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CONTROLLABILITY OF SECOND-ORDER SEMILINEAR INFINITE-DIMENSIONAL DYNAMICAL SYSTEMS

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In the paper, the approximate controllability of semilinear abstract second-order infinite-dimensional dynamical systems is considered. It is proved by using the frequency-domain and functional-analysis methods that the approximate controllability of second-order semilinear dynamical system can be verified by the approximate controllability conditions for a simplified suitably-defined first-order linear dynamical system. General results are then applied to a semilinear mechanical flexible-structure vibratory dynamical system. Some special cases are also considered. Moreover, remarks and comments on the relationships between different concepts of controllability are given. The paper extends the results presented in (Klamka, 1992; Triggiani, 1978) to a more general class of second-order abstract dynamical systems.

1. Introduction

Controllability is one of the fundamental concepts in mathematical control theory (Klamka, 1991). Roughly speaking, controllability generally means that it is possible to steer a dynamical system from an arbitrary initial state to an arbitrary final state by using a given set of admissible controls. In the literature, there are many different definitions of controllability which depend on the class of considered dynamical systems (Bensoussan *et al.*, 1993; Klamka, 1991; 1993a; 1993b; Kobayashi, 1992; Narukawa, 1982; 1984; Triggiani, 1975a; 1976; 1978). For infinite-dimensional dynamical systems it is necessary to distinguish between the notions of approximate and exact controllability (Klamka, 1991; 1993b; Triggiani, 1975a; 1975b; 1976; 1977; 1978). This follows directly from the fact that in infinite-dimensional spaces there exist linear subspaces which are not closed . The controllability theory for various types of abstract linear control systems is well-known (Chen and Triggiani, 1990b). On the other hand, in the case of semilinear infinite-dimensional control systems there exist rather restrictive and complicated sufficient conditions for exact or approximate controllability (Chen and Triggiani, 1989; 1990a; Huang, 1988; Narukowa, 1982; O'Brien, 1979; Rubio, 1995; Zhou, 1984).

The main purpose of the present paper is to study the approximate controllability of semilinear infinite-dimensional second-order dynamical systems with damping using general results given in the papers (Naito, 1987; 1989; Naito and Park, 1989;

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Seidman, 1987). For such dynamical systems direct verification of approximate controllability is a rather difficult and complicated task (Klamka, 1992). Therefore, using the frequency-domain method (Kobayashi, 1992) and fixed-point theorems (Seidman, 1987), it is shown that the approximate controllability of second-order semilinear dynamical system can be checked by the approximate controllability conditions for a suitably-defined simplified first-order linear dynamical system. The general results are then applied for studying the approximate controllability of a semilinear mechanical flexible-structure vibratory dynamical system (Kunimatsu and Ito, 1988).

2. System Description

Let V and U denote separable Hilbert spaces. Let $A\colon V \supset D(A) \to V$ be a linear, possibly unbounded, self-adjoint and positive-definite linear operator with a domain $D(A)$ which is dense in V and a compact resolvent $R(s; A)$ for all s in the resolvent set $\rho(A)$. Then A has the following properties (Bensoussan *et al.*, 1993; Klamka, 1991; Kunimatsu and Ito, 1988; Triggiani, 1975a; 1976):

1. It has only a pure discrete point spectrum $\sigma_p(A)$ consisting entirely of isolated real positive eigenvalues

$$0 < s_1 < s_2 < \cdots < s_i < \cdots, \qquad \lim_{i\to\infty} s_i = +\infty$$

   Each eigenvalue s_i has finite multiplicity $n_i < \infty$, $i = 1, 2, 3, \ldots$ equal to the dimension of the corresponding eigenmanifold.

2. The eigenvectors $v_{ik} \in D(A)$, $i = 1, 2, 3, \ldots$, $k = 1, 2, 3, \ldots, n_i$ form a complete orthonormal set in the separable Hilbert space V.

3. A has spectral representation

$$Av = \sum_{i=1}^{i=\infty} s_i \sum_{k=1}^{k=n_i} \langle v, v_{ik}\rangle_V v_{ik} \quad \text{for} \quad v \in D(A)$$

4. The fractional powers A^α, $0 < \alpha \le 1$ of the operator A can be defined as

$$A^\alpha v = \sum_{i=1}^{i=\infty} s_i^\alpha \sum_{k=1}^{k=n_i} \langle v, v_{ik}\rangle_V v_{ik} \quad \text{for} \quad v \in D(A^\alpha)$$

   where $D(A^\alpha) = \left\{ v \in V\colon \sum_{i=1}^{i=\infty} s_i^{2\alpha} \sum_{k=1}^{k=n_i} \left| \langle v, v_{ik}\rangle_V \right|^2 < \infty \right\}.$

5. The operators A^α, $0 < \alpha \le 1$ are self-adjoint, positive-definite with dense domains in V and generate analytic semigroups on V.

Let us consider a semilinear infinite-dimensional control system described by the abstract second-order differential equation with dumping

$$\ddot{v}(t) + 2\left(c_2A + c_1A^{1/2}\right)\dot{v}(t) + \left(d_2A + d_1A^{1/2}\right)v(t) + h\big(v(t)\big) = Bu(t) \quad (1)$$

In the sequel, we shall also consider the corresponding linear infinite-dimensional control system described by the following abstract second-order differential equation:

$$\ddot{v}(t) + 2\left(c_2A + c_1A^{1/2}\right)\dot{v}(t) + \left(d_2A + d_1A^{1/2}\right)v(t) = Bu(t) \quad (2)$$

where $c_1 \geq 0$, $c_2 \geq 0$, d_1 (unrestricted in sign) and $d_2 > 0$ are given real constants.

It is assumed that the operator $B\colon\ U \to V$ is linear and its adjoint operator $B^*\colon\ V \to U$ is $A^{1/2}$-bounded (Bensoussan *et al.*, 1993; Chen and Russell, 1982; Kobayashi, 1992), i.e. $D(B^*) \supset D(A^{1/2})$, and there is a positive real number M such that

$$\|B^*v\|_U \leq M\left(\|v\|_V + \left\|A^{1/2}v\right\|_V\right) \quad \text{for} \quad v \in D\left(A^{1/2}\right)$$

Moreover, it is assumed that the admissible controls $u \in L^2_{\mathrm{loc}}([0,\infty),U)$.

It is well-known (Chen and Russell, 1982; Chen and Triggiani, 1989; 1990a; 1990b; Fujii and Sakawa, 1974) that the abstract ordinary differential equation (1) with initial conditions

$$v(0) \in D(A), \quad \dot{v}(0) \in V$$

has for each $t_1 > 0$ a unique solution $v(t; v(0), \dot{v}(0), u) \in C^{(2)}([0,t_1],V)$ such that $v(t) \in D(A)$ and $\dot{v}(t) \in D(A)$ for $t \in (0,t_1]$. Moreover, for $v(0) \in V$ there exists the so-called "mild solution" for (1) in the product space $W = D(A^{1/2}) \times V$ with the inner product and norm respectively defined as

$$\langle w,q\rangle_W = \langle w_1,q_1\rangle_V + \langle w_2,q_2\rangle_V, \quad \|w\|_W = \|w_1\|_V + \|w_2\|_V$$

for each $w = (w_1,w_2) \in W$ and $q = (q_1,q_2) \in W$.

Let us assume that the nonlinear operator $h\colon\ V \to V$ satisfies the following conditions (Yamamoto and Park, 1990):

(i) It is uniformly bounded on $V_\alpha = D(A)^\alpha \subset V$ for some $\alpha \subset [0,1)$, i.e. there exists a positive constant M such that $\|h(v)\|_{V_\alpha} \leq M$ for each $v \in V_\alpha$.

(ii) It is Lipschitz-continuous in v, i.e. there exists a positive constant L such that $\|h(v_1) - h(v_2)\|_{V_\alpha} \leq L\|v_1 - v_2\|_{V_\alpha}$ for each $v_1 \in V_\alpha$ and $v_2 \in V_\alpha$.

In order to transform the second-order equation (1) into a first-order equation in the Hilbert space W, let us substitute (Bensoussan *et al.*, 1993; Chen and Russell, 1982; Chen and Triggiani, 1989; 1990a; 1990b; Huang, 1988; Triggiani, 1978):

$$v(t) = w_1(t), \quad \dot{v}(t) = w_2(t)$$

Then the semilinear equation (1) becomes

$$\dot{w}(t) = Fw(t) + Gu(t) + H\big(w(t)\big) \quad (3)$$

where $F: W \supset D(F) \to W,\ G : U \to W,\ H : W_\alpha \to W_\alpha,\ W_\alpha = V_\alpha \times V_\alpha,$

$$w(t) = \begin{vmatrix} w_1(t) \\ w_2(t) \end{vmatrix}, \qquad F = \begin{vmatrix} 0 & I \\ -d_2A - d_1A^{1/2} & -2c_2A - 2c_1A^{1/2} \end{vmatrix}$$

$$G = \begin{vmatrix} 0 \\ B \end{vmatrix}, \qquad H(w) = \begin{vmatrix} 0 \\ -h(w_1) \end{vmatrix}.$$

Similarly, the linear equation (2) can be expressed in the following form:

$$\dot{w}(t) = Fw(t) + Gu(t) \tag{4}$$

Since the operators A and $A^{1/2}$ are self-adjoint, we can obtain for the operator F its adjoint operator F^* as follows:

$$F^* = \begin{vmatrix} 0 & -d_2A^* - d_1\left(A^{1/2}\right)^* \\ I & -2\left(c_2A^* + c_1\left(A^{1/2}\right)^*\right) \end{vmatrix} = \begin{vmatrix} 0 & -d_2A - d_1A^{1/2} \\ I & -2\left(c_2A + c_1A^{1/2}\right) \end{vmatrix}$$

Remark 1. It should be pointed out that if $c_1^2 + c_2^2 > 0$, then the operators F and F^* generate analytic semigroups of bounded linear operators on the Hilbert space $D(A^{1/2}) \times V$ (Chen and Russell, 1982; Chen and Triggiani, 1989; 1990a; 1990b; Huang, 1988; Kunimatsu and Ito, 1988). However, in case $c_1 = c_2 = 0$, the operator F generates a group of bounded linear operators which cannot be analytic since the linear operator F is unbounded (Triggiani, 1976). These statements are important for controllability investigations.

In the sequel, for comparison we shall consider instead of the linear second-order equation (2) also the simplified first-order linear differential equation

$$\dot{v}(t) = A^\alpha v(t) + Bu(t) \quad \text{for} \quad 0 < \alpha < \infty \tag{5}$$

In the next sections, we shall also study a semilinear first-order dynamical system with finite-dimensional control space $U = \mathbb{R}^m$.

For convenience, we shall introduce the following notation:

$$B = [b_1, b_2, \ldots, b_j, \ldots, b_m], \quad u(t) = \left[u_1(t), u_2(t), u_3(t), \ldots, u_j(t), \ldots, u_m(t)\right]^T$$

where $b_j \in V$, $u_j(t) \in L^2_{\text{loc}}\left([0,\infty), \mathbb{R}\right)$ for $j = 1, 2, 3, \ldots, m$. Let us observe that in this special case the linear operator B is finite-dimensional and therefore it is compact (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1975b; 1977).

Using the eigenvectors v_{ik}, $i = 1, 2, 3, \dots$, $k = 1, 2, 3, \dots, n_i$, we introduce for the finite-dimensional operator B the following notation (Klamka, 1991; Triggiani, 1976).

$$B_i = \begin{vmatrix} \langle b_1, v_{i1}\rangle_V & \langle b_2, v_{i1}\rangle_V & \cdots & \langle b_j, v_{i1}\rangle_V & \cdots & \langle b_m, v_{i1}\rangle_V \\ \langle b_1, v_{i2}\rangle_V & \langle b_2, v_{i2}\rangle_V & \cdots & \langle b_j, v_{i2}\rangle_V & \cdots & \langle b_m, v_{i2}\rangle_V \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle b_1, v_{ik}\rangle_V & \langle b_2, v_{ik}\rangle_V & \cdots & \langle b_j, v_{ik}\rangle_V & \cdots & \langle b_m, v_{ik}\rangle_V \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \langle b_1, v_{in_i}\rangle_V & \langle b_2, v_{in_i}\rangle_V & \cdots & \langle b_j, v_{in_i}\rangle_V & \cdots & \langle b_m, v_{in_i}\rangle_V \end{vmatrix} \tag{6}$$

B_i, $i = 1, 2, 3, \dots$ are $n_i \times m$-dimensional constant matrices which play an important role in controllability investigations (Klamka, 1991; 1992; 1993b; Triggiani, 1976). In the case when the eigenvalues s_i are simple, i.e. $n_i = 1$ for $i = 1, 2, 3, \dots$, B_i are m-dimensional row vectors.

3. Approximate Controllability

For infinite-dimensional dynamical systems we may introduce two general kinds of controllability, i.e. approximate (weak) controllability and exact (strong) controllability (Bensoussan *et al.*, 1993; Klamka, 1991; 1993b; Triggiani, 1975a; 1976). In the paper (Klamka, 1995) necessary and sufficient conditions for approximate controllability of the linear-second order dynamical system (2) were formulated and proved using frequency-domain methods.

In the present paper, we shall concentrate on approximate controllability for the semilinear second-order dynamical system (1).

Definition 1. (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1976) The dynamical system (1) is said to be *approximately controllable in the time interval* $[0, t_1]$ if for any initial condition $w(0) \in V \times V$, any given final condition $w_f \in V \in V$ and each positive real number ε there exists an admissible control $u \in L^2([0, t_1], U)$ such that

$$\left\| w(t_1;\, w(0), u) - w_f \right\|_{V\times V} \le \varepsilon \tag{7}$$

Definition 2. (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1975a) The dynamical system (1) is said to be *approximately controllable in finite time* (or briefly *approximately controllable*) if for any initial condition $w(0) \in V \times V$, any given final condition $w_f \in V \times V$ and each positive real number ε, there exist a finite time $t_1 < \infty$ (depending generally on $w(0)$ and w_f) and an admissible control $u \in L^2([0, t_1], U)$ such that the inequality (7) holds.

Remark 2. It should be stressed that in the case where the semigroup associated with the dynamical system (2) is analytic, approximate controllability in finite time coincides with approximate controllability in each time interval $[0, t_1]$ (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1975a; 1976).

Remark 3. It should be mentioned that in the case where the semigroup associated with the dynamical system (2) is compact or the control operator is compact, the linear dynamical system (2) is never exactly controllable in an infinite-dimensional state space (Bensoussan *et al.*, 1993; Klamka, 1991; Triggiani, 1975b; 1977).

Now, let us recall some well-known lemmas (Kobayashi, 1992; O'Brien; 1979) concerning the approximate controllability of linear infinite-dimensional dynamical systems, which will be useful in the sequel.

Lemma 1. (Kobayashi, 1992) *The linear first-order dynamical system (4) is approximately controllable if and only if for any complex number z there exists no non-zero $w \in D(F^*)$ such that*

$$\begin{vmatrix} F^* - zI \\ G^* \end{vmatrix} w = 0 \tag{8}$$

Similarly, the linear first-order dynamical system (5) is approximately controllable if and only if for any complex number s there exists no non-zero $v \in D(A^\alpha) \subset V$ such that

$$\begin{vmatrix} F^\alpha - sI \\ B^* \end{vmatrix} v = 0 \tag{9}$$

Lemma 2. (O'Brien, 1979) *The linear first-order dynamical system (5) is approximately controllable if and only if it is approximately controllable for some $\alpha \in (0, \infty)$.*

Lemma 3. (Klamka, 1995) *The linear second-order dynamical system with dumping (2) is approximately controllable if and only if the first-order dynamical system (5) is approximately controllable for some $\alpha \in (0, \infty)$.*

Lemma 4. (Yamamoto and Park, 1990) *Let us assume that the nonlinear operator H is uniformly bounded on $V \times V$ and Lipschitz-continuous in w. Then the semilinear first-order dynamical system (3) is approximately controllable if and only if the linear first-order dynamical system (4) is approximately controllable.*

Now, we shall formulate and prove a necessary and sufficient condition for the approximate controllability of the semilinear infinite-dimensional dynamical system (1), which constitutes the main result of the present paper.

Theorem 1. *The semilinear dynamical system (1) is approximately controllable if and only if the linear dynamical system (4) is approximately controllable for some $\alpha \in (0, \infty)$.*

Proof. First of all, let us observe that the semilinear dynamical systems (1) and (3) are equivalent. Since, by assumption, the nonlinear term $h(v)$ in eqn. (1) is uniformly bounded on the space V and Lipschitz-continuous in v, taking into account the form of the nonlinear operator $H(w)$ and assumptions (i) and (ii) concerning the nonlinear operator $h(v)$, we see that $H(w)$ is uniformly bounded on W_α and Lipschitz-continuous in w. Indeed, we have

$$\|H(w)\|_{W_\alpha} = \|(0, h(v))\|_{W_\alpha} = \|h(v)\|_{V_\alpha} \leq M \quad \text{for each} \quad w \in W_\alpha$$

$$\|H(w_1) - H(w_2)\|_{W_\alpha} = \|(0, h(v_1)) - (0, h(v_2))\|_{W_\alpha} = \|h(v_1) - h(v_2)\|_{V_\alpha} \leq L\|v_1 - v_2\|_{V_\alpha}$$

Therefore, by Lemma 4, the semilinear first-order dynamical system (3) is approximately controllable if and only if the linear first-order dynamical system (4) is approximately controllable. Moreover, by Lemmas 2 and 3, the dynamical system (4) is approximately controllable if and only if the dynamical system (5) is controllable for some $\alpha \in (0, \infty)$. Therefore our theorem follows. ∎

Corollary 1. *Suppose that $c_1^2 + c_2^2 > 0$. Then the semilinear dynamical system (1) is approximately controllable in any time interval $[0, t_1]$ if and only if the linear dynamical system (4) is approximately controllable in finite time.*

Proof. Since in case $c_1^2 + c_2^2 > 0$ the operator F generates an analytic semigroup, by (Naito and Park, 1989) the approximate controllability of the semilinear dynamical system (3), and hence that of the dynamical system (1) too, is equivalent to its approximate controllability in any time interval $[0, t_1]$. Therefore the corollary follows from Theorem 1. ∎

Corollary 2. *Suppose that the space of control values is finite-dimensional, i.e. $U = \mathbb{R}^m$, and the operator F generates an analytic semigroup. Then the semilinear dynamical system (1) is approximately controllable in any time interval $[0, t_1]$ if and only if*

$$\operatorname{rank} B_i = n_i \quad \textit{for} \quad i = 1, 2, 3, \ldots \tag{10}$$

Proof. The corollary is a direct consequence of Theorem 1, Corollary 1 and some well-known results (Klamka, 1991; Triggiani, 1975a; 1975b; 1976) concerning the approximate controllability of infinite-dimensional dynamical systems with finite-dimensional controls. ∎

Corollary 3. *Suppose that $U = \mathbb{R}^m$, the operator F generates an analytic semigroup and, moreover, $n_i = 1$ for $i = 1, 2, 3, \ldots$. Then the semilinear dynamical system (1) is approximately controllable in any time interval $[0, t_1]$ if and only if*

$$\sum_{j=1}^{j=m} \langle b_j, v_i \rangle_V^2 \neq 0 \quad \textit{for} \quad i = 1, 2, 3, \ldots \tag{11}$$

Proof. From Corollary 2 it follows immediately that in the case where $n_i = 1$, $i = 1, 2, 3, \ldots$ the dynamical system (1) is approximately controllable in any time interval if and only if the m-dimensional row vectors

$$B_i = \left| \langle b_1, v_i \rangle_V \;\; \langle b_2, v_i \rangle_V \;\; \cdots \;\; \langle b_j, v_i \rangle_V \;\; \cdots \;\; \langle b_m, v_i \rangle_V \right| \neq 0 \quad \text{for} \quad i=1,2,3,\ldots \tag{12}$$

Thus the corollary follows. ■

In the next section we shall use the general controllability results in order to check the approximate controllability of a semilinear mechanical flexible structure vibratory dynamical system.

4. Approximate Controllability of a Vibratory System

In this section we shall consider a vibratory dynamical system described by the following non-linear partial-differential equation (Kunimatsu and Ito, 1988):

$$\begin{aligned} w_{tt}(t,x) \; &- \; 2c_1 w_{txx}(t,x) + 2c_2 w_{txxxx}(t,x) - d_1 w_{xx}(t,x) + d_2 w_{xxxx}(t,x) \\ &- \; d\left(\int_0^L \left| v_x(t,x) \right|^2 \mathrm{d}x \right) v_{xx}(t,x) = \sum_{j=1}^{j=r} b_j(x) u_j(t) \end{aligned} \tag{13}$$

defined for $x \in [0, L]$ and $t \in [0, \infty)$, with initial conditions

$$v(0,x) = v_0(x), \quad v_t(0,x) = v_1(x) \quad \text{for } x \in [0, L] \tag{14}$$

and boundary conditions

$$v(t,0) = v(t,L) = v_{xx}(t,0) = v_{xx}(t,L) = 0 \quad \text{for } t \in [0, \infty) \tag{15}$$

Equation (13) describes the transverse motion of an elastic beam which occupies the interval $[0, L]$ in a reference and stress-free state. The function $w(t,x)$ denotes the displacement from the reference state at time t and position x. On the left-hand side of eqn. (13), the second and the third terms represent internal structural viscous dampings, and the fourth term represents the effect of the axial force on the beam (Kunimatsu and Ito, 1988). The boundary conditions (15) corresponds to hinged ends.

Let $V = L^2[0, L]$ be a separable Hilbert space of all square-integrable functions on $[0, L]$ with standard norm and inner product (Bensoussan *et al.*, 1993; Klamka, 1991). In order to regard the vibratory system (13)–(15) in the general framework considered in the previous sections, let us define the unbounded linear differential operator (Klamka, 1995) $A\colon V \supset D(A) \to V$,

$$Av(x) = v_{xxxx}(x) \quad \text{for} \quad v(x) \in D(A) \tag{16}$$

$$D(A) = \left\{ v(x) \in H^4[0, L]; \quad v(0) = v(L) = v_{xx}(0) = v_{xx}(L) = 0 \right\}$$

where $H^4[0, L]$ denotes the fourth-order Sobolev space on $[0, L]$.

The linear unbounded operator A has the following properties (Bensoussan *et al.*, 1993; Klamka, 1991; Kunimatsu, 1988; Triggiani, 1975a):

1. It is a self-adjoint and positive-definite operator whose domain $D(A)$ is dense in V.
2. There exists a compact inverse A^{-1} and, consequently, the resolvent $R(s;A)$ of A is a compact operator for all $s \in \rho(A)$.
3. The operator A has a spectral representation

$$Av = \sum_{i=1}^{i=\infty} s_i \langle v, v_i \rangle_H v_i \quad \text{for} \quad v \in D(A)$$

   where $s_i > 0$ and $v_i \in D(A)$, $i = 1, 2, 3, \ldots$ are simple ($n_i = 1$) eigenvalues and the corresponding eigenfunctions of A, respectively. Moreover,

$$s_i = \left(\frac{\pi i}{L}\right)^4, \quad v_i(x) = \sqrt{\frac{2}{L}} \sin\left(\frac{\pi i x}{L}\right) \quad \text{for } x \in [0, L]$$

   and the set $\{v_i(x),\ i = 1, 2, 3, \ldots\}$ forms a complete orthonormal system in V.
4. The fractional powers A^α, $0 < \alpha \leq 1$ can be defined by

$$A^\alpha v = \sum_{i=1}^{i=\infty} s_i^\alpha \langle v, v_i \rangle_V v_i \quad \text{for} \quad v \in D(A^\alpha) \quad \text{and} \quad 0 \leq \alpha \leq 1$$

   which are also self-adjoint and positive-definite operators with dense domains in H.

In particular, for $\alpha = 1/2$ we have (Kunimatsu and Ito, 1988)

$$A^{1/2} v = -v_{xx}$$

with the domain $D(A^{1/2}) = \{v \in H^2[0, L]\colon\ v(0) = v(L) = 0\}$. Moreover (Kunimatsu and Ito, 1988),

$$\|A^{1/4} v\|_V^2 = \int_0^L |v_x(x)|^2 \, dx$$

Now, we can consider the partial-differential equation (13) with conditions (14) and (15) as the following second-order semilinear evolution equation in the Hilbert space V (Kunimatsu and Ito, 1988):

$$\ddot{v}(t) + 2\big(c_2 A + c_1 A^{1/2}\big)\dot{v}(t) + \big(d_2 A + d_1 A^{1/2}\big) v(t) + d\|A^{1/4} v(t)\|_V^2 A^{1/2} v(t) = \sum_{j=1}^{j=m} b_j u_j(t) \tag{17}$$

where

$$v(t) = v(t,\cdot) \in V, \quad \dot{v}(t) = v_t(t,\cdot) \in V, \quad \ddot{v}(t) = v_{tt}(t,\cdot) \in V$$

$$b_j = b_j(\cdot) \in V, \quad j = 1,2,\ldots,m$$

Set

$$h(v) = d\|A^{1/4}v\|_V^2 A^{1/2}v$$

It is well-known (Kunimatsu and Ito, 1988) that the nonlinear operator $h(v)$ is uniformly bounded and locally Lipschitz-continuous on $D(A^{1/2})$.

Let the initial conditions be of the following form:

$$v(0) = v_0 \in D(A), \quad \dot{v}(0) = v_1 \in V$$

Then there exists a unique solution to the partial-differential equation (13) (Kunimatsu and Ito, 1988).

Theorem 2. *The semilinear vibratory dynamical system (13) is approximately controllable in any time interval* $[0, t_1]$ *if and only if*

$$\sum_{j=1}^{j=m} \left(\int_0^L \sqrt{\frac{2}{L}} b_j(x) \sin\left(\frac{\pi i x}{L}\right) \mathrm{d}x \right)^2 \neq 0 \quad \textit{for} \quad i = 1,2,3,\ldots \tag{18}$$

Proof. Let us observe that the semilinear dynamical system (13) satisfies all the assumptions of Corollary 1. Therefore, taking into account the analytic formula for the eigenvectors $v_i(x)$, $i = 1,2,3,\ldots$ and the form of the inner product in the separable Hilbert space $L^2([0,L],\mathbb{R})$, from (11) we directly obtain (18). ■

5. Conclusions

The present paper contains some results concerning the approximate controllability of semilinear second-order abstract infinite-dimensional dynamical systems. By using the general method given in the papers (Naito, 1987; 1989; Naito and Park, 1989; Seidman, 1987) and some methods of functional analysis, especially the theory of unbounded linear operators, necessary and sufficient conditions for approximate controllability in any time interval have been formulated and proved. Moreover, some special cases have also been investigated and discussed. Then the general controllability conditions have been applied to investigation of the approximate controllability of a mechanical flexible structure vibratory dynamical system. The results presented in the present paper are a generalization of the controllability conditions given in the literature (Bensoussan *et al.*, 1993; Klamka, 1992; Kobayashi, 1992; O'Brien, 1979; Triggiani, 1975a; 1975b) to semilinear second-order abstract dynamical systems with damping terms. Finally, it should be pointed out that the obtained results can be extended to cover the case of more complicated semilinear second-order abstract dynamical systems (Chen and Triggiani, 1989; 1990a; 1990b; Fujii and Sakawa, 1974).

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