldsymbol{x}_1, \boldsymbol{0}) \boldsymbol{f}_{2,j+1}(\boldsymbol{x}_1, \boldsymbol{0}) \leq 0 \tag{40}$$

for all $\boldsymbol{x}_1 \neq \boldsymbol{0}$ *in a neighbourhood of* $\boldsymbol{x}_1 = \boldsymbol{0}$, *then the equilibrium point (3) is stable. If, additionally, Krasovskii's condition (Barbašin and Krasovskii, 1952)*

$$\dot{v}(\boldsymbol{x}_1) \equiv 0 \quad \textit{if} \quad \boldsymbol{x}_1(t) \equiv \boldsymbol{0} \tag{41}$$

is fulfilled, then the equilibrium point (3) is asymptotically stable.

A characteristic of this result is that the Lyapunov function and its time-derivative depend only on a part of the variables ($\boldsymbol{x}_1$) and do not depend on $\boldsymbol{x}_2$. This fact is stimulating to apply the theory of stability with respect to a part of variables (Müller, 1982; Oziraner and Rumiantsev, 1972) to the stability problem of descriptor systems. Comparing our problem with the definition and the results of partial stability, essentially we have to look for stability with respect to the variables $\boldsymbol{x}_{1e}$ while the behaviour of $\boldsymbol{x}_{1r}$, $\boldsymbol{x}_2$ is not of explicit interest. But the practical problem is that usually $\boldsymbol{x}_{1e}$ is not explicitly known. Therefore, in applications the variables $\boldsymbol{x}_1$ which embed $\boldsymbol{x}_{1e}$ are considered. Nevertheless, the properties of definiteness will be essentially related to $\boldsymbol{x}_{1e}$ by the following statement.

Lemma 1. *If there exist two functions* $v_0(\boldsymbol{x}_1)$ *and*

$$v_1\big(\| \boldsymbol{f}_{21}(\boldsymbol{x}_1, \boldsymbol{0}) \|, \dots, \| \boldsymbol{f}_{2,k-2}(\boldsymbol{x}_1, \boldsymbol{0}) \| \big) > 0 \tag{42}$$

such that

$$v(\boldsymbol{x}_1) = v_0(\boldsymbol{x}_1) + v_1\big(\| \boldsymbol{f}_{21}(\boldsymbol{x}_1, \boldsymbol{0}) \|, \dots, \| \boldsymbol{f}_{2,k-2}(\boldsymbol{x}_1, \boldsymbol{0}) \| \big) > 0 \tag{43}$$

is positive definite, then

$$v(\boldsymbol{x}_1) = v\left(\boldsymbol{P}^{-1} \begin{bmatrix} \boldsymbol{x}_{1e} \\ \boldsymbol{x}_{1r} \end{bmatrix} \right) \geq v\left(\boldsymbol{P}^{-1} \begin{bmatrix} \boldsymbol{x}_{1e} \\ \boldsymbol{g}_{1r}(\boldsymbol{x}_{1e}) \end{bmatrix} \right) = v_e(\boldsymbol{x}_{1e}) > 0 \tag{44}$$

where $v_e(\boldsymbol{x}_{1e})$ *is positive definite with respect to* $\boldsymbol{x}_{1e}$.

Therefore, by using the invariants (26) as first integrals, the construction of suitable Lyapunov functions is simplified and definiteness properties can essentially be directed to a consistent solution manifold.

Now we are able to transfer some of the results of partial stability (Müller, 1982; Oziraner and Rumiantsev, 1972) to the stability problem of nonlinear descriptor systems.

Theorem 3. (Müller, 1996b) *If there exist functions* v_0 *and* v_1 *of Lemma 1 such that*

$$\dot{v}(\boldsymbol{x}_1, \boldsymbol{x}_2) = \frac{\partial v(\boldsymbol{x}_1)}{\partial \boldsymbol{x}_1^T} \boldsymbol{f}_1(\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{0}) \le 0 \tag{45}$$

then the equilibrium point (3) is stable with respect to $\boldsymbol{x}_{1e} = \boldsymbol{0}$. *If, additionally,*

$$\dot{v}(\boldsymbol{x}_1, \boldsymbol{x}_2) \le -w_2(\boldsymbol{x}_1) < 0 \tag{46}$$

holds for a certain positive definite function $w_2(\boldsymbol{x}_1)$, *then* $\boldsymbol{x}_{1e} = \boldsymbol{0}$ *is asymptotically stable. If (46) is replaced by*

$$\dot{v}(\boldsymbol{x}_1, \boldsymbol{x}_2) \le -w_2(\boldsymbol{x}_1) \le 0 \tag{47}$$

for a positive definite function $w_2(\boldsymbol{x}_1)$, *then asymptotic stability is guaranteed if Krasovskii's condition*

$$\dot{v}(\boldsymbol{x}_1, \boldsymbol{x}_2) \equiv 0 \quad \textit{if} \quad \boldsymbol{x}_{1e} \equiv \boldsymbol{0} \tag{48}$$

is additionally met.

Theorem 3 shows how the theory of stability with respect to a part of variables can successfully be applied to the stability problem of descriptor systems.

5.2. Thomson-Tait-Chetaev Theorem

In the case of the linear time-invariant holonomic mechanical system

$$\boldsymbol{M}\ddot{\boldsymbol{z}}(t) + (\boldsymbol{D} + \boldsymbol{G})\dot{\boldsymbol{z}}(t) + \boldsymbol{K}\boldsymbol{z}(t) = \boldsymbol{F}^T\boldsymbol{\lambda}(t) \tag{49}$$

$$\boldsymbol{F}\boldsymbol{z}(t) = \boldsymbol{0}, \tag{50}$$

where the matrices $\boldsymbol{D} = \boldsymbol{D}^T$, $\boldsymbol{G} = -\boldsymbol{G}^T$, $\boldsymbol{K} = \boldsymbol{K}^T$ are respectively related to damping, gyroscopic and stiffness forces, the result of Theorem 2 can be simplified. Without the constraints (50) the stability behaviour of the unconstrained system (49) ($\boldsymbol{\lambda} \equiv \boldsymbol{0}$) was investigated in detail in (Müller, 1977). A very famous stability result is the theorem of Thomson and Tait which has been proved by Chetaev by applying Lyapunov's theory. A generalization with respect to pervasive damping instead of complete damping was presented in (Müller, 1977). Now it is possible to generalize these stability results to constrained mechanical systems (49), (50), q.v. (Müller, 1993; 1994).

Theorem 4. (Müller, 1993) *If the system (49), (50) satisfies the conditions*

$$\boldsymbol{M} = \boldsymbol{M}^T > \boldsymbol{0}, \quad \boldsymbol{D} + \alpha_2\boldsymbol{F}^T\boldsymbol{F} \ge \boldsymbol{0}, \quad \boldsymbol{K} + \alpha_1\boldsymbol{F}^T\boldsymbol{F} > \boldsymbol{0} \tag{51}$$

for some numbers $\alpha_1 \ge 0$ *and* $\alpha_2 \ge 0$, *then the mechanical system is asymptotically stable if the condition*

$$\operatorname{rank}\begin{bmatrix} \boldsymbol{M}s^2 + (\boldsymbol{D}+\boldsymbol{G})s + \boldsymbol{K} & -\boldsymbol{F}^T \\ \boldsymbol{F} & \boldsymbol{0} \\ (\boldsymbol{I} - \boldsymbol{F}^+\boldsymbol{F})\boldsymbol{D}s & \boldsymbol{0} \end{bmatrix} = f + p \tag{52}$$

is met for all $s \in \mathbb{C}$.

The condition (52) is an explicit form of Krasovskii's requirement (41). The matrix $\boldsymbol{F}^+$ represents the Moore-Penrose inverse matrix of $\boldsymbol{F}$.

Theorem 4 shows explicitly, by the constants α_1 and α_2, the stabilization effect of the holonomic constraints. The unconstrained system ($\boldsymbol{F} = \boldsymbol{0}$) may be unstable but the constraints may lead to a stable constrained system. Then the instabilities are located in the space of the constrained modes. The theorem represents a proper generalization of the well-known stability theorem of Thomson, Tait, and Chetaev including also the effect of pervasive damping due to (52).

The results of Theorems 2 and 3 can also be applied to nonlinear constrained mechanical systems (4), (5) by using the Hamiltonian as a Lyapunov function. The stability theorems were reported in (Müller, 1994).

6. Optimal Control

6.1. General Theory

The purpose of this section is to derive conditions for the design of optimal control of descriptor systems. If we look for the system description (1), (12), (13) instead of (1), (2), then an obvious problem appears. Generally, the system behaviour may depend not only on the control input $\boldsymbol{u}$, but also on its time-derivatives $\dot{\boldsymbol{u}}, \ddot{\boldsymbol{u}}, \dots, \boldsymbol{u}^{(k)}$. This is an inconvenient problem statement, so we have defined the property of causality in Section 4 to distinguish between the cases where time-derivatives appear or not. Therefore, in the following we have to distinguish between causal and non-causal descriptor systems (1), (2). In the first case we can apply Pontryagin's maximum principle without any problem whereas in the other we have to discuss a more complicated optimization problem. According to the discussions in (Müller, 1997a; 1996a) the following results are presented for the optimal control design of the descriptor system (1), (2) with respect to the performance criterion (11).

Theorem 5. (Müller, 1996a; 1997a) *For causal descriptor systems (1), (2) the Pontryagin maximum principle can be applied as usual. Necessarily there are nontrivial adjoint vectors $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ such that with the Hamiltonian*

$$H = \boldsymbol{\lambda}_1^T \boldsymbol{f}_1 + \boldsymbol{\lambda}_2^T \boldsymbol{f}_2 - f_0 \tag{53}$$

the adjoint differential-algebraic equations

$$\dot{\boldsymbol{\lambda}}_1 = -\frac{\partial H}{\partial \boldsymbol{x}_1} \tag{54}$$

$$\boldsymbol{0} = -\frac{\partial H}{\partial \boldsymbol{x}_2} \tag{55}$$

are satisfied and the optimal control fulfils

$$\max_{\boldsymbol{u} \in U} H = H_{\text{opt}} \tag{56}$$

For simplicity, the boundary conditions of the adjoint variables are not considered explicitly, but they have to be appropriately chosen.

The optimal control problem looks quite different for non-causal descriptor systems because of the influence of the time-derivatives of the control input. Actually, the following procedure is recommended. Having regard to $\dot{u}, \ldots, u^{(k-1)}$ the representation (1), (12), (13) of the descriptor system is preferred. By introducing the vectors

$$\xi_1 = u, \quad \xi_2 = \dot{u}, \quad \ldots, \quad \xi_k = u^{(k-1)}, \quad v = u^{(k)} \tag{57}$$

an extended state vector

$$x_e = \begin{bmatrix} x_1^T & x_2^T & \xi_1^T & \xi_2^T & \cdots & \xi_k^T \end{bmatrix}^T \tag{58}$$

is defined which satisfies the ordinary differential equation

$$\dot{x}_e = \begin{bmatrix} f_1(x_1, x_2, \xi_1) \\ \bar{f}_2(x_1, x_2, \xi_1, \xi_2, \ldots, \xi_k, v) \\ \xi_2 \\ \vdots \\ \xi_k \\ v \end{bmatrix} \tag{59}$$

and the algebraic equations

$$L^j(f_2) \equiv f_{2j}(x_1, x_2, \xi_1, \xi_2, \ldots, \xi_{j+1}) = 0, \quad j = 0, \ldots, k-1 \tag{60}$$

The original control constraints $u \in U$ appear now as state constraints $\xi_1 \in U$. Additionally, the question arises whether the control problem is stated properly. For a reasonable problem statement, additional constraints on $\xi_2 = \dot{u}, \ldots, \xi_k = u^{(k-1)}$ and particularly on $v = u^{(k)}$ may be efficient. Now, the corresponding Hamilton function is introduced:

$$H = \lambda_1^T f_1 + \lambda_2^T \bar{f}_2 - f_0 + \psi_1^T \xi_2 + \psi_2^T \xi_3 + \cdots + \psi_{k-1}^T \xi_k + \psi_k^T v \tag{61}$$

For the extended system (59) and the Hamilton function (61) the procedures of the calculus of variations or Pontryagin's maximum principle can be applied leading to an optimal control design of non-causal descriptor systems. But again the constraints $\xi_1 \in U$ of the extended states must be taken into account, which highly complicates the optimization procedure.

Note that the algebraic constraints (60) do not have to be included in (61) explicitly because they are taken into account by the second set of differential equations, i.e. by (13). But the initial conditions of the system (59) have to be chosen consistently, i.e. they have to satisfy the algebraic constraints (60) according to the requirements (23).

6.2. Illustrative Examples

In the following, three small academic examples are discussed to illustrate the questions of causality and optimization. The first two examples are essentially related to the distinction of causal and non-causal mechanical descriptor systems.

6.2.1. Causal Mechanical Descriptor System

The system considered consists of two spring-mass-oscillators where the masses are connected by a rigid bar (Fig. 1). Additionally, the first oscillator is controlled. Lagrange's equations of first kind (when neglecting the static forces due to the equilibrium position z_{10}, z_{20} and regarding only the dynamic behaviour) are as follows:

$$m_1\ddot{z}_1 + c_1 z_1 = \lambda + u \tag{62}$$

$$m_2\ddot{z}_2 + c_2 z_2 = -\lambda \tag{63}$$

$$z_1 - z_2 = 0 \tag{64}$$

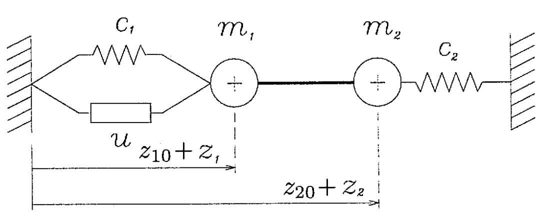

Fig. 1. Two spring-mass oscillators connected by a rigid bar.

The description (1), (2) is obtained by:

$$\boldsymbol{x}_1^T = \begin{bmatrix} z_1 & z_2 & \dot{z}_1 & \dot{z}_2 \end{bmatrix}, \quad \boldsymbol{x}_2 = [\lambda] \tag{65}$$

$$\boldsymbol{f}_1 = \begin{bmatrix} x_3 \\ x_4 \\ -\dfrac{c_1}{m_1}x_1 + \dfrac{\lambda}{m_1} + \dfrac{u}{m_1} \\ -\dfrac{c_2}{m_2}x_2 - \dfrac{\lambda}{m_2} \end{bmatrix}, \quad f_2 = x_1 - x_2 \tag{66}$$

It follows from (32) and (35) that the system has index 3 and that it is causal.

There is a more general result behind. Holonomic mechanical descriptor systems (4), (5) have uniform index $k = 3$ and they are causal iff $\boldsymbol{R} = \boldsymbol{0}$, i.e. if the constraints (5) are not controlled. If at least one constraint is controlled, i.e. it depends explicitliy on $\boldsymbol{u}$, then the system (4), (5) is non-causal.

The optimal control design is performed according to Theorem 5. In (Müller, 1997a) the linear-quadratic optimal regulator problem was discussed. In (Müller, 1996a) the time-optimal control problem was solved. In both the cases well-known results were obtained, but based on a DAE-approach and not on a classical state space approach which arises from the ordinary differential equation

$$(m_1 + m_2)\ddot{z}_1 + (c_1 + c_2)z_1 = u \tag{67}$$

where z_1 is chosen as a generalized coordinate.

6.2.2. Non-Causal Mechanical Descriptor System

The second example differs from the first one by introducing the control to the constraint such that we have the typical problem of a controlled mechanism:

$$m_1\ddot{z}_1 + c_1 z_1 = \lambda \tag{68}$$

$$m_2\ddot{z}_2 + c_2 z_2 = -\lambda \tag{69}$$

$$z_1 - z_2 + u = 0 \tag{70}$$

The description (1), (2) is obtained from (65) and

$$\boldsymbol{f}_1 = \begin{bmatrix} x_3 \\ x_4 \\ -\dfrac{c_1}{m_1}x_1 + \dfrac{\lambda}{m_1} \\ -\dfrac{c_2}{m_2}x_2 - \dfrac{\lambda}{m_2} \end{bmatrix}, \quad f_2 = x_1 - x_2 + u \tag{71}$$

The condition (35) is violated ($\boldsymbol{B}_2 \neq \boldsymbol{0}$) and therefore this system is non-causal. This fact can be shown explicitly if we write down Lagrange's equation of second kind with respect to the generalized coordinate z_1 and the equation determining the constraint force λ:

$$(m_1 + m_2)\ddot{z}_1 + (c_1 + c_2)z_1 = -c_2 u - m_2\ddot{u} \tag{72}$$

$$\lambda = \frac{m_1 m_2}{m_1 + m_2}\left\{\left(\frac{c_1}{m_1} - \frac{c_2}{m_2}\right)z_1 - \frac{c_2}{m_2}u - \ddot{u}\right\} \tag{73}$$

The representation (72) shows that introduction of extended state variables (57) is necessary to handle properly the optimization procedure. In the same way the solution $\boldsymbol{x}_2 = [\lambda]$ depend on $\ddot{u}$ such that the system (68)–(70) is non-causal. In (Müller, 1997a; 1996a) some discussions of the optimal control design are presented.

6.2.3. Non-Causal Academic Example

In order to illustrate the difficulties and the surprising results of the optimal control design of non-causal descriptor systems, we consider the following simple academic example:

$$\dot{x}_2 = x_1 + b_1 u \tag{74}$$

$$\dot{x}_3 = x_2 + b_2 u \tag{75}$$

$$0 = x_3 + b_3 u \tag{76}$$

The index is $k = 3$ and the system is causal for $b_2 = 0,\ b_3 = 0$. The underlying ordinary differential equations on a manifold are written down according to (1), (12), (13):

$$\dot{x}_1 = -b_1\dot{u} - b_2\ddot{u} - b_3\dddot{u} \tag{77}$$

$$\dot{x}_2 = x_1 + b_1 u \tag{78}$$

$$\dot{x}_3 = x_2 + b_2 u \tag{79}$$

with the invariants

$$x_1 + b_1 u + b_2\dot{u} + b_3\ddot{u} = 0 \tag{80}$$

$$x_2 + b_2 u + b_3\dot{u} = 0 \tag{81}$$

$$x_3 + b_3 u = 0 \tag{82}$$

The invariants represent the explicit solution of the system (74)–(76). According to (80), (81) the solutions of $x_1,\ x_2$ depend on $\dot{u}$ and $\ddot{u}$ if the coefficients $b_2,\ b_3$ do not vanish.

Looking for an optimal control with respect to the quadratic performance criterion

$$J = \frac{1}{2}\int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + r u^2\right) dt \longrightarrow \text{minimum} \tag{83}$$

where u is unbounded, and $r > 0,\ q_i \geq 0,\ i = 1, 2, 3$, the calculus of variations leads to the optimal feedback control

$$u = \frac{1}{r + q_2 b_1 b_3}\left[(b_1 q_1 - b_3 q_2)x_1 + b_2 q_2 x_2 + b_3 q_3 x_3 - b_2 q_1 \dot{x}_1 + b_3 q_1 \ddot{x}_1\right] \tag{84}$$

While a simple proportional feedback

$$u_c = \frac{1}{r} b_1 q_1 x_1 \tag{85}$$

is designed for the causal system, for the non-causal one a proportional and a (twice) time-derivative feedback (84) arises. The related closed-loop control system is a standard system of fourth order ($b_3 \neq 0$, $q_1 > 0$):

$$b_3^2 q_1 \ddot{x}_1 - b_3 b_2 q_1 \dot{x}_1 - b_3(b_3 q_2 - b_1 q_1)x_1 + b_2 b_3 q_2 x_2 + (r + b_1 b_3 q_2 + b_3^2 q_3)x_3 = 0$$

$$\dot{x}_2 - x_1 + \frac{b_1}{b_3}x_3 = 0, \qquad \dot{x}_3 - x_2 + \frac{b_2}{b_3}x_3 = 0 \tag{86}$$

Arbitrary values of x_{10}, $\dot{x}_{10}$, x_{20}, x_{30} can be chosen as initial conditions.

It is easy to prove that the characteristic polynomial of the closed-loop system (86) is bi-quadratic such that the system is unstable. This is a consequence of an irregularly stated optimization problem. The performance criterion (83) considers the square of the control but it is not related to the time-derivatives of the input. To regularize the problem, the weighting of $\ddot{u}$ has to be introduced corresponding to the solution (80). Then a stable optimal feedback control can be expected.

According to (57) the extended variables

$$\xi_1 = u, \quad \xi_2 = \dot{u}, \quad v = \ddot{u} \tag{87}$$

are introduced. The criterion (83) is modified into

$$J = \frac{1}{2}\int_0^\infty \left(q_1 x_1^2 + q_2 x_2^2 + q_3 x_3^2 + r_1 u^2 + r_2 \dot{u}^2 + r_3 \ddot{u}^2\right) \mathrm{d}t \tag{88}$$

Since the solution (80)–(82) of the descriptor system (74)–(76) is known, the resulting optimization problem reads as follows. For the dynamical system

$$\dot{\xi}_1 = \xi_2, \quad \dot{\xi}_2 = v \tag{89}$$

the performance criterion

$$\begin{aligned} J = \frac{1}{2}\int_0^\infty \Big\{ & q_1(b_1\xi_1 + b_2\xi_2 + b_3 v)^2 + q_2(b_2\xi_1 + b_3\xi_2)^2 \\ & + q_3 b_3^2 \xi_1^2 + r_1 \xi_1^2 + r_2 \xi_2^2 + r_3 v^2 \Big\} \mathrm{d}t \end{aligned} \tag{90}$$

has to be minimized with respect to v. This is a classical linear-quadratic optimal regulator problem which can be solved by the usual Riccati approach. After some lengthy calculations we have

$$v = -\frac{1}{q_1 b_3^2 + r_3}\Big\{(q_1 b_1 b_3 + P_{12})\xi_1 + (q_1 b_2 b_3 + P_{22})\xi_2\Big\} \tag{91}$$

with

$$P_{12} = -q_1 b_1 b_3 + \sqrt{q_2 b_2 b_3 (q_1 b_3^2 + r_3) + q_1 b_1 (q_1 b_1 b_3^2 + b_2 r_3)} \tag{92}$$

$$P_{22} = -q_1 b_2 b_3 + \sqrt{(q_1 b_2^2 + q_2 b_3^2 + r_2 + 2P_{12})(q_1 b_3^2 + r_3)} \tag{93}$$

Having in mind (87), the result (91) represents an asymptotically stable differential equation of second order with respect to u. Therefore this control leads to an asymptotically stable behaviour of the descriptor system (74)–(76) for consistent initial conditions.

It should be mentioned that the solution of the linear-quadratic optimal control problem can be discussed by introducing a generalized Riccati equation. But this only makes sense in the case of causal systems, because the feedback control is then a proportional feedback of the descriptor-variables. But even in this case the generalized Riccati equation has to be handled very carefully. A detailed discussion was presented in (Hou, 1995; Schüpphaus, 1995) also including numerical aspects. For non-causal systems the Riccati equation approach does not meet the problem, because the typical time-derivative feedback part is not represented by this approach directly. Only when the extension (57)–(60) is performed, there may exist a certain chance to define a Riccati equation approach. But there is still a problem, because the extended state space system (59) is not completely controllable because of the invariants (60). Therefore this approach is a completely open problem which has to be discussed in future.

Summarizing, this simple example clearly shows the difficulties of the optimal control design of non-causal descriptor systems. To solve such problems carefully, the extended state vector (58) according to the time-derivatives (57) has to be introduced but also the performance criterion has to be re-considered by introducing penalty functions of time-derivatives of the control input to regularize the optimal control problem.

7. Conclusions

The description of nonlinear dynamical systems by the so-called descriptor systems becomes more and more popular. Therefore tools for the analysis and control design of such systems are needed. In this contribution some results on the stability analysis and optimal control design have been reported. The stability analysis is essentially based on Lyapunov's stability theory with respect to a part of the variables. In the case of linear mechanical descriptor systems, the famous result of Thomson, Tait, and Chetaev has been generalized. For the optimal control design it was necessary to introduce the notion of causality and to distinguish between causal and non-causal descriptor systems. In the first case the calculus of variations and Pontryagin's maximum principle can be applied as usual, but in the second case one has to take care and it is recommended to solve the problem on the basis of the so-called underlying ordinary differential equation on a manifold. Two examples of constrained spring-mass oscillators illustrate the occurence of causal and non-causal mechanical descriptor systems. The third academic example shows the severe difficulties of the optimal control design of non-causal descriptor systems. Surprising results appeared demonstrating that many problems are still open. In particular, efficient numerical methods are missing, especially for non-causal systems.

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