Int. J. Appl. Math. and Comp. Sci., 2000, Vol.10, No.2, 309–320

EXISTENCE AND COMPUTATION OF THE SET OF POSITIVE SOLUTIONS TO POLYNOMIAL MATRIX EQUATIONS

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Necessary and sufficient conditions are established for the existence of positive solutions to polynomial diophatine equations. A method of computing of the set of positive solutions to a polynomial diophatine equation based on extreme points and extreme directions is proposed. The effectiveness of the method is demonstrated on a numerical example.

Keywords: computation, set of positive solutions, polynomial matrix equation

1. Introduction

Polynomial matrix equations have been considered in many papers and books (Emre and Silverman, 1981; Feinstein and Barness, 1984; Kaczorek, 1986a; 1986b; 1987; 1992; Kučera, 1972; 1979; 1994; Kučera and Zagalak, 1999; Qianhua and Zhongjun, 1987; Solak, 1985; Šebek, 1980; 1983; 1989; Šebek and Kučera, 1981; Wolovich, 1987). Recently the positive systems theory has become a field of great interest and research (Kaczorek, 1997; Maeda and Kodama, 1981; Maeda *et al.*, 1977; Ohta *et al.*, 1984; van den Hof, 1997). Some automatic-control problems can be reduced to finding positive polynomial matrix solutions to suitable polynomial matrix equations (Kaczorek, 1992, Kučera, 1979). In (Kaczorek, 1998) necessary and sufficient conditions for the existence of positive solutions to polynomial matrix equations have been established and two methods for computation of the positive solutions have been proposed. The main subject of this paper is to present a method of computing of the set of positive solutions to the polynomial diophatine equation.

2. Preliminaries and Problem Statement

Let $\mathbb{R}^{q\times p}$ be the set of $q\times p$ real matrices and $\mathbb{R}^q := \mathbb{R}^{q\times 1}$. The set of $q\times p$ polynomial real matrices in the variable s will be denoted by $\mathbb{R}^{q\times p}[s]$ and $\mathbb{R}^q[s] := \mathbb{R}^{q\times 1}[s]$.

Consider a polynomial matrix of the form

$$A(s) = A_n s^s + A_{n-1}s^{n-1} + \cdots + A_1 s + A_0 \in \mathbb{R}^{p\times q}[s]. \tag{1}$$

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If A_n is a non-zero matrix, then a non-negative integer n is called the degree of $A(s)$ and it will be denoted by $\deg A(s)$. The polynomial matrix (1) is called regular if $p = q$ and $\det A_0 \neq 0$. Let $\mathbb{R}_+^{q\times p}$ be the set of $q \times p$ real matrices with non-negative entries.

Definition 1. The polynomial matrix (1) is called positive if $A_i \in \mathbb{R}_+^{q\times p}$ for $i = 0, 1, \dots, n$.

Consider the well-known diophantine equation

$$A(s)X(s) + B(s)Y(s) = C(s), \tag{2}$$

where

$$A(s) \in \mathbb{R}^{k\times t}[s], \quad B(s) \in \mathbb{R}^{k\times v}[s], \quad C(s) \in \mathbb{R}^{k\times q}[s]$$

are given and

$$X(s) \in \mathbb{R}^{t\times q}[s], \quad Y(s) \in \mathbb{R}^{v\times q}[s]$$

are unknown. Equation (2) can be written as

$$D(s)Z(s) = C(s), \tag{3}$$

where

$$D(s) := \big[A(s), B(s)\big] \in \mathbb{R}^{k\times p}[s], \quad Z(s) := \begin{bmatrix} X(s) \\ Y(s) \end{bmatrix} \in \mathbb{R}^{p\times q}[s], \quad p = t + v.$$

A pair of positive polynomial matrices $X(s)$ and $Y(s)$ (resp. $Z(s)$) satisfying (2) (resp. (3)) is called a positive solution to (2) (resp. (3)). The problem under consideration can be stated as follows: Given polynomial matrices $A(s)$, $B(s)$ and $C(s)$, establish conditions under which there exists a positive solution $X(s)$, $Y(s)$ to (2) and give a procedure for computation of the set of positive solutions to (2) (if, of course, this set is not empty).

3. Existence of a Positive Solution

Let

$$D(s) = D_n s^n + D_{n-1}s^{n-1} + \cdots + D_1 s + D_0 \in \mathbb{R}^{k\times p}[s] \tag{4a}$$

and

$$C(s) = C_m s^m + C_{m-1}s^{m-1} + \cdots + C_1 s + C_0 \in \mathbb{R}^{k\times q}[s]. \tag{4b}$$

From (3) it follows that the minimal degree of $Z(s)$ is equal to $n - m = r$. Substituting (4) and

$$Z(s) = Z_r s^r + Z_{r-1}s^{r-1} + \cdots + Z_1 s + Z_0 \in \mathbb{R}^{p\times q}[s] \tag{5}$$

into (3) and comparing the coefficients at the same powers of s, we obtain for $r+n \geq m$ and $n > r$

$$\begin{aligned} D_0 Z_0 &= C_0, \\ D_1 Z_0 + D_0 Z_1 &= C_1, \\ &\vdots \\ D_r Z_0 + D_{r-1} Z_1 + \cdots + D_0 Z_r &= C_r, \\ &\vdots \\ D_n Z_0 + D_{n-1} Z_1 + \cdots + D_{n-r} Z_r &= C_n, \\ &\vdots \\ D_n Z_{m-n} + D_{n-1} Z_{m-n+1} + \cdots + D_{m-r} Z_r &= C_m. \end{aligned} \tag{6}$$

Equations (6) for $r+n=m$ can be written as

$$\bar{D}\bar{Z} = \bar{C}, \tag{7}$$

where

$$\bar{D} := \begin{bmatrix} D_0 & 0 & 0 & \cdots & 0 & 0 \\ D_1 & D_0 & 0 & \cdots & 0 & 0 \\ D_2 & D_1 & D_0 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_r & D_{r-1} & D_{r-2} & \cdots & D_1 & D_0 \\ D_{r+1} & D_r & D_{r-1} & \cdots & D_2 & D_1 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ D_n & D_{n-1} & D_{n-2} & \cdots & D_{n-r-1} & D_{n-r} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 0 & D_n \end{bmatrix} \in \mathbb{R}^{w \times t},$$

$$\bar{Z} := \begin{bmatrix} Z_0 \\ Z_1 \\ Z_2 \\ \vdots \\ Z_r \end{bmatrix} \in \mathbb{R}^{t\times q}, \qquad \bar{C} := \begin{bmatrix} C_0 \\ C_1 \\ \vdots \\ C_r \\ \vdots \\ C_n \\ \vdots \\ C_m \end{bmatrix} \in \mathbb{R}^{w\times q},$$

$$w := (m+1)k, \qquad t := (r+1)p.$$

The problem of finding a positive polynomial solution $Z(s) \in \mathbb{R}_+^{p\times q}$ to (3) has been reduced to finding a suitable positive real matrix $\bar{Z} \in \mathbb{R}_+^{t\times q}$ of (7) for given real matrices $\bar{D}$ and $\bar{C}$.

A vector $b \in \mathbb{R}^n$ is called a positive linear combination of vectors $a_i \in \mathbb{R}^n$, $i = 1,\dots,k$ if there exist non-zero scalars $\eta_i \geq 0$ such that $b = \sum_{i=1}^{k} \eta_i a_i$.

Definition 2. (Cohen and Rothblum, 1993) The smallest non-negative integer t is called the nonnegative column rank of $A \in \mathbb{R}^{p\times q}$ (denoted by $\operatorname{rank}_+ A$) if there exist t columns in A such that each column of A is a positive linear combination of columns of A.

Theorem 1. *(Cohen and Rothblum, 1993) Let $A \in \mathbb{R}^{m\times n}$ be a non-negative matrix. Then $\operatorname{rank} A \leq \operatorname{rank}_+ A \leq \min(m,n)$.*

Let $\operatorname{cone} A$ be the cone generated by the columns of the matrix A, i.e. the set of all positive linear combinations of the columns of A.

Theorem 2. *The polynomial equation (3) (or equivalently (2)) has a positive solution $Z(s)$ if and only if one of the following conditions is satisfied:*

1. $\operatorname{rank}_+ \lfloor \bar{D}, \bar{C} \rfloor = \operatorname{rank}_+ \bar{D}$, *or*
2. $\bar{c}_i \in \operatorname{cone} \bar{D}$ *for* $i = 1,\dots,q$, *where* $\bar{c}_i$ *is the* i*-th column of* $\bar{C}$.

Proof. From Definition 2 it follows that eqn. (7) has a positive solution $\bar{Z} \in \mathbb{R}_+^{t\times q}$ if and only if every column of $\bar{C}$ is a positive linear combination of the columns of $\bar{D}$ and this holds if and only if the first condition is satisfied. Note that the second condition is equivalent to the fact that each column of $\bar{C}$ is a positive linear combination of the columns of $\bar{D}$. ∎

Theorem 3. *(Rudin, 1998) If M is a closed subspace of H, then there exists only one pair of transformations P and Q such that P transforms the space H into the* *subspace M, Q transforms the space H into the subspace $M^{\perp}$ (which is orthogonal to the subspace M), and*

$$x = Px + Qx$$

for any $x \in M$. Moreover, the transformations P and Q have the following properties:

1. *If $x \in M$, then $Px = x$, $Qx = 0$; if $x \in M^{\perp}$ then $Px = 0$, $Qx = x$.*
2. $||x - Px|| = \inf\{||x - y||: \ y \in M, \ x \in H\}$.
3. $||x||^2 = ||Px||^2 + ||Qx||^2$.
4. *P and Q are linear transformations.*

Theorem 4. *(Rudin, 1998) If $\{u_1, \ldots, u_k\}$ is an orthonormal set in H and $x \in H$, then for arbitrary scalars $\lambda_1, \ldots, \lambda_k$ we have*

$$||x - \sum_{j=1}^{k}(x, u_j)u_j|| \leq ||x - \sum_{j=1}^{k} \lambda_j u_j||$$

and the equality occurs if and only if $\lambda_j = (x, u_j)$ for $j = 1, \ldots, k$, where $(\cdot, \cdot)$ denotes the inner product. The vector

$$\sum_{j=1}^{k}(x, u_j)u_j$$

is the orthogonal projection of the vector x onto the subspace $[u_1, \ldots, u_k]$. If δ represents the distance between x and $[u_1, \ldots, u_k]$, then

$$\sum_{j=1}^{k} |(x, u_j)|^2 = ||x||^2 - \delta^2.$$

Consider

$$\lfloor w_1 \ \ldots \ w_p \rfloor K = w_q,$$

where $\lfloor w_1 \ \ldots \ w_p \rfloor$ is the set of p columns taken from a matrix W, w_q is the q-th column of the matrix W, $q \notin \{1, \ldots, p\}$ and $K \in \mathbb{R}^p$. An orthonormal basis U_w of the matrix $\lfloor w_1 \ \ldots \ w_p \rfloor$ can be obtained via SVD (Singular Value Decomposition, cf. Golub and van Loan, 1989). By Theorem 4, if the condition $||U_w^T w_q|| = ||w_q||$ is satisfied, then w_q can be expressed as a linear combination of the columns $\lfloor w_1 \ \ldots \ w_p \rfloor$. The desired coefficient matrix is given by $K = [w_1 \ \ldots \ x w_p]^+ w_q$, where the upper index + denotes the pseudoinverse which can be calculated based on SVD.

From Theorems 1, 2 and 3 we have the following algorithm for the computation of the positive rank of W. The algorithm will be presented based on the notation used in MATLAB.

Algorithm:

Step 1. $\text{rank}_+ W := 1$; $i = 0$; assume matrix X to be empty $(X := [\,])$.

Step 2. If $i - 1 = \min(m, n)$, then the algorithm ends.

Step 3. $i := i + 1$.

Step 4. Augment X by the i-th column of the matrix $W(X := [X\ w_i])$.

Step 5. As w_q choose the $(i + 1)$-th column of W.

Step 6. Calculate SVD of $X([U, \Sigma, V] = \text{svd}(X);\ X = U\Sigma V^T)$.

Step 7. Extract U_w from $U(U_w := U(:, 1 : r),\ r = \text{rank}\, X)$.

Step 8. If $||U_w^T w_q|| \neq ||w_q||$ (w_q cannot be expressed as a linear combination of columns of X), then $\text{rank}_+ W := \text{rank}_+ W + 1$ and go to Step 2.

Step 9. Calculate $K = X^+ w_q$, $X^+ = V\Sigma^{-1}U^T$.

Step 10. If the components of the vector K are positive, then go to Step 2 (w_q can be expressed as a positive linear combination of the columns of X).

Step 11. Vector w_q cannot be expressed as a positive linear combination of the columns of X. Therefore $\text{rank}_+ W := \text{rank}_+ W + 1$. Go to Step 2.

4. Determination of the Set of Positive Solutions

For (2) we usually have $t \geq w$. It is assumed that the matrix $\bar{D}$ has full row rank w. Let $A \otimes B$ be the Kronecker product of the matrices $A = [a_{ij}] \in \mathbb{R}^{m \times n}$ and B defined by

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \dots & \dots & \dots & \dots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}. \tag{8}$$

Using the Kronecker product we may rewrite (7) in the form

$$Ax = b, \tag{9a}$$

where

$$A := \bar{D} \otimes I_q \in \mathbb{R}^{\bar{m}} \times \bar{n}, \qquad x := \left[\bar{z}_1, \bar{z}_2, \dots, \bar{z}_t\right]^T \in \mathbb{R}^{\bar{n}},$$
$$b := \left[\bar{c'}_1, \bar{c'}_2, \dots, \bar{c'}_w\right]^T \in \mathbb{R}^{\bar{m}}, \qquad \bar{m} := wq, \qquad \bar{n} := tq \tag{9b}$$

and $\bar{z}_i$ and $\bar{c'}_j$ are the i-th and j-th rows of $\bar{Z}$ and $\bar{C}$, respectively. Therefore the problem of determination of the set of positive solutions to (2) has been reduced to the problem: Given A and b, find a set of $x \geq 0$ satisfying (9a). Note that if $t \geq w$ and $\operatorname{rank} \bar{D} = w$, then the matrix A also has full row rank.

Let S be a non-empty set defined by $S := \{x : \ Ax = b, \ \ x \geq 0\}$ (Bazaraa *et al.*, 1993). A point x is an extreme point of S if and only if $A \in \mathbb{R}^{\bar{m} \times \bar{n}}$ can be decomposed into $[B, N]$ such that $\det B \neq 0$ and

$$x = \begin{bmatrix} B^{-1}b \\ 0 \end{bmatrix} \geq 0. \tag{10}$$

If $\operatorname{rank} A = \bar{m}$, then S has at least one extreme point. The number of extreme points is less than or equal to $\bar{n}!/\bar{m}!(\bar{n} - \bar{m})!$.

A vector d is an extreme direction of S if and only if A can be decomposed into $[B, N]$ such that $\det B \neq 0$ and

$$d = \begin{bmatrix} -B^{-1}a_j \\ e_j \end{bmatrix} \geq 0, \tag{11}$$

where a_j is the i-th column of N and e_j is an $\bar{n} - \bar{m}$ vector of zeros except for unity in position j. The set S has at least one extreme direction if and only if it is unbounded. The maximum number of extreme directions is bounded by $\bar{n}!/\bar{m}!(\bar{n} - \bar{m} - 1)!$.

Let $x_1, x_2, \ldots, x_k$ be the extreme points of S and $d_1, d_2, \ldots, d_l$ be the extreme directions of S. It is well-known (Bazaraa *et al.*, 1993) that every $x \in S$ can be written as

$$x = \sum_{j=1}^{k} \lambda_j x_j + \sum_{i=1}^{l} \mu_i d_i, \tag{12}$$

where $\lambda_j \geq 0, \ j = 1, \ldots, k, \ \sum_{j=1}^{k} \lambda_j = 1, \ \mu_i \geq 0, \ i = 1, \ldots, l$. From the above considerations we have the following procedure for computation of the set of positive solutions to (2).

Procedure:

Step 1. Given polynomial matrices $A(s)$, $B(s)$ and $C(s)$, find the coefficient matrices $\bar{D}$, $\bar{C}$ and next, using (9b), the matrix A and vector b.

Step 2. Decomposing the matrix A into $\lfloor B_j, N_j \rfloor$, $j = 1, \ldots, k$ such that $\det B_j \neq 0$, find the sequences $B_1, B_2, \ldots, B_k$ and $N_1, N_2, \ldots, N_k$.

Step 3. Find

$$x_j = \begin{bmatrix} B_j^{-1}b \\ 0 \end{bmatrix} \geq 0 \quad \text{for} \quad j = 1, \ldots, k \tag{13}$$

and

$$d_{ji} = \begin{bmatrix} -B_j^{-1} a_{ji} \\ e_j \end{bmatrix} \geq 0 \quad \text{for} \quad \begin{cases} j = 1, \ldots, k, \\ i = 1, \ldots, \bar{n} - \bar{m}, \end{cases} \tag{14}$$

where a_{ji} is the i-th column of N_j.

Step 4. Find the desired set

$$S = \left\{ x : \; x = \sum_{j=1}^{k} \lambda_j x_j + \sum_{i=1}^{l} \mu_i d_i, \; \lambda_j \geq 0, \; \sum_{j=1}^{k} \lambda_j = 1, \; \mu_i \geq 0 \right\}. \tag{15}$$

Example 1. Consider (2) with

$$A = \begin{bmatrix} s & 1 & 0 \\ 0 & s+1 & 1 \end{bmatrix}, \quad B = \begin{bmatrix} s \\ 1 \end{bmatrix}, \quad C = \begin{bmatrix} 2s^2 \\ 2s+2 \end{bmatrix}. \tag{16}$$

Using the foregoing procedure, we perform their consecutive steps.

Step 1. In this case, we have

$$D(s) = \big[A(s), B(s)\big] = \begin{bmatrix} s & 1 & 0 & s \\ 0 & s+1 & 1 & 1 \end{bmatrix} = D_1 s + D_0,$$

$$D_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad D_1 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$C(s) = \begin{bmatrix} 2s^2 \\ 2s+2 \end{bmatrix} = C_2 s^2 + C_1 s + C_0,$$

$$C_2 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}, \quad C_1 = C_0 = \begin{bmatrix} 0 \\ 2 \end{bmatrix},$$

$$A = \bar{D} = \begin{bmatrix} D_0 & 0 \\ D_1 & D_0 \\ 0 & D_1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{bmatrix},$$

$$b = \begin{bmatrix} C_0 \\ C_1 \\ C_2 \end{bmatrix} = \begin{bmatrix} 0 & 2 & 0 & 2 & 2 & 0 \end{bmatrix}^T.$$

Step 2. Let a_i be the i-th column of the matrix A. Decomposing A into $\lfloor B_j, N_j \rfloor$, $j = 1, \ldots, 28$, we obtain the result listed in Table 1.

Table 1. Decomposition of the matrix A in Step 2 of the Example.

j	B_j	$\det B_j$
1	$[a_1, a_2, a_3, a_4, a_5, a_6]$	0
2	$[a_1, a_2, a_3, a_4, a_5, a_7]$	0
3	$[a_1, a_2, a_3, a_4, a_5, a_8]$	0
4	$[a_1, a_2, a_3, a_4, a_6, a_7]$	0
5	$[a_1, a_2, a_3, a_4, a_6, a_8]$	0
6	$[a_1, a_2, a_3, a_4, a_7, a_8]$	0
7	$[a_1, a_2, a_3, a_5, a_6, a_7]$	$\neq 0$
8	$[a_1, a_2, a_3, a_5, a_6, a_8]$	$\neq 0$
9	$[a_1, a_2, a_3, a_5, a_7, a_8]$	0
10	$[a_1, a_2, a_3, a_6, a_7, a_8]$	$\neq 0$
11	$[a_1, a_2, a_4, a_5, a_6, a_7]$	0
12	$[a_1, a_2, a_4, a_5, a_6, a_8]$	0
13	$[a_1, a_2, a_4, a_5, a_7, a_8]$	0
14	$[a_1, a_2, a_4, a_6, a_7, a_8]$	0

j	B_j	$\det B_j$
15	$[a_1, a_2, a_5, a_6, a_7, a_8]$	0
16	$[a_1, a_3, a_4, a_5, a_6, a_7]$	0
17	$[a_1, a_3, a_4, a_5, a_6, a_8]$	0
18	$[a_1, a_3, a_4, a_5, a_7, a_8]$	0
19	$[a_1, a_3, a_4, a_6, a_7, a_8]$	0
20	$[a_1, a_3, a_5, a_6, a_7, a_8]$	0
21	$[a_1, a_4, a_5, a_6, a_7, a_8]$	0
22	$[a_2, a_3, a_4, a_5, a_6, a_7]$	$\neq 0$
23	$[a_2, a_3, a_4, a_5, a_6, a_8]$	$\neq 0$
24	$[a_2, a_3, a_4, a_5, a_7, a_8]$	0
25	$[a_2, a_3, a_4, a_6, a_7, a_8]$	$\neq 0$
26	$[a_2, a_3, a_5, a_6, a_7, a_8]$	0
27	$[a_2, a_4, a_5, a_6, a_7, a_8]$	0
28	$[a_3, a_4, a_5, a_6, a_7, a_8]$	0

Step 3. Using (13), (14) and Table 1, we obtain

$$\begin{aligned}
&[a_1, a_2, a_3, a_5, a_6, a_7]^{-1}[\,0\ 2\ 0\ 2\ 2\ 0\,]^T \Longrightarrow x^1 = [\,0\ 0\ 2\ 0\ 2\ 0\ 2\ 0\,]^T,\\
&[a_1, a_2, a_3, a_5, a_6, a_8]^{-1}[\,0\ 2\ 0\ 2\ 2\ 0\,]^T \Longrightarrow x^2 = [\,0\ 0\ 2\ 0\ 0\ 0\ 0\ 2\,]^T,\\
&[a_1, a_2, a_3, a_6, a_7, a_8]^{-1}[\,0\ 2\ 0\ 2\ 2\ 0\,]^T \Longrightarrow x^3 = [\,0\ 0\ 2\ 0\ 0\ 0\ 0\ 2\,]^T,\\
&[a_2, a_3, a_4, a_5, a_6, a_7]^{-1}[\,0\ 2\ 0\ 2\ 2\ 0\,]^T \Longrightarrow x^4 = [\,0\ 0\ 2\ 0\ 2\ 0\ 2\ 0\,]^T,\\
&[a_2, a_3, a_4, a_5, a_6, a_8]^{-1}[\,0\ 2\ 0\ 2\ 2\ 0\,]^T \Longrightarrow x^5 = [\,0\ 0\ 2\ 0\ 0\ 0\ 0\ 2\,]^T,\\
&[a_2, a_3, a_4, a_6, a_7, a_8]^{-1}[\,0\ 2\ 0\ 2\ 2\ 0\,]^T \Longrightarrow x^6 = [\,0\ 0\ 2\ 0\ 0\ 0\ 0\ 2\,]^T.
\end{aligned} \tag{17}$$

Note that in this case the set of extreme directions is empty, $x^1 = x^4$, $x^2 = x^3 = x^5 = x^6$ and the set S is bounded.

Step 4. In this case the desired set (15) has the form

$$S = \left\{ x: \quad x = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} \lambda + \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} (1-\lambda) = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 2\lambda \\ 0 \\ 2\lambda \\ 2(1-\lambda) \end{bmatrix}, \quad \lambda \in [0,1] \right\}. \tag{18}$$

Hence

$$Z(s) = \begin{bmatrix} 2\lambda s \\ 0 \\ 2\lambda s + 2 \\ 2(1-\lambda)s \end{bmatrix}$$

and

$$X(s) = \begin{bmatrix} 2\lambda s \\ 0 \\ 2\lambda s + 2 \end{bmatrix}, \quad Y(s) = 2(1-\lambda)s \quad \text{for} \quad 0 \le \lambda \le 1.$$

5. Conclusions

Necessary and sufficient conditions for the existence of positive solutions to polynomial matrix equations have been established. An algorithm for computation of the positive rank of a given real matrix has been presented. A method of computation of the set of positive solutions to the polynomial diophatine equation (2) based on extreme points and directions has been proposed.

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Received: 5 January 2000