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GLOBAL ASYMPTOTIC CONVERGENCE RESULTS FOR MULTITYPE MODELS

LINDA RASS*, JOHN RADCLIFFE*

This paper proves global asymptotic convergence in a multitype epidemic model which encompasses both the $S \to I$ and $S \to I \to S$ epidemics. Systems are considered where the infection matrix may be reducible, and for which the system may be closed or can be open with a stable population size. New global asymptotic convergence results are obtained.

Keywords: global asymptotic convergence, multitype $S \to I \to S$ epidemics, stable population size, reducible systems

1. Introduction

This paper considers multitype epidemics of both $S \to I$ and $S \to I \to S$ type with stable population size. This includes both closed systems and open systems in which the birth and immigration rates into the system are balanced by the death rates from the system.

The non-reducible model for a closed system in which infectious individuals in each population can infect susceptible individuals in any population, possibly through a sequence of infections, has been considered by Hethcote and Thieme (1985) and Lajmanovich and Yorke (1976). Denote the infection matrix by $\mathbf{\Gamma}$ and its Perron-Frobenius root by $\rho(\mathbf{\Gamma})$. The models they considered included a term for a return to the susceptible state for all populations, so that $\mathbf{\Gamma}$ has all finite entries. When $\rho(\mathbf{\Gamma}) \leq 1$ there is only one equilibrium. This equilibrium corresponds to no infectious individuals of any type. When $\rho(\mathbf{\Gamma}) > 1$ there are two equilibria, one corresponding to no infectious individuals, and the second to an endemic state with a positive proportion of infectious individuals in each population. Hethcote and Thieme (1985) examined the local asymptotic stability. They showed that the equilibrium corresponding to no infection is locally asymptotically stable if $\rho(\mathbf{\Gamma}) \leq 1$, otherwise it is unstable. When $\rho(\mathbf{\Gamma}) > 1$ the endemic equilibrium is locally asymptotically stable. Lajmanovich and Yorke (1976) considered the global asymptotic stability. When $\rho(\mathbf{\Gamma}) \leq 1$ the unique equilibrium with no infection present was shown to be globally asymptotically stable. When $\rho(\mathbf{\Gamma}) > 1$ the equilibrium corresponding to the endemic state is globally asymptotically stable provided the system starts with some infection present.

* School of Mathematical Sciences, Queen Mary and Westfield College, Mile End Road, London E1 4NS, England, e-mail: L.Rass@qmw.ac.uk

Subsequent work on multi-type models has also concentrated on the non-reducible case and has used monotonicity and sublinearity methods based on those of Krasnosel'skii (1964). Aronsson and Melander (1980) looked at time periodic forcing; this work being extended to arbitrary time-dependent forcing by Thieme (1988, Section 4). A population with distributed infection risk is considered in Thieme (1985). Convergence results for a chronological age structure are proved in Busenberg *et al.* (1991). The use of Liapunov functions in multi-type models with many infection stages is explored by Simon and Jacquez (1992).

In this paper we consider a multi-type system with stable population size when the infection matrix is not restricted to be non-reducible. The models are discussed in Sections 2 and 3. A fundamental theorem on the non-negative solutions of a system of equations involving a convex function is stated in Section 4. This is used to determine the equilibrium solutions for the reducible as well as the non-reducible case. The proof of this theorem is given in Radcliffe and Rass (1984), and is based on the methods of Krasnosel'skii (1964).

Preliminary results are obtained in Sections 5 and 6 for general non-reducible models in which additional terms are included. Section 5 discusses possible equilibria and Section 6 derives global asymptotic convergence results. The additional terms are needed to facilitate the proof in the reducible case. When these terms are zero, the model encompasses both $S \to I$ and $S \to I \to S$ epidemics in open and closed systems.

These are then used in Section 7, Theorem 5, to establish the main result of this paper, namely the global convergence for a general multitype $S \to I(\to S)$ epidemic when no non-reducibility constraints are placed on the infection matrix. This shows that every solution of a multitype $S \to I(\to S)$ model with stable population size converges towards an equilibrium, even if the infection matrix is reducible.

Although we use what are essentially Liapunov functions in proofs of the convergence results, proofs are accomplished using simple analysis without the need to appeal to Liapunov theory.

2. The $S{\to}I{\to}S$ Model

Consider n populations, each consisting of susceptible and infectious individuals. The rate of infection of a susceptible individual in population i by an infectious individual in population j is λ_{ij}. Infectious individuals in population i return to the susceptible state at rate $\beta_i \geq 0,\ i = 1, \ldots, n$. Note that $\beta_i > 0,\ i = 1, \ldots, n$, corresponds to the $S \to I \to S$ epidemic and $\beta_i = 0,\ i = 1, \ldots, n$, corresponds to the $S \to I$ epidemic.

Denote by $S_i(t)$ and $I_i(t)$ the numbers of susceptible and infectious individuals in the i-th population at time t. Then the model is described by the following system of equations:

$$\frac{\mathrm{d}S_i(t)}{\mathrm{d}t} = -\sum_{j=1}^{n} \lambda_{ij} S_i(t) I_j(t) + \beta_i I_i(t),$$

$$\frac{\mathrm{d}I_i(t)}{\mathrm{d}t} = \sum_{j=1}^{n} \lambda_{ij} S_i(t) I_j(t) - \beta_i I_i(t), \quad \text{for} \quad i = 1, \ldots, n. \tag{1}$$

Let σ_i be the size of the i-th population. Thus $\sigma_i = S_i(t) + I_i(t)$. Equations (1) can be re-written as

$$\frac{\mathrm{d}I_i(t)}{\mathrm{d}t} = \sum_{j=1}^{n} \lambda_{ij} I_j(t)\big(\sigma_i - I_i(t)\big) - \beta_i I_i(t), \quad \text{for} \quad i = 1, \ldots, n,$$

where $0 \le I_i(t) \le \sigma_i$. Denote by $y_i(t)$ the proportion of individuals in population i who are infectious at time t, i.e. $y_i(t) = I_i(t)/\sigma_i$. Then we obtain the system of equations

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = \big(1 - y_i(t)\big) \sum_{j=1}^{n} \sigma_j \lambda_{ij} y_j(t) - \beta_i y_i(t), \quad \text{for} \quad i = 1, \ldots, n. \tag{2}$$

3. The Open $S \to I \to S$ Model

In this section an open version of the $S \to I \to S$ model of Section 2 is considered. Individuals enter the population by birth and/or immigration and leave by death. The equations for this model turn out to be identical to those of the $S \to I \to S$ model but with different parameters. Thus we can analyse both the models simultaneously.

Consider n populations, each consisting of susceptible and infectious individuals. The parameters are specified as in Section 2, with the additional parameters v_i, u_i and α_i representing, for population i, the rate at which susceptibles are born, the rate at which they immigrate into the population and the death rate. The extended model is described by the following system of equations:

$$\frac{\mathrm{d}S_i(t)}{\mathrm{d}t} = v_i\big(S_i(t) + I_i(t)\big) + u_i - \alpha_i S_i(t) + \beta_i I_i(t) - \sum_{j=1}^{n} \lambda_{ij} S_i(t) I_j(t),$$

$$\frac{\mathrm{d}I_i(t)}{\mathrm{d}t} = \sum_{j=1}^{n} \lambda_{ij} S_i(t) I_j(t) - \alpha_i I_i(t) - \beta_i I_i(t), \quad \text{for} \quad i = 1, \ldots, n. \tag{3}$$

We consider the case when the size, σ_i, of the i-th population remains constant. Thus $\sigma_i = S_i(t) + I_i(t)$ and hence $(v_i\sigma_i + u_i) = \alpha_i\sigma_i$. Note that the model of Section 2 i.e. a disease in a closed population with no births, deaths or immigration corresponds to the special case $v_i = u_i = \alpha_i = 0$. Equations (3) can be re-written as

$$\frac{\mathrm{d}I_i(t)}{\mathrm{d}t} = \sum_{j=1}^{n} \lambda_{ij} I_j(t)\big(\sigma_i - I_i(t)\big) - \big(\alpha_i + \beta_i\big) I_i(t), \quad \text{for} \quad i = 1, \ldots, n,$$

where $0 \le I_i(t) \le \sigma_i$. Let $\mu_i = (\alpha_i + \beta_i)$ and denote the proportion of individuals in population i who are infectious at time t by $y_i(t)$, i.e. $y_i(t) = I_i(t)/\sigma_i$. Then we obtain the system of equations

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = \big(1 - y_i(t)\big) \sum_{j=1}^{n} \sigma_j \lambda_{ij} y_j(t) - \mu_i y_i(t), \quad \text{for} \quad i = 1, \ldots, n. \tag{4}$$

In the closed system of Section 2 $\alpha_i = 0$, so that $\mu_i = \beta_i$, $i = 1, \ldots, n$. Thus (4) describes both closed and open systems where $\mu_i = \beta_i$ in a closed system and $\mu_i = (\alpha_i + \beta_i)$ in an open system. In general, $\mu_i = 0$ for all i models the situation when the duration of the epidemic is short relative to the life cycle of the individuals and no return to the susceptible state is possible; otherwise positive values of the μ_i will be required. Terms with some μ_i's zero and some positive occur when the populations under consideration have quite varied life cycle lengths.

4. The Non-Negative Solutions of a System of Non-Linear Equations

The theorem stated in this section concerns the non-negative solutions to a system of equations involving a convex function $f(y)$. It is used to obtain the equilibrium solutions and prove global stability. It is a simple generalisation of Theorem 1 in Radcliffe and Rass (1984), which proves the result for a specific function $f(y) = -\log(1-y)$. The proof, however, only requires certain properties of $-\log(1-y)$. The proof therefore may be simply adapted for a general function $f(y)$ satisfying those properties, which are given in the statement of Theorem 1 below. The method of proof is based on the monotone techniques of Krasnosel'skii (1964). Since the adaptation is simple the results are merely stated. In this paper we use Theorem 1 with $f(y) = y/(1-y)$.

We are concerned in this section with a non-reducible matrix. However, it is necessary in Section 5 to partition the non-reducible infection matrix, which can result in a reducible submatrix. Also in Section 8 a reducible infection matrix is used. Theorem 1 therefore not only gives solutions when the matrix $\boldsymbol{B}$ is non-reducible, but also gives solutions of a particular form in the reducible case. There are other solutions possible when $\boldsymbol{B}$ is reducible which are irrelevant to the mathematical analysis of this paper. These correspond to replacing appropriate positive vectors $\boldsymbol{\eta}_i(\boldsymbol{B}, \boldsymbol{a})$ in Part 3 by vectors of zeros. The solutions specified in Part 3 of Theorem 1 are precisely those required to give the limiting results in Part 4 of that Theorem.

Theorem 1. *Let* $\boldsymbol{B} = (\beta_{ij})$ *be a non-negative* $n \times n$ *matrix and let* $a_i \geq 0$ *for all* $i = 1, \ldots, n$. *Let* $f(y)$ *be a function on* $[0,1)$ *such that* $f(0) = 0$, $f'(0) = 1$, $f''(y) > 0$ *and* $\lim_{y \uparrow 1} f(y) = \infty$.

Consider the possible solutions to the system of equations

$$f(y_i) = \sum_{j=1}^{n} \beta_{ij} y_j + a_i, \tag{5}$$

for $i = 1, \ldots, n$, *where* $0 \leq y_i < 1$.

The matrix $\boldsymbol{B}$ is written in normal form (Gantmacher, 1959, p.75) and the n-dimensional vectors $\boldsymbol{a}$ and $\boldsymbol{y}$, with $\{\boldsymbol{a}\}_i = a_i$ and $\{\boldsymbol{y}\}_i = y_i$, are partitioned so that

$$\boldsymbol{B} = \begin{pmatrix} \boldsymbol{B}_{11} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{B}_{22} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{B}_{ss} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{B}_{s+1,1} & \boldsymbol{B}_{s+1,2} & \dots & \boldsymbol{B}_{s+1,s} & \boldsymbol{B}_{s+1,s+1} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{B}_{g1} & \boldsymbol{B}_{g2} & \dots & \boldsymbol{B}_{gs} & \boldsymbol{B}_{g,s+1} & \dots & \boldsymbol{B}_{gg} \end{pmatrix},$$

$$\boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \\ \vdots \\ \vdots \\ \boldsymbol{a}_g \end{pmatrix} \quad \textit{and} \quad \boldsymbol{y} = \begin{pmatrix} \boldsymbol{y}_1 \\ \boldsymbol{y}_2 \\ \vdots \\ \vdots \\ \boldsymbol{y}_g \end{pmatrix}.$$

Here $\boldsymbol{B}_{ii}$ is a non-reducible square matrix of order r_i and $\boldsymbol{a}_i$ and $\boldsymbol{y}_i$ are r_i-dimensional vectors for $i = 1, \dots, g$. In addition, if $s < g$, at least one $\boldsymbol{B}_{i1}, \dots, \boldsymbol{B}_{i,i-1}$ is non-zero for each i such that $s+1 \le i \le g$.

1. *If $\boldsymbol{a}_i \neq 0$ for all $i = 1, \dots, s$, then eqns. (5) have a unique solution $\boldsymbol{y} = \eta(\boldsymbol{B}, \boldsymbol{a}) > \mathbf{0}$.*
2. *When $\mathbf{B}$ is non-reducible (i.e. $s = g = 1$) and $\boldsymbol{a} = \mathbf{0}$, then eqns. (5) admit the trivial solution $\boldsymbol{y} = \mathbf{0}$. If $\rho(\boldsymbol{B}) > 1$ there exists a unique non-trivial solution $\boldsymbol{y} = \eta(\boldsymbol{B}, \mathbf{0}) > \mathbf{0}$. When $\rho(\boldsymbol{B}) \le 1$ no non-trivial solution exists. In this case we define $\eta(\boldsymbol{B}, \mathbf{0}) = \mathbf{0}$.*
3. *When $\boldsymbol{B}$ is reducible with at least one $\boldsymbol{a}_i = \mathbf{0}$ for $i = 1, \dots, s$, there exists a solution $\boldsymbol{y}$ to eqns. (5) of a particular form. For each $i = 1, \dots, s$, this form has $\boldsymbol{y}_i > \mathbf{0}$ if $\rho(\boldsymbol{B}_{ii}) > 1$ and/or $\boldsymbol{a}_i \neq \mathbf{0}$, and $\boldsymbol{y}_i = \mathbf{0}$ otherwise. Then successively (if $s < g$) for $i = s+1, \dots, g$, it has $\boldsymbol{y}_i > \mathbf{0}$ if $\rho(\boldsymbol{B}_{ii}) > 1$ and/or $\sum_{j<i} \boldsymbol{B}_{ij}\boldsymbol{y}_j + \boldsymbol{a}_i \neq \mathbf{0}$. Again $\boldsymbol{y}_i = \mathbf{0}$ otherwise.*

   The solution is the unique solution of this form. We denote it by $\boldsymbol{y} = \eta(\boldsymbol{B}, \boldsymbol{a})$, and partition it so that

$$\eta(\boldsymbol{B}, \boldsymbol{a}) = \begin{pmatrix} \eta_1(\boldsymbol{B}, \boldsymbol{a}) \\ \eta_2(\boldsymbol{B}, \boldsymbol{a}) \\ \vdots \\ \eta_g(\boldsymbol{B}, \boldsymbol{a}) \end{pmatrix}.$$

The components of the solution are specified in terms of solutions based on a non-reducible matrix as follows. For each $i = 1, \ldots, s$, $\boldsymbol{\eta}_i(\boldsymbol{B}, \boldsymbol{a}) = \boldsymbol{\eta}(\boldsymbol{B}_{ii}, \boldsymbol{a}_i)$. *Then, successively for each* $i = s+1, \ldots, g$, $\boldsymbol{\eta}_i(\boldsymbol{B}, \boldsymbol{a}) = \boldsymbol{\eta}(\boldsymbol{B}_{ii}, \boldsymbol{b}_i)$, *where* $\boldsymbol{b}_i = \sum_{j<i} \boldsymbol{B}_{ij}\boldsymbol{\eta}_j(\boldsymbol{B}, \boldsymbol{a}) + \boldsymbol{a}_i$.

Note that if $\boldsymbol{a} = \boldsymbol{0}$ *and* $\rho(\boldsymbol{B}) \leq 1$ *only the trivial solution is possible. In this case* $\boldsymbol{\eta}(\boldsymbol{B}, \boldsymbol{0}) = \boldsymbol{0}$.

4. *In all cases, if* $\boldsymbol{b} \geq \boldsymbol{a} \geq \boldsymbol{0}$, *then* $\boldsymbol{\eta}(\boldsymbol{B}, \boldsymbol{b}) \geq \boldsymbol{\eta}(\boldsymbol{B}, \boldsymbol{a})$. *Also, for any* $\boldsymbol{a} \geq \boldsymbol{0}$,

$$\lim_{\boldsymbol{b} \downarrow \boldsymbol{a}} \boldsymbol{\eta}(\boldsymbol{B}, \boldsymbol{b}) = \boldsymbol{\eta}(\boldsymbol{B}, \boldsymbol{a}).$$

5. Preliminary Results Concerning Equilibria

We consider the solution to a generalisation of eqns. (4) for a non-reducible system, and then show convergence to a specified equilibrium for this system. The generalisation consists of the addition of terms a_i, for $i = 1, \ldots, n$, to the model. If we set $a_i = 0$ for all i, then we obtain the global result for the solution $y_i(t)$, for $i = 1, \ldots, n$, to eqns. (4) for the non-reducible case.

The introduction of the additional terms has no biological interpretation. The modification is merely a mathematical device which allows intermediate results to be obtained. These are then utilised in Section 7 to enable us to obtain global convergence results for reducible systems.

Define $\boldsymbol{\Lambda}$ to be the matrix with $\{\boldsymbol{\Lambda}\}_{ij} = \sigma_j\lambda_{ij}$ and let $\{\boldsymbol{\mu}\}_i = \mu_i$ and $\{\boldsymbol{a}\}_i = a_i$. The matrix $\boldsymbol{\Lambda}$ is assumed to be non-reducible, i.e. for each i, j there exists a sequence $i_1, \ldots, i_r$ with $i_1 = i$ and $i_r = j$ such that $\lambda_{i_s i_{s+1}} \neq 0$ for $s = 1, \ldots, r-1$.

Consider the system of equations

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = \big(1 - y_i(t)\big)\left(\sum_{j=1}^{n} \sigma_j\lambda_{ij}y_j(t) + a_i\right) - \mu_i y_i(t), \tag{6}$$

for $i = 1, \ldots, n$, where $\boldsymbol{a} \geq \boldsymbol{0}$ and $\boldsymbol{\mu} \geq \boldsymbol{0}$. Take $0 \leq y_i(0) \leq 1$ for all i so that $0 \leq y_i(t) \leq 1$ for all i and all $t \geq 0$.

The possible equilibrium solutions $y_i(t) = y_i$ to eqns. (6) then satisfy the equations

$$(1 - y_i)\left(\sum_{j=1}^{n} \sigma_j\lambda_{ij}y_j + a_i\right) = \mu_i y_i,$$

for $i = 1, \ldots, n$. Define $\{\boldsymbol{y}\}_i = y_i$. This may be then be rewritten as

$$\mathrm{diag}(\boldsymbol{1} - \boldsymbol{y})(\boldsymbol{\Lambda}\boldsymbol{y} + \boldsymbol{a}) = \mathrm{diag}(\boldsymbol{\mu})\boldsymbol{y}. \tag{7}$$

A complete description of the possible equilibrium solutions $\boldsymbol{y}$ to eqn. (7) is easily obtained from Theorem 1 and is given in Theorem 2.

Theorem 2. *The equilibrium* $\boldsymbol{y} = \boldsymbol{0}$ *occurs precisely when* $\boldsymbol{a} = \boldsymbol{0}$. *This is the only possible equilibrium solution when* $\boldsymbol{a} = \boldsymbol{0}$, $\boldsymbol{\mu} > \boldsymbol{0}$ *and* $\rho((\mathrm{diag}(\boldsymbol{\mu}))^{-1}\boldsymbol{\Lambda}) \leq 1$. *In this case we define* $\boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) = \boldsymbol{0}$.

In all other cases there is a unique non-zero equilibrium $\boldsymbol{y} = \boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) > \boldsymbol{0}$. *This is specified as follows:*

1. *When* $\boldsymbol{\mu} > \boldsymbol{0}$ *and either* $\boldsymbol{a} \neq \boldsymbol{0}$ *or* $\rho((\mathrm{diag}(\boldsymbol{\mu}))^{-1}\boldsymbol{\Lambda}) > 1$, *then*

$$\boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) = \boldsymbol{\eta}\Big(\big(\mathrm{diag}(\boldsymbol{\mu})\big)^{-1}\boldsymbol{\Lambda}, \big(\mathrm{diag}(\boldsymbol{\mu})\big)^{-1}\boldsymbol{a}\Big).$$

2. *When* $\boldsymbol{\mu} = \boldsymbol{0}$, *then* $\boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) = \boldsymbol{1}$.

3. *The remaining case corresponds to a partitioning of* $\boldsymbol{\Lambda}$, $\boldsymbol{\mu}$, $\boldsymbol{a}$ *and* $\boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a})$ *(by permutation of the indices) into*

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{11} & \boldsymbol{\Lambda}_{12} \\ \boldsymbol{\Lambda}_{21} & \boldsymbol{\Lambda}_{22} \end{pmatrix}, \quad \boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \end{pmatrix}, \quad \boldsymbol{a} = \begin{pmatrix} \boldsymbol{a}_1 \\ \boldsymbol{a}_2 \end{pmatrix}$$

and

$$\boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) = \begin{pmatrix} \boldsymbol{\zeta}_1(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) \\ \boldsymbol{\zeta}_2(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) \end{pmatrix}$$

where $\boldsymbol{\mu}_1 = \boldsymbol{0}$ *and* $\boldsymbol{\mu}_2 > \boldsymbol{0}$. *In this case* $\boldsymbol{\zeta}_1(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) = \boldsymbol{1}$ *and*

$$\boldsymbol{\zeta}_2(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) = \boldsymbol{\eta}\Big(\big(\mathrm{diag}(\boldsymbol{\mu}_2)\big)^{-1}\boldsymbol{\Lambda}_{22}, \big(\mathrm{diag}(\boldsymbol{\mu}_2)\big)^{-1}(\boldsymbol{\Lambda}_{21}\boldsymbol{1} + \boldsymbol{a}_2)\Big).$$

Proof. Clearly, when $\boldsymbol{a} = \boldsymbol{0}$, $\boldsymbol{y} = \boldsymbol{0}$ is a solution to (7). It is also easily seen that if $y_i = 0$ for some i then $a_i = 0$ and, from the non-reducibility of $\boldsymbol{\Lambda}$, that $y_j = 0$ and $a_j = 0$ for all j. Hence $\boldsymbol{y} = \boldsymbol{0}$ is a solution if and only if $\boldsymbol{a} = \boldsymbol{0}$. Any solution with $\boldsymbol{y} \neq \boldsymbol{0}$ must have $\boldsymbol{y} > \boldsymbol{0}$. The non-existence result follows as for case 1 below.

We now obtain the positive solution, and show uniqueness, for each of the three specified cases:

1. Since $\boldsymbol{\mu} > \boldsymbol{0}$, eqn. (7) can be rewritten in the form

$$f(y_i) = \frac{y_i}{1 - y_i} = \sum_{j=1}^{n} \frac{\sigma_j \lambda_{ij}}{\mu_i} y_j + \frac{a_i}{\mu_i},$$

for $i = 1, \ldots, n$. Since either $a_i/\mu_i > 0$ for some i or $\rho(\boldsymbol{B}) > 1$, where $\boldsymbol{B} = (\mathrm{diag}(\boldsymbol{\mu}))^{-1}\boldsymbol{\Lambda}$, the result then follows immediately from Theorem 1, Part 1.

2. Since $\boldsymbol{\mu} = \boldsymbol{0}$, $\boldsymbol{y} > \boldsymbol{0}$ and $\boldsymbol{\Lambda}$ is non-reducible it immediately follows that $\boldsymbol{\Lambda y} > \boldsymbol{0}$ so that the only solution to (7) is $\boldsymbol{y} = \boldsymbol{1}$.

3. As for Case 2, since $\boldsymbol{\Lambda}$ is non-reducible, any solution $\boldsymbol{y} > \mathbf{0}$ to (7) has $(\boldsymbol{\Lambda}_{11}\boldsymbol{y}_1 + \boldsymbol{\Lambda}_{12}\boldsymbol{y}_2) > \mathbf{0}$. As $\boldsymbol{\mu}_1 = \mathbf{0}$ it then follows from eqns. (7) that $\boldsymbol{y}_1 = \mathbf{1}$. Then $\boldsymbol{y}_2$ must satisfy the following equation:

$$\left(\mathrm{diag}(\mathbf{1} - \boldsymbol{y}_2)\right)^{-1}\boldsymbol{y}_2 = \left(\mathrm{diag}(\boldsymbol{\mu}_2)\right)^{-1}\boldsymbol{\Lambda}_{22}\boldsymbol{y}_2 + \left(\mathrm{diag}(\boldsymbol{\mu}_2)\right)^{-1}(\boldsymbol{\Lambda}_{21}\mathbf{1} + \boldsymbol{a}_2), \quad (8)$$

where $\boldsymbol{\Lambda}_{21}\mathbf{1} \neq \mathbf{0}$ since $\boldsymbol{\Lambda}$ is non-reducible.

In Theorem 1, take $f(y) = y/(1-y)$, $\boldsymbol{B} = (\mathrm{diag}(\boldsymbol{\mu}_2))^{-1}\boldsymbol{\Lambda}_{22}$ and $\boldsymbol{a} = (\mathrm{diag}(\boldsymbol{\mu}_2))^{-1}(\boldsymbol{\Lambda}_{21}\mathbf{1} + \boldsymbol{a}_2)$. If $\boldsymbol{B}$ is non-reducible, since $\boldsymbol{\Lambda}$ is non-reducible and $\boldsymbol{a} \geq \boldsymbol{\Lambda}_{21}\mathbf{1} \neq \mathbf{0}$ the result follows from Theorem 1, Part 1. When $\boldsymbol{B}$ is reducible and written in normal form, the non-reducibility of $\boldsymbol{\Lambda}$ implies that the first s components of $\boldsymbol{a}$ corresponding to that normal form are all non-zero. Hence from Theorem 1, Part 3 there is a unique positive solution to (8) given by $\boldsymbol{y}_2 = \eta((\mathrm{diag}(\boldsymbol{\mu}_2))^{-1}\boldsymbol{\Lambda}_{22}, (\mathrm{diag}(\boldsymbol{\mu}_2))^{-1}(\boldsymbol{\Lambda}_{22}\mathbf{1} + \boldsymbol{a}_2))$. ■

6. The Global Asymptotic Stability of the Equilibria

In this section, we look at the global asymptotic stability of the equilibrium $\boldsymbol{\eta} = \zeta(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a})$ when $\boldsymbol{\Lambda}$ is non-reducible. Define $\{\boldsymbol{y}(t)\}_i = y_i(t)$. We show that if $\boldsymbol{y}(0) \neq \mathbf{0}$, then $\boldsymbol{y}(t)$ tends to $\boldsymbol{\eta}$ as t tends to infinity. Note that when $\boldsymbol{a} = \mathbf{0}$, $\boldsymbol{\mu} > \mathbf{0}$ and $\rho((\mathrm{diag}(\boldsymbol{\mu}))^{-1}\boldsymbol{\Lambda}) \leq 1$, then the equilibrium $\boldsymbol{\eta} = \mathbf{0}$, otherwise it is the unique positive equilibrium. When $\boldsymbol{\mu} > \mathbf{0}$ define $\boldsymbol{\Gamma} = (\mathrm{diag}(\boldsymbol{\mu}))^{-1}\boldsymbol{\Lambda}$, so that $\{\boldsymbol{\Gamma}\}_{ij} = \sigma_j\lambda_{ij}/\mu_i$.

Global Asymptotic Stability when $\boldsymbol{a}$=0, $\boldsymbol{\mu}$>0 and $\rho(\boldsymbol{\Gamma})\leq$1. Let $\boldsymbol{u}$ be the right eigenvector and $\boldsymbol{v}$ be the left eigenvector of $\boldsymbol{\Gamma}$ corresponding to $\rho(\boldsymbol{\Gamma})$. When $\rho(\boldsymbol{\Gamma}) \leq 1$ we present a proof of the global asymptotic stability based on the function $a(t) = \max_i(y_i(t)/u_i)$ where $u_i = \{\boldsymbol{u}\}_i$. A proof of this result using a different Liapunov function $V(t) = \boldsymbol{v}'(\mathrm{diag}(\boldsymbol{\mu}))^{-1}\boldsymbol{y}(t)$ is given in Lajmanovich and Yorke (1976).

Theorem 3. *When $\boldsymbol{a} = \mathbf{0}$, $\boldsymbol{\mu} > \mathbf{0}$ and $\rho(\boldsymbol{\Gamma}) \leq 1$, $\boldsymbol{y} = \mathbf{0}$ is globally asymptotically stable on* $[0,1]^n$.

Proof. Suppose that $y_i(t) > 0$ for at least one i. Define $a(t) = \max_i(y_i(t)/u_i)$. For a given t, choose i such that $y_i(t)/u_i = a(t)$. There may be more than one such i. However, i can be choosen so that, for sufficiently small $\epsilon > 0$, $a(s) = y_i(s)/u_i$ for $s \in [t, t+\epsilon]$. Then $y_i(t) = u_i a(t)$ and $y_j(t) \leq u_j a(t)$ for $j \neq i$. The function $a(t)$ so defined is a continuous function of t with a right hand derivative given by

$$\begin{aligned} u_i\frac{\mathrm{d}_+a(t)}{\mathrm{d}t} &= \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \\ &= \mu_i\left[(1 - y_i(t))\sum_{j=1}^{n}\frac{\sigma_j\lambda_{ij}}{\mu_i}y_j(t) - y_i(t)\right] \\ &= \mu_i\left[(1 - u_i a(t))\sum_{j=1}^{n}\frac{\sigma_j\lambda_{ij}}{\mu_i}y_j(t) - u_i a(t)\right] \end{aligned}$$

$$\leq \mu_i\left[\left(1-u_ia(t)\right)a(t)\sum_{j=1}^{n}\frac{\sigma_j\lambda_{ij}}{\mu_i}u_j - u_ia(t)\right]$$

$$= \mu_i\Big[\left(1-u_ia(t)\right)a(t)\rho(\mathbf{\Gamma})u_i - u_ia(t)\Big]$$

$$= \mu_i a(t)u_i\Big[\rho(\mathbf{\Gamma})\left(1-u_ia(t)\right) - 1\Big]$$

$$= -\mu_i a(t)u_i\Big[\left(1-\rho(\mathbf{\Gamma})\right) + u_ia(t)\rho(\mathbf{\Gamma})\Big].$$

Let $\alpha = (1-\rho(\mathbf{\Gamma}))\min_i(\mu_i)$ and $\theta = \rho(\mathbf{\Gamma})\min_i(u_i\mu_i)$. If $\rho(\mathbf{\Gamma}) < 1$ then $\mathrm{d}_+a(t)/\mathrm{d}t \leq -\alpha a(t)$, where $\alpha > 0$. If $\rho(\mathbf{\Gamma}) = 1$ then $\mathrm{d}_+a(t)/\mathrm{d}t \leq -\theta(a(t))^2$, where $\theta > 0$. Thus $a(t)$ is monotone decreasing and is bounded below by 0, and so must tend to a limit. It is simple to show that this limit is 0.

Consider first the case where $\rho(\mathbf{\Gamma}) < 1$. Let $a(0) = a_0$ and suppose that $a(t) \downarrow a_1 > 0$ as $t \to \infty$. Now $\mathrm{d}_+a(t)/\mathrm{d}t \leq -\alpha a_1/2$ provided $a(t) > a_1/2$. Thus if $a(t) > a_1/2$ the graph of $y = a(t)$ lies below the line $y = a_0 - \alpha a_1 t/2$. It follows that $a(t)$ will decrease to $a_1/2$ at least by time $(2a_0 - a_1)/\alpha a_1$. This contradicts the statement that $a(t) \downarrow a_1 > 0$. If $\rho(\mathbf{\Gamma}) = 1$, a similar argument shows that $a(t)$ decreases to $a_1/2$ before time $(4a_0 - 2a_1)/\theta a_1^2$. Thus in both cases $a(t) \downarrow 0$ as $t \to \infty$ and the theorem is proved. ■

Global Asymptotic Stability in All Other Cases. Now consider the remaining cases when $\boldsymbol{a} \neq \mathbf{0}$ and/or one of the following occurs; either $\mu_i = 0$ for some i or $\boldsymbol{\mu} > \mathbf{0}$ and $\rho(\mathbf{\Gamma}) > 1$. Suppose that $y_i(0) > 0$ for at least one i. Then it can be shown that $y_i(t) > 0$ for all $i = 1, \ldots, n$, and all $t > 0$. This is then used to establish global asymptotic stability. The following lemma is a first step in proving this result. Write $y^{(r)}(t) = \mathrm{d}^r y(t)/\mathrm{d}t^r$.

Lemma 1. *If* $\boldsymbol{y}(0) \in [0,1]^n \backslash \{\mathbf{0}\}$, *then* $\exists\, T > 0$ *such that* $y_i(t) > 0,\ i = 1, \ldots, n$ *for* $t \in (0, T]$.

Proof. Let $S = \{i | y_i(0) = 0\}$ and $S' = \{i | y_i(0) > 0\}$. If $i \in S'$ it follows by continuity that $\exists\, T_i > 0$ such that $y_i(t) > 0$ for $t \in [0, T_i]$.

Now consider $i \in S$. Then $y_i^{(1)}(0) = \sum_{j=1}^{n}\{\mathbf{\Lambda}\}_{ij}y_j(0) + a_i$. If $a_i > 0$ or if $\exists\, j \in S'$ such that $\{\mathbf{\Lambda}\}_{ij}y_j(0) > 0$, then $y_i^{(1)}(0) > 0$. If so stop. Otherwise differentiate the i-th equation of (6). Then

$$y_i^{(2)}(0) = \sum_{j=1}^{n}\{\mathbf{\Lambda}\}_{ij}y_j^{(1)}(0) = \sum_{j=1}^{n}\Big(\{\mathbf{\Lambda}^2\}_{ij}y_j(0) + \{\mathbf{\Lambda}\}_{ij}a_j\Big).$$

If either $\exists\, j$ such that $\{\mathbf{\Lambda}\}_{ij} > 0$ and $a_j > 0$, or if $\exists\, j \in S'$ such that $\{\mathbf{\Lambda}^2\}_{ij} > 0$, then $y_i^{(2)}(0) > 0$. If so stop. Otherwise differentiate the i-th equation of (6) twice and continue. Since $\mathbf{\Lambda}$ is non-reducible, for any $j \in S'$ $\exists\, r \leq n-1$ such that $\{\mathbf{\Lambda}^r\}_{ij} > 0$ and so the process must terminate after at most r steps. Thus for any $i \in S$ there exists an s such that $1 \leq s \leq n-1$ with $y_i^{(m)}(0) = 0$ for $1 \leq m \leq s$ and $y_i^{(s)}(0) > 0$. Here s is the smallest positive integer such that either $\{\mathbf{\Lambda}^{s-1}\boldsymbol{a}\}_i > 0$ or $\{\mathbf{\Lambda}^s \boldsymbol{y}(0)\}_i > 0$. Hence, for each $i \in S \; \exists \, T_i > 0$ such that $y_i(t) > 0$ for $t \in (0, T_i]$. The theorem then follows by taking $T = \min_i T_i$. ■

Theorem 4. *When $\boldsymbol{a} \neq \mathbf{0}$ and/or one of the following occurs: either $\mu_i = 0$ for some i or $\boldsymbol{\mu} > \mathbf{0}$ and $\rho(\mathbf{\Gamma}) > 1$, then*

1. *$\boldsymbol{y}(t) > \mathbf{0}$ for all $t > 0$ if $\boldsymbol{y}(0) \in [0,1]^n \backslash \{\mathbf{0}\}$.*

2. *$\boldsymbol{y} = \boldsymbol{\eta}$ is globally asymptotically stable on $[0,1]^n \backslash \{\mathbf{0}\}$.*

Proof. Observe that here $\boldsymbol{\eta} = \zeta(\mathbf{\Lambda}, \boldsymbol{\mu}, \boldsymbol{a}) > \mathbf{0}$. Let $\eta_i = \{\boldsymbol{\eta}\}_i$. Define $b(t) = \min_i(y_i(t)/\eta_i)$. For a given t, choose i such that $y_i(t)/\eta_i = b(t)$. There may be more than one such i. However, i can be choosen so that, for sufficiently small $\epsilon > 0$, $b(s) = (y_i(s)/\eta_i)$ for $s \in [t, t+\epsilon]$. Then $y_i(t) = \eta_i b(t)$ and $y_j(t) \geq \eta_j b(t)$ for $j \neq i$. The function $b(t)$ so defined is a continuous function of t with a right hand derivative given by

$$
\begin{aligned}
\eta_i \frac{\mathrm{d}_+ b(t)}{\mathrm{d}t} &= \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \\
&= \big(1 - y_i(t)\big)\left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} y_j(t) + a_i\right) - \mu_i y_i(t) \\
&= \big(1 - \eta_i b(t)\big)\left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} y_j(t) + a_i\right) - \mu_i \eta_i b(t) \\
&\geq \big(1 - \eta_i b(t)\big)\left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} \eta_j b(t) + a_i\right) - \mu_i \eta_i b(t).
\end{aligned}
$$

Now consider the cases (i) $\mu_i = 0$ and (ii) $\mu_i > 0$ separately.

(i) From the relation $(1 - \eta_i)(\sum_{j=1}^n \sigma_j \lambda_{ij} \eta_j + a_i) = \mu_i \eta_i$ it follows that $\mu_i = 0$ implies that $\eta_i = 1$. Thus if $\mu_i = 0$, necessarily $b(t) = y_i(t) \leq 1$ and

$$
\begin{aligned}
\frac{\mathrm{d}_+ b(t)}{\mathrm{d}t} &\geq \big(1 - b(t)\big)\left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} \eta_j b(t) + a_i b(t)\right) \\
&= b(t)\big(1 - b(t)\big)\left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} \eta_j + a_i\right).
\end{aligned}
$$

(ii) If $\mu_i > 0$ and if $b(t) \le 1$, then

$$\eta_i \frac{\mathrm{d}_+ b(t)}{\mathrm{d}t} \ge (1-\eta_i b(t))\left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} \eta_j + a_i\right) b(t) - \mu_i \eta_i b(t)$$

$$= \mu_i \left[(1-\eta_i b(t)) b(t) \frac{\eta_i}{1-\eta_i} - \eta_i b(t) \right],$$

and so

$$\frac{\mathrm{d}_+ b(t)}{\mathrm{d}t} \ge \frac{\mu_i \eta_i}{(1-\eta_i)} b(t)(1-b(t)).$$

Thus in both case (i) and case (ii), if $b(t) \le 1$,

$$\frac{\mathrm{d}_+ b(t)}{\mathrm{d}t} \ge A b(t)(1-b(t)),$$

where $A = \min\left(\min\limits_{i:\mu_i=0} \left(\sum\limits_{j=1}^{n} \sigma_j \lambda_{ij} \eta_j + a_i \right), \min\limits_{i:\mu_i>0} \frac{\mu_i \eta_i}{1-\eta_i} \right)$.

From Lemma 1 $\boldsymbol{y}(t) > \mathbf{0}$ and hence $b(t) > 0$ for $t \in (0,T]$ for some T. Now $\mathrm{d}_+ b(t)/\mathrm{d}t > 0$ if $0 < b(t) < 1$ and $\mathrm{d}_+ b(t)/\mathrm{d}t \ge 0$ if $b(t) = 1$. It follows that $b(t) \ge b(T) > 0$ for $t > T$ and hence that $\boldsymbol{y}(t) \ge b(t)\boldsymbol{\eta} > \mathbf{0}$ for $t > 0$. This proves Part 1 of the theorem.

Let $m(t) = \min(b(t), 1)$. Then $m(t)$ is monotone non-decreasing and bounded above by 1, and so must tend to a limit. We show that this limit is 1. If $b(T) \ge 1$ then $m(t) = 1$ for $t \ge T$ and the limit is 1. Consider the case where $0 < b(T) = b_0 < 1$. Suppose $m(t) \uparrow b_1 < 1$ as $t \to \infty$. Now

$$\frac{\mathrm{d}_+ m(t)}{\mathrm{d}t} = \frac{\mathrm{d}_+ b(t)}{\mathrm{d}t} \ge \theta,$$

where $\theta = A\min[b_0(1-b_0), b_1(1-b_1)]$. So the the graph of $y = m(t)$ lies above the line $y = b_0 - \theta(t-T)$. Thus $m(t)$ will increase to $(1+b_1)/2$ at least by time $T + (1 + b_1 - 2b_0)/2\theta$. This contradicts the statement that $m(t) \uparrow b_1 < 1$ as $t \to \infty$. Thus $m(t) \uparrow 1$ as $t \to \infty$.

Define $c(t) = \max_i(y_i(t)/\eta_i)$. Then $c(t)$ is a continuous function of t. We can choose i so that, for sufficiently small $\epsilon > 0$, $c(s) = (y_i(s)/\eta_i)$ for $s \in [t, t+\epsilon]$. Then $y_i(t) = \eta_i c(t)$ and $y_j(t) \le \eta_j c(t)$ for $j \ne i$.

We need to consider the cases (i) $\mu_i = 0$ and (ii) $\mu_i > 0$ separately.

(i) If $\mu_i = 0$ then $\eta_i = 1$ and so $c(t) = y_i(t) \le 1$ and the right hand derivative of $c(t)$ is given by

$$\frac{\mathrm{d}_+ c(t)}{\mathrm{d}t} = \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} = (1-c(t))\left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} y_j(t) + a_i\right).$$

In this case if $c(t) \ge 1$, then necessarily $c(t) = 1$ and $\mathrm{d}_+ c(t)/\mathrm{d}t = 0$.

(ii) When $\mu_i > 0$ then, if $c(t) \geq 1$,

$$\begin{aligned}
\eta_i \frac{\mathrm{d}_+ c(t)}{\mathrm{d}t} &= \frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \\
&= \left[(1 - y_i(t)) \left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} y_j(t) + a_i \right) - \mu_i y_i(t) \right] \\
&\leq \left[(1 - \eta_i c(t)) \left(\sum_{j=1}^{n} \sigma_j \lambda_{ij} \eta_j c(t) + c(t) a_i \right) - \mu_i \eta_i c(t) \right] \\
&= \mu_i \left[(1 - \eta_i c(t)) c(t) \frac{\eta_i}{1 - \eta_i} - \eta_i c(t) \right] \\
&= -\frac{\mu_i \eta_i^2}{(1 - \eta_i)} c(t) \big(c(t) - 1 \big).
\end{aligned}$$

In both cases for $c(t) \geq 1$,

$$\frac{\mathrm{d}_+ c(t)}{\mathrm{d}t} \leq -c(t) \big(c(t) - 1 \big) B,$$

where $B = \min_i \dfrac{\mu_i \eta_i}{1 - \eta_i}$.

From case (ii) it follows that $\mathrm{d}_+ c(t)/\mathrm{d}t < 0$ if $c(t) > 1$ and from cases (i) and (ii) it follows that $\mathrm{d}_+ c(t)/\mathrm{d}t \leq 0$ if $c(t) = 1$. Let $M(t) = \max(c(t), 1)$. Then $M(t)$ is monotone non-increasing and bounded below by 1, and so must tend to a limit. We show that this limit is 1. If $c(0) \leq 1$ then $M(0) = 1$ and the limit is 1. Consider the case where $c(0) = c_0 > 1$. Suppose $M(t) \downarrow c_1 > 1$ as $t \to \infty$. Now

$$\frac{\mathrm{d}_+ M(t)}{\mathrm{d}t} = \frac{\mathrm{d}_+ c(t)}{\mathrm{d}t} \leq -\alpha,$$

where $\alpha = B \min[c_0(c_0 - 1), c_1(c_1 - 1)]$. So the the graph of $y = M(t)$ lies below the line $y = c_0 - \alpha t$. Thus $M(t)$ will decrease to $(1 + c_1)/2$ before time $(2c_0 - 1 - c_1)/2\alpha$. This contradicts the statement that $M(t) \downarrow c_1 > 1$. Thus $M(t) \downarrow 1$ as $t \to \infty$.

Now for all i,

$$m(t) \leq b(t) \leq \frac{y_i(t)}{\eta_i} \leq c(t) \leq M(t).$$

Since $m(t) \to 1$ and $M(t) \to 1$ as $t \to \infty$, it follows that $y_i(t) \to \eta_i$ as $t \to \infty$ for $i = 1, \ldots, n$. ■

The global asymptotic stability for models of epidemics of $S \to I$ and $S \to I \to S$ type in both open and closed systems, described by eqns. (4), can be immediately obtained from Theorems 3 and 4 by setting $\boldsymbol{a} = \mathbf{0}$. It is assumed that $\boldsymbol{\Lambda}$ is non-reducible. The result is given in the following Corollary.

Corollary 1. *When $\boldsymbol{\mu} > \mathbf{0}$ and $\rho((\mathrm{diag}(\boldsymbol{\mu}))^{-1}\boldsymbol{\Lambda}) \leq 1$, then $\boldsymbol{y}(t) \to \mathbf{0}$ as $t \to \infty$ regardless of the amount of initial infection present. The infection dies out in this case.*

When either $\mu_i = 0$ for some i, or $\boldsymbol{\mu} > \mathbf{0}$ and $\rho((\mathrm{diag}(\boldsymbol{\mu}))^{-1}\boldsymbol{\Lambda}) > 1$, then the endemic equilibrium $\boldsymbol{\zeta}(\boldsymbol{\Lambda}, \boldsymbol{\mu}, \mathbf{0}) > \mathbf{0}$ is globally asymptotically stable on $[0,1]^n \backslash \{\mathbf{0}\}$. Provided there is some initial infection, the proportion of infectives $\boldsymbol{y}(t)$ in the populations will tend to the endemic level as $t \to \infty$.

7. Reducible Epidemics

Now consider the solution $\boldsymbol{y}(t)$ to eqn. (4) when the infection matrix $\boldsymbol{\Lambda}$ is reducible. We need only to consider the case when the initial infection can cause infection amongst all types. If certain types cannot be infected these should be omitted from the set of types and a model based only upon the remaining types considered.

The equivalent results to Corollary 1 are now derived when the infection matrix $\boldsymbol{\Lambda}$ is reducible. The proof makes use of Theorem 4 with $\boldsymbol{a} \neq \mathbf{0}$.

Theorem 5. *Permute the types so that $\boldsymbol{\Lambda}$ is in normal form (Gantmacher, 1959) and partition the vectors $\boldsymbol{\mu}$ and $\boldsymbol{y}(t)$ in a corresponding manner so that*

$$\boldsymbol{\Lambda} = \begin{pmatrix} \boldsymbol{\Lambda}_{11} & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Lambda}_{22} & \dots & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \mathbf{0} & \mathbf{0} & \dots & \boldsymbol{\Lambda}_{ss} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{\Lambda}_{s+1,1} & \boldsymbol{\Lambda}_{s+1,2} & \dots & \boldsymbol{\Lambda}_{s+1,s} & \boldsymbol{\Lambda}_{s+1,s+1} & \dots & \mathbf{0} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \boldsymbol{\Lambda}_{g1} & \boldsymbol{\Lambda}_{g2} & \dots & \boldsymbol{\Lambda}_{gs} & \boldsymbol{\Lambda}_{g,s+1} & \dots & \boldsymbol{\Lambda}_{gg} \end{pmatrix},$$

$$\boldsymbol{\mu} = \begin{pmatrix} \boldsymbol{\mu}_1 \\ \boldsymbol{\mu}_2 \\ \vdots \\ \boldsymbol{\mu}_g \end{pmatrix} \quad \text{and} \quad \boldsymbol{y}(t) = \begin{pmatrix} \boldsymbol{y}_1(t) \\ \boldsymbol{y}_2(t) \\ \vdots \\ \boldsymbol{y}_g(t) \end{pmatrix}.$$

If $y_i(0) \neq \mathbf{0}$ for all $i = 1, \ldots, s$, then $y(t) \to \eta$ as $t \to \infty$, where η is partitioned so that

$$\eta = \begin{pmatrix} \eta_1 \\ \eta_2 \\ \vdots \\ \eta_g \end{pmatrix},$$

with the components defined as follows. For each $i = 1, \ldots, s$, $\eta_i = \zeta(\Lambda_{ii}, \mu_i, \mathbf{0})$. Then successively for each $i = s+1, \ldots, g$, $\eta_i = \zeta(\Lambda_{ii}, \mu_i, b_i)$, where $b_i = \sum_{j<i} \Lambda_{ij}\eta_j$.

Proof. The results for $i = 1, \ldots, s$ follow immediately from Corollary 1. Also note that if $\eta_i > \mathbf{0}$ then $y_i(t) > \mathbf{0}$ for all $t > 0$.

Now we successively prove that $y_j(t) \to \eta_j$ as $t \to \infty$ for each $j = s+1, \ldots, g$. In addition we show that, for each such j, if $\eta_j > \mathbf{0}$ then there exists a $T > 0$ such that $y_j(t) > \mathbf{0}$ for $t > T$.

Suppose these results hold for all $j < i$, where $s+1 \leq i \leq g$. We now show that the results hold for $y_i(t)$. Since $y_j(t) \to \eta_j$ as $t \to \infty$ for all $j < i$, for every $\varepsilon > 0$ there exists a $T > 0$ such that

$$\eta_j - \varepsilon\mathbf{1} \leq y_j(t) \leq \eta_j + \varepsilon\mathbf{1}$$

for all $t \geq T$ and $j < i$. Define $\{\theta_j\}_s = \min(\{\eta_j + \varepsilon\mathbf{1}\}_s, 1)$ and $\{\phi_j\}_s = \max(\{\eta_j - \varepsilon\mathbf{1}\}_s, 0)$. Then

$$\phi_j \leq y_j(t) \leq \theta_j$$

for all $t > T$ and all $j < i$. Also note that for each $j < i$ such that $\eta_j > \mathbf{0}$ we can choose T sufficiently large so that also $y_j(t) > \mathbf{0}$ for $t > T$.

Then the following two inequalities hold:

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \leq \operatorname{diag}(\mathbf{1} - y_i(t))\left(\Lambda_{ii}y_i(t) + a\right) - \operatorname{diag}(\mu_i)y_i(t) \tag{9}$$

for all $t > T$, where $a = \sum_{j<i} \Lambda_{ij}\theta_j$. Also

$$\frac{\mathrm{d}y_i(t)}{\mathrm{d}t} \geq \operatorname{diag}(\mathbf{1} - y_i(t))\left(\Lambda_{ii}y_i(t) + b\right) - \operatorname{diag}(\mu_i)y_i(t) \tag{10}$$

for all $t > T$, where $b = \sum_{j<i} \Lambda_{ij}\phi_j$. Note that if $\sum_{j<i} \Lambda_{ij}\eta_j = \mathbf{0}$, then $b = \mathbf{0}$, and (10) holds for all $t \geq 0$.

We divide the proof into two parts, the first part uses (9) to show that $\limsup_{t\to\infty} y_i(t) \leq \zeta(\Lambda_{ii}, \mu_i, a)$. Since this holds for all $\varepsilon > 0$, continuity of $\zeta(\Lambda_{ii}, \mu_i, a)$ in a implies that $\limsup_{t\to\infty} y_i(t) \leq \eta_i$.

This is all that is required when $\sum_{j<i} \Lambda_{ij}\eta_j = \mathbf{0}$, $\mu_i > \mathbf{0}$ and $\rho((\operatorname{diag}(\mu))^{-1}\Lambda) \leq 1$, since in this case $\eta_i = \mathbf{0}$.

In all other cases (i.e. when $\boldsymbol{\eta}_i > \mathbf{0}$) we need also to obtain a lower bound for $\boldsymbol{y}_i(t)$. We use (10) to show that $\boldsymbol{y}_i(t) > \mathbf{0}$ for $t > T$ and to prove that $\liminf_{t\to\infty} \boldsymbol{y}_i(t) \geq \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})$. Again this holds for all $\varepsilon > 0$, so that continuity of $\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})$ in $\boldsymbol{b}$ implies that $\liminf_{t\to\infty} \boldsymbol{y}_i(t) \geq \boldsymbol{\eta}_i$.

When $\boldsymbol{\eta}_i > \mathbf{0}$ the results from the two parts imply that $\boldsymbol{y}_i(t)$ tends to $\boldsymbol{\eta}_i$ as t tends to infinity. Also there exists a $T > 0$ such that $\boldsymbol{y}_i(t) > \mathbf{0}$ for $t > T$.

To prove the first part, consider inequality (9). Define $c(t) = \max_s \{\boldsymbol{y}_i(t)\}_s / \{\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\}_s$. Then $c(t)$ is a continuous function of t. Also for a given $t > T$, there exists an s and $t^* > t$ such that $\{\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\}_s c(\tau) = \{\boldsymbol{y}_i(\tau)\}_s$ for $t \leq \tau < t^*$. Now use (9) with this specific s. If $c(t) \geq 1$, then

$$
\begin{aligned}
&\{\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\}_s \frac{\mathrm{d}_+ c(t)}{\mathrm{d}t} \\
&\quad \leq \Big(1 - c(t)\{\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\}_s\Big)\Big\{\boldsymbol{\Lambda}_{ii} c(t) \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a}) + \boldsymbol{a}\Big\}_s \\
&\qquad - \{\boldsymbol{\mu}_i\}_s c(t) \Big\{\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\Big\}_s \\
&\quad \leq \Big(1 - c(t)\{\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\}_s\Big) c(t) \Big\{\boldsymbol{\Lambda}_{ii} \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a}) + \boldsymbol{a}\Big\}_s \\
&\qquad - \{\boldsymbol{\mu}_i\}_s c(t) \Big\{\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\Big\}_s \\
&\quad \leq c(t) \Big\{\mathrm{diag}\big(\boldsymbol{I} - \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\big)\big[\boldsymbol{\Lambda}_{ii} \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a}) + \boldsymbol{a}\big] \\
&\qquad - \mathrm{diag}(\boldsymbol{\mu}_i) \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})\Big\}_s = 0.
\end{aligned}
$$

Note that the inequality is strict when $c(t) > 1$. Hence, provided $t > T$, if $c(t) > 1$ then $\mathrm{d}_+ c(t)/\mathrm{d}t < 0$ and if $c(t) = 1$ then $\mathrm{d}_+ c(t)/\mathrm{d}t \leq 0$.

Now define $M(t) = \max(1, c(t))$. Then $\mathrm{d}M(t)/\mathrm{d}t \leq 0$ with the inequality strict if $M(t) \neq 1$. Since $M(t)$ is monotone, non-decreasing and bounded below by one, it must tend to a limit. As in the proof of Theorem 4 it is easily shown that this limit is one. Hence $M(t) \to 1$ as $t \to \infty$. But $\boldsymbol{y}_i(t) \leq M(t) \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})$ for $t > T$. Hence $\limsup_{t\to\infty} \boldsymbol{y}_i(t) \leq \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a})$.

Now this result holds for any $\varepsilon > 0$, and $\boldsymbol{a} = \sum_{j<i} \boldsymbol{\Lambda}_{ij} \boldsymbol{\eta}_j + \varepsilon \sum_{j<i} \boldsymbol{\Lambda}_{ij} \mathbf{1}$. Therefore $\boldsymbol{a} \downarrow \sum_{j<i} \boldsymbol{\Lambda}_{ij} \boldsymbol{\eta}_j$ as $\varepsilon \downarrow 0$. Consequently, from Theorems 1 and 2, $\boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{a}) \downarrow \boldsymbol{\zeta}(\boldsymbol{\Lambda}_{ii}, \boldsymbol{\mu}_i, \sum_{j<i} \boldsymbol{\Lambda}_{ij} \boldsymbol{\eta}_j) = \boldsymbol{\eta}_i$ as $\varepsilon \downarrow 0$. Hence $\limsup_{t\to\infty} \boldsymbol{y}_i(t) \leq \boldsymbol{\eta}_i$. When $\boldsymbol{\eta}_i = \mathbf{0}$ this implies that $\boldsymbol{y}_i(t) \to \mathbf{0}$ as $t \to \infty$ so the proof is complete in this case.

To prove the second part when $\boldsymbol{\eta}_i > \mathbf{0}$, consider inequality (10). In the case when $\sum_{j<i} \boldsymbol{\Lambda}_{ij} \boldsymbol{\eta}_j = \mathbf{0}$, for any $\varepsilon > 0$, $\boldsymbol{b} = 0$. Hence inequality (10) holds for all $t > 0$. In this case take $T = 0$. Since $\boldsymbol{y}_j(0) \neq \mathbf{0}$ for $j = 1, \ldots, s$, as in Lemma 1 this implies that there exists a $T^* > 0$ such that $\boldsymbol{y}_i(t) > \mathbf{0}$ for $t \in (0, T^*]$. When $\sum_{j<i} \boldsymbol{\Lambda}_{ij} \boldsymbol{\eta}_j \neq 0$ inequality (10) only holds for $t \geq T$. Now there exists a $j < i$ such that $\mathbf{\Lambda}_{ij} \neq \mathbf{0}$ and $\eta_j > \mathbf{0}$. Hence $\boldsymbol{y}_j(t) > \mathbf{0}$ for $t > T$. Then, as in Lemma 1, there exists a $T^* > T$ such that $\boldsymbol{y}_i(t) > \mathbf{0}$ for $t \in (T, T^*]$.

Now take $b(t) = \min_s(\{\boldsymbol{y}_i(t)\}_s / \{\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\}_s)$ for $t \geq T^*$. Note that $b(t)$ is a continuous function of t. Also for a given $t \geq T^*$, there exists an s and $t^* > t$ such that $\{\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\}_s b(\tau) = \{\boldsymbol{y}_i(\tau)\}_s$ for $t \leq \tau < t^*$. Now use (10) with this specific s. If $b(t) \leq 1$ then

$$\begin{aligned}
&\{\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\}_s \frac{\mathrm{d}_+ b(t)}{\mathrm{d}t} \\
&\qquad \geq \Big(1 - b(t)\{\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\}_s\Big)\Big\{\mathbf{\Lambda}_{ii} b(t) \zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b}) + \boldsymbol{b}\Big\}_s \\
&\qquad\quad - \{\boldsymbol{\mu}_i\}_s b(t) \Big\{\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\Big\}_s \\
&\qquad \geq \Big(1 - b(t)\{\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\}_s\Big) b(t) \Big\{\mathbf{\Lambda}_{ii} \zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b}) + \boldsymbol{b}\Big\}_s \\
&\qquad\quad - \{\boldsymbol{\mu}_i\}_s b(t) \Big\{\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\Big\}_s \\
&\qquad \geq b(t) \Big\{\mathrm{diag}\big(\boldsymbol{I} - \zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\big)\big[\mathbf{\Lambda}_{ii} \zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b}) + \boldsymbol{b}\big] \\
&\qquad\quad - \mathrm{diag}(\boldsymbol{\mu}_i) \zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})\Big\}_s = 0.
\end{aligned}$$

Note that the inequality is strict when $0 < b(t) < 1$. Hence, provided $t > T$, if $0 < b(t) < 1$, then $\mathrm{d}_+ b(t)/\mathrm{d}t > 0$ and if $b(t) = 1$ then $\mathrm{d}_+ b(t)/\mathrm{d}t \geq 0$. Since $b(T^*) > 0$ it follows that $b(t) \geq \min(1, b(T^*)) > 0$ for $t \geq T^*$. Therefore $\boldsymbol{y}_i(t) > \mathbf{0}$ for $t \geq T^*$.

If we now define $m(t) = \min(1, b(t))$ then $m(t)$ is monotone and non-decreasing for $t \geq T^*$ and is bounded above by one, so must tend to a limit as t tends to infinity. As in the proof of Theorem 4, it is easily shown that $m(t) \uparrow 1$ as $t \to \infty$. Hence $\boldsymbol{y}_i(t) \geq m(t)\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})$ and therefore $\lim_{t^* \to \infty} \inf_{t \geq t^*} \boldsymbol{y}_i(t) \geq \zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})$.

This result holds for any $\varepsilon > 0$. If $\sum_{j<i}\{\mathbf{\Lambda}_{ij}\eta_j\}_s = 0$, then $\{\boldsymbol{b}\}_s = 0$. Otherwise, for ε sufficiently small, $\{\boldsymbol{b}\}_s = \sum_{j<i}\{\mathbf{\Lambda}_{ij}\eta_j\}_s - \varepsilon \sum_{j<i}\{\mathbf{\Lambda}_{ij}\mathbf{1}\}_s$. In either case, if we let ε tend to zero, then $\boldsymbol{b}$ tends to $\sum_{j<i} \mathbf{\Lambda}_{ij}\eta_j$. Using the continuity of $\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b})$ in $\boldsymbol{b}$, from Theorems 1 and 2, it then follows that $\zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \boldsymbol{b}) \downarrow \zeta(\mathbf{\Lambda}_{ii}, \boldsymbol{\mu}_i, \sum_{j<i} \mathbf{\Lambda}_{ij}\eta_j) = \eta_i$ as $\varepsilon \to \infty$. Hence $\liminf_{t\to\infty} \boldsymbol{y}_i(t) \geq \eta_i$.

Therefore in the case when $\eta_i > \mathbf{0}$, $\eta_i \leq \liminf_{t\to\infty} \boldsymbol{y}_i(t) \leq \limsup_{t\to\infty} \boldsymbol{y}_i(t) \leq \eta_i$. Hence $\boldsymbol{y}_i(t) \to \eta_i$ as $t \to \infty$. Note that we have also shown that there exists a $T > 0$ such that $\boldsymbol{y}_i(t) > \mathbf{0}$ for $t > T$.

This completes the proof of the Theorem. ■

Hence when the infection matrix is reducible convergence of the proportion of infectives to the appropriate equilibrium is established.

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