

## DISCRETE-TIME PREDICTIVE CONTROL WITH OVERPARAMETERIZED DELAY-PLANT MODELS AND AN IDENTIFIED CANCELLATION ORDER

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The paper presents several solutions to the discrete-time generalized predictive (GPC) controller problem, including an anticipative filtration mechanism, which are suitable for plants with nonzero transportation delays. Necessary modifications of the GPC design procedure required for controlling plants based on their non-minimal models are discussed in detail. Although inevitably invoking the troublesome pole-zero cancellation problem, such models can be used in adaptive systems as a remedy for the uncertainty or variability of the model order. The purpose of this paper is to present a complete GPC controller design for delay plants that is robust to the overparameterization of the plant model. Refined conditions for the existence and stability of GPC control solutions in terms of pertinent design parameters are given, and explicit forms of closed-loop characteristic polynomials are provided. The issue of identifying the model cancellation order is also considered, and practical solutions are proposed. The presented ideas are illustrated numerically.

**Keywords:** delay plants, overparameterization, system design, predictive control, robust control synthesis, adaptive control

### 1. Introduction

Control system design, in both adaptive and non-adaptive variants, can be effectively based on a long-range model-based predictive control (MBPC) approach. The control strategies applied generally rely on an ‘emulator’ paradigm that consists in emulating the operations of signal prediction, which are physically unrealizable, with the aid of non-parametric or parametric models (Gawthrop, 1987; Gawthrop *et al.*, 1996). The original MBPC strategies are principally founded on the input-output models of controlled plants. Among them, the GPC algorithm developed in (Clarke and Mohtadi, 1989; Clarke *et al.*, 1987) is one of the most highly approved MBPC techniques (Clarke, 1988; Grimbale, 1992; Landau *et al.*, 1998). This algorithm (in its simplest form) also appears to be a promising substitute for the ubiquitous PID controller. This is mainly due to its ability to deal with various types of difficult objects, such as unstable and non-minimum phase systems, dynamic plants of unknown orders, and either unknown or variable transportation delays (Clarke, 1988; Grimbale, 1992). What is more, the GPC strategy is easy to implement and has great flexibility, which results from several tuning knobs tailoring the algorithm to specific application needs.

Some generic guidelines for selecting the design tuning knobs/parameters of the basic GPC algorithm (namely, discrete-time horizon parameters:  $N_1$ ,  $N_2$  and

$N_u$ ) are given in (Clarke, 1988; Clarke and Mohtadi, 1989; Clarke *et al.*, 1987; Mohtadi and Clarke, 1986). A comprehensive presentation of the results on tuning GPC controllers can be found in Appendix A, where the stability issue is also reviewed.

Common factors may appear in plant-model polynomials for several reasons connected with fundamental parameterization and identification issues (Söderström, 1974; Söderström and Stoica, 1989). Such cancellations may, for instance, occur in adaptive systems using an overestimated plant-model order while attempting to model the dynamic behavior of an unknown plant (Arent *et al.*, 1998). It is a known fact that many standard pole-assignment techniques for designing digital controllers (Landau *et al.*, 1998; Wellstead and Zarrop, 1991; Ogata, 1995) have the disadvantage consisting in the fact that if there is a (near or exact) pole-zero cancellation in the plant model, the system of algebraic equations solved within a control synthesis procedure becomes singular. This may result in large-valued controller coefficients and/or in a degraded closed-loop performance. A method of designing pole-placement controllers with an improved noise performance (over the standard designs), in cases when the plant model includes such an exact or near pole-zero cancellation, is presented in (Halpern, 1988). As demonstrated by (Lozano-Leal and Goodwin, 1985; Kreisselmeier, 1986; Ossman and Kamen, 1987; Lozano-Leal, 1989; Lozano-Leal and Zhao, 1994), even for simple cases ‘safe’ regions

in the parameter space which exclude pole-zero cancellations have very irregular geometry. A simplified methodology for avoiding such non-minimal models in adaptive control was considered by (van der Kooji and Polderman, 1993; Arent *et al.*, 1995; Arent *et al.*, 1998), where truncated (approximate) all-pole representations of the plant are employed to overcome the pole-zero cancellation problem.

The question of how such overparameterized plant models with common factors affect predictive control algorithms was considered by Rossiter *et al.* (1997). It was shown that, in general, in the presence of common factors it is not possible to satisfy finite-horizon stable-predictive-control terminal constraints. This work concentrated on two predictive algorithms guaranteeing stability (see also Appendix A): SIORHC (Mosca and Zhang, 1992) and SGPC (Kouvaritakis *et al.*, 1992). They showed that any common factor makes the matrices of the respective linear design equations rank deficient. In order to preserve design solvability, two approaches based on the concept of pseudoinverse were proposed with the assumption that the common factor is either known or unknown.

This paper concerns the standard GPC design strategy equipped with a simple mechanism of the so-called anticipative filtering (AF) of the control error (Kowalczyk *et al.*, 1996; Kowalczyk and Suchomski, 2002), which allows abating the command signal fictitiously. With such an easy-to-implement mechanism we can minimize the control effort associated with the GPC and facilitate  $\lambda$ -tuning procedure. As was shown in (Kowalczyk *et al.*, 1996; Kowalczyk and Suchomski, 2002), with AF-GPC designed for non-delay systems, under certain conditions, the solution of the GPC design always exists and the design yields stable control systems. It was also confirmed there that some bounds on preliminary GPC design parameters have to be taken into account. The effectiveness of this generic approach is related to the possibility of using a design procedure for determining the control horizons, the order of plant cancellation, and the controller gain of the GPC algorithm.

The principal aim of this paper is to present a discrete-time GPC controller design suitable for plants with a non-zero transportation delay. Some prior results on the conditions of solvability of the GPC synthesis problem and on the corresponding closed-loop transfer function descriptions were shown in (Kowalczyk and Suchomski, 1997). A consequent idea is to propose a necessary modification of the GPC control design procedure that is required for controlling plants based on their non-minimal models.

As was mentioned above, overparameterization, which can be utilized in adaptive systems as a remedy for the uncertainty or variability of the model order, inevitably results in pole-zero cancellations, which are trou-

blesome in control system designing. Moreover, there is a modeling-and-identification reality of the existing approximate systemic and/or numerical cancellations, which can be easily portrayed numerically by ‘almost’ non-minimal plant models, which are also difficult for GPC control design procedures. Furthermore, the elaborated rules of tuning design parameters can be used as a starting point for adaptive versions of the analyzed unconstrained GPC methodology. Thus the ultimate purpose of this paper is to present the AF-GPC controller for plants with a non-zero transportation delay that is robust to the overparameterization of the plant model.

In particular, new refined conditions for the existence and stability of the solution of the GPC design problem in terms of pertinent design parameters will be given, and explicit forms of closed-loop characteristic polynomials will be supplied. The examination of the space of principal GPC design parameters (i.e., the *observation*, or *prediction horizons* and the *control horizon*) from the GPC solvability viewpoint will allow us to distinguish four regions *admitted* for GPC solutions and one *denied* region. Simple analytical rules will then be offered that, for a given model of the plant, result in a set of controllers, which are conveniently parameterized with respect to these design ‘knobs’ and lead to the characteristic polynomials of a required low degree. The polynomials of the zero, first, and second degrees are of a practical interest to the designer due to the ease of assessing stable GPC systems.

With uncertainty or a lack of knowledge about the cancellation order, it is generally impossible to obtain the characteristic polynomial of a desired degree. This brings about fundamental difficulties in guaranteeing the stability of GPC closed-loop systems. It is also clear that our parameterization can be effectively utilized provided that the cancellation order is known.

Therefore, two approaches to the detecting of the detection order will be considered. One method can be regarded as a variant of the standard SVD-based algorithm for estimating the numerical rank of an appropriately defined matrix. The effectiveness of this method (and others of this type) highly depends on a ‘decision threshold’ that can scarcely be chosen effortlessly. Therefore, in the present paper we offer a new numerically robust algorithm for the estimation of the cancellation order based on a measure of the distance between subsequent solutions of a properly defined set of Diophantine equations. With this, it is important that the obtained knowledge about the true system order not only facilitates the ‘principal’ GPC design but also duly conditions the procedure of  $\lambda$ -tuning.

In brief, the content of this paper is the following: The AF-GPC principles are presented in Section 2 in the form of a short lecture on fundamentals necessary for understanding the principal material of the paper. Several design aspects are thoroughly considered

in Section 3, where we present some refined and precisely proved fundamental statements concerning the issue of solvability (in terms of GPC parameter solvability regions), only partially discussed in our previous reports (Kowalczyk and Suchomski, 2002; Suchomski and Kowalczyk, 2002a), and propose certain new conditions for characteristic closed-loop polynomials of a definite low degree. Another novelty of this paper is the treatment of the overparameterization issue given in Section 4. We propose there a calculus of the cancellation order and next, having identified this model parameter, we concentrate on the design of GPC control systems for non-minimal models. This allows us to present a modification of GPC tuning rules suitable for the case of overparameterization and pole-zero cancellation. The presented analytical deliberations are illustrated with two numerical examples in Section 5.

## 2. AF-GPC Fundamentals

Let a discrete-time plant be represented by a polynomial model of the CARIMA type:

$$A(q^{-1})y(t) = B(q^{-1})u(t) + \Delta^{-1}C(q^{-1})v(t), \quad (1)$$

where  $\{u(t)\}$ ,  $\{y(t)\}$ , and  $\{v(t)\}$  are the input, output, and zero-mean white-noise signals, respectively, while  $q^{-1}$  is the backward shift operator and  $\Delta = 1 - q^{-1}$  portrays the (non-scaled) two-point difference. The polynomials are defined as

$$A(q^{-1}) = \sum_{i=0}^{N_A} a_i q^{-i}, \quad a_0 = 1, \quad (2)$$

$$B(q^{-1}) = \sum_{i=n_B}^{N_B} b_i q^{-i} = q^{-n_B} B^0(q^{-1}), \quad 1 \leq n_B \leq N_B, \quad (3)$$

$$C(q^{-1}) = \sum_{i=0}^{N_C} c_i q^{-i}, \quad c_0 = 1, \quad 0 \leq N_C \leq N_A + 1, \quad (4)$$

where  $n_b = n_B - 1$  denotes the transportation delay of the system considered. In the plant model (1) an incremental-control channel and a disturbance channel can readily be distinguished as

$$y(t) = \frac{\bar{B}(q^{-1})}{\hat{A}(q^{-1})} \Delta u(t-1) + \frac{C(q^{-1})}{\hat{A}(q^{-1})} v(t), \quad (5)$$

where  $\hat{A}(q^{-1}) = \Delta A(q^{-1}) = \sum_{i=0}^{N_A+1} \hat{a}_i q^{-i}$ ,  $\hat{a}_0 = 1$ , and  $\bar{B}(q^{-1}) = qB(q^{-1})$ . In our development an additional restriction is assumed, namely,  $n_B \leq N_B \leq$

$N_A + 1$ , assuring that the dimension of the minimal state-space representation associated with the incremental-control channel  $B(q^{-1})/\hat{A}(q^{-1})$  is equal to  $N_A + 1$  (a detailed discussion can be found in (Clarke and Mohtadi, 1989) and Appendix B).

### 2.1. Diophantine Basis

Let  $(\hat{A}(q^{-1}), B(q^{-1}))$  be coprime. The Diophantine basis defined for  $i \geq 1$  consists of the following two pairs of coupled Diophantine equations:

$$D1: \quad \hat{A}(q^{-1})E_i(q^{-1}) + q^{-i}F_i(q^{-1}) = C(q^{-1}), \quad (6)$$

$$D2: \quad C(q^{-1})H_i(q^{-1}) + q^{-i}G_i(q^{-1}) = \bar{B}(q^{-1})E_i(q^{-1}), \quad (7)$$

$$D3: \quad \hat{A}(q^{-1})H_i(q^{-1}) + q^{-i}L_i(q^{-1}) = \bar{B}(q^{-1}), \quad (8)$$

$$D4: \quad \hat{A}(q^{-1})G_i(q^{-1}) + \bar{B}(q^{-1})F_i(q^{-1}) = C(q^{-1})L_i(q^{-1}). \quad (9)$$

The coefficients of the quotient polynomial  $H_i(q^{-1}) = \sum_{j=0}^{i-1} h_j q^{-j}$  are Markov parameters of the control channel  $\bar{B}(q^{-1})/\hat{A}(q^{-1}) = \sum_{i=0}^{\infty} h_i q^{-i}$ . Note that here  $h_{n_b} = b_{n_B}$  is the first non-zero Markov parameter. Similarly, the coefficients of the second quotient polynomial  $E_i(q^{-1}) = \sum_{j=0}^{i-1} e_j q^{-j}$  with  $e_0 = 1$  are Markov parameters of the disturbance channel  $C(q^{-1})/\hat{A}(q^{-1}) = \sum_{j=0}^{\infty} e_j q^{-j}$ . The residues of the Diophantine basis take the following forms:  $F_i(q^{-1}) = \sum_{j=0}^{N_A} f_{i,j} q^{-j}$ ,  $G_i(q^{-1}) = \sum_{j=0}^{N_G} g_{i,j} q^{-j}$ , and  $L_i(q^{-1}) = \sum_{j=0}^{N_A} l_{i,j} q^{-j}$ , where  $N_G = \max\{N_B - 2, N_C - 1\}$ . With  $(N_B \leq 1, N_C = 0)$ , we assume that  $G_i(q^{-1}) = 0$  for every  $i$ . Likewise, the zeroing of  $G_i(q^{-1})$  appears when  $\bar{B}(q^{-1}) = C(q^{-1})$ . Moreover, in the sequel, only the non-trivial case of  $N_G \geq 0$  will be considered.

It is a simple matter to check that in the case of a non-zero transportation delay  $n_b > 0$  and for  $i \geq n_b + 1$  we have  $H_i(q^{-1}) = q^{-n_b} H_i^0(q^{-1})$  with  $H_i^0(q^{-1}) = \sum_{j=n_b}^{i-1} h_j q^{-(j-n_b)}$ , while  $H_i(q^{-1}) = 0$  and  $H_i^0(q^{-1}) = 0$  if  $1 \leq i \leq n_b$ . This then results in the following residual polynomials for  $1 \leq i \leq n_b$ :  $G_i(q^{-1}) = q^i \bar{B}(q^{-1})E_i(q^{-1})$  and  $L_i(q^{-1}) = q^i \bar{B}(q^{-1})$ .

### 2.2. Design with Anticipative Filtering

The minimum-variance  $i$ -step ahead predictor of the plant output has the following form derived from the model (1)

for  $i \geq 1$ :

$$\hat{y}(t+i) = H_i(q^{-1})\Delta u(t+i-1) + \hat{y}(t+i|t), \quad (10)$$

where  $\hat{y}(t+i|t)$  denotes the free component that can be resolved recursively from (Gorez *et al.*, 1987; Clarke and Mohtadi, 1989):

$$C(q^{-1})\hat{y}(t+i|t) = F_i(q^{-1})y(t) + G_i(q^{-1})\Delta u(t-1). \quad (11)$$

A vector

$$\Delta \mathbf{u}(t) = [\Delta u(t) \ \cdots \ \Delta u(t+N_u-1)]^T,$$

where  $N_u$  denotes the control horizon, is sought after by the minimization of the following quadratic cost function:

$$\begin{aligned} \hat{J}(\Delta \mathbf{u}(t)) = & \sum_{i=N_1}^{N_2} (\hat{e}(t+i) - \delta \hat{y}(t+i))^2 \\ & + \lambda \sum_{i=1}^{N_u} (\Delta u(t+i-1))^2, \quad (12) \end{aligned}$$

where  $N_1$  and  $N_2$  are bottom and top prediction horizon parameters, respectively. The function  $\hat{e}(t+i) = \hat{e}(t+i|t) = r_i e(t)$  represents a filtered error sequence defined for a current control error  $e(t) = w(t) - y(t)$ , and  $[N_1, N_2]$ , with  $N_2 \geq N_1 \geq 1$ , determines an observation interval. For the constant future set points  $w(t+i) = w(t)$ , the quantities  $r_i$  are the coefficients of the step response of the anticipation filter (AF), and  $\delta \hat{y}(t+i) = \hat{y}(t+i) - y(t)$  describes the trajectory of the incrementally predicted plant output (Kowalczyk *et al.*, 1996). A similar concept of such a ‘reference trajectory’ was also considered by Camacho and Bordons (1999) and Maciejowski (2002).

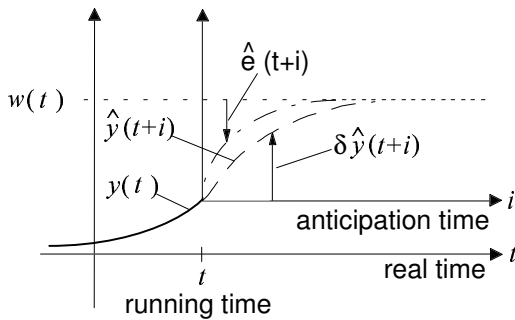


Fig. 1. Anticipative perspective on the plant output signal.

As is shown in Fig. 1, with the anticipative approach, the generalized-predictive control signal ought to drive the predicted output  $\delta \hat{y}(t+i)$  to the reference signal  $\hat{e}(t+i)$ , taking into account the cost of control within the time interval  $[1, N_u]$ . In practice, at each step, only the first element of a currently computed optimal control sequence

$\Delta \mathbf{u}^*(t)$  minimizing the index (12) is applied to the controlled plant input

$$\Delta \mathbf{u}^*(t) = \mathbf{k}^T (\hat{e}(t|t) - \delta \hat{y}(t|t)), \quad (13)$$

where, for an effective observation horizon  $N_0 = N_2 - N_1 + 1$ , the vectors  $\hat{e}(t|t) \in \mathbb{R}^{N_0}$  and  $\delta \hat{y}(t|t) \in \mathbb{R}^{N_0}$  are respectively defined as  $\hat{e}(t|t) = (w(t) - y(t)) [r_{N_1} \ \cdots \ r_{N_2}]^T$  and  $\delta \hat{y}(t|t) = [\hat{y}(t+N_1|t) - y(t) \ \cdots \ \hat{y}(t+N_2|t) - y(t)]^T$ , while  $\mathbf{k}^T = [k_1 \ \cdots \ k_{N_0}]$ ,  $\mathbf{k} \in \mathbb{R}^{N_0}$ , is the first row of the gain matrix  $\mathbf{K} \in \mathbb{R}^{N_u \times N_0}$ :

$$\mathbf{K} = \left[ (\mathbf{H}_{N_1, N_2}^{N_u})^T \mathbf{H}_{N_1, N_2}^{N_u} + \lambda \mathbf{I}_{N_u} \right]^{-1} (\mathbf{H}_{N_1, N_2}^{N_u})^T, \quad (14)$$

where the matrix  $\mathbf{H}_{N_1, N_2}^{N_u} \in \mathbb{R}^{N_0 \times N_u}$  of Markov parameters has the Toeplitz structure

$$\mathbf{H}_{N_1, N_2}^{N_u} = \begin{bmatrix} h_{N_1-1} & \cdots & h_{N_1-N_u} \\ \vdots & \vdots & \vdots \\ h_{N_2-1} & \cdots & h_{N_2-N_u} \end{bmatrix}, \quad (15)$$

with  $h_k = 0$  for  $k < 0$ .

Note that our synthetic approach using Toeplitz matrices is dual (or rather complementary) to the classical Kalman-Ho identification-based system theory methodology using Hankel matrices (see Appendix C).

Certain fundamental properties of the above matrix are presented in Appendix D.

### 2.3. Analysis of the Closed-Loop System

From (11) and (13) it follows that the AF-GPC algorithm can be written down in the form

$$\begin{aligned} C(q^{-1})\Delta \mathbf{u}^*(t) = & gC(q^{-1})(w(t) - y(t)) \\ & - G(q^{-1})\Delta \mathbf{u}^*(t) - \tilde{F}(q^{-1})y(t), \quad (16) \end{aligned}$$

where

$$g = \sum_{i=1}^{N_0} k_i r_{N_1+i-1} \quad (17)$$

and, furthermore,  $G(q^{-1}) = q^{-1} \sum_{i=1}^{N_0} k_i G_{N_1+i-1}(q^{-1})$ ,  $\tilde{F}(q^{-1}) = F(q^{-1}) - C(q^{-1}) \sum_{i=1}^{N_0} k_i$  in which we have  $F(q^{-1}) = \sum_{i=1}^{N_0} k_i F_{N_1+i-1}(q^{-1})$ . Then the resulting closed-loop AF-GPC control system takes its observer structure depicted in Fig. 2.

Taking into account that

$$\begin{aligned} \hat{A}(q^{-1})G(q^{-1}) + B(q^{-1})F(q^{-1}) \\ = q^{-1}C(q^{-1})L(q^{-1}), \quad (18) \end{aligned}$$

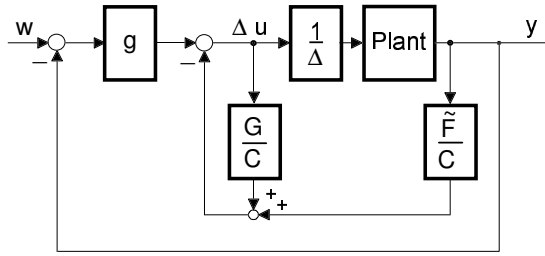


Fig. 2. Observer structure of the AF-GPC system.

where  $L(q^{-1}) = \sum_{i=1}^{N_0} k_i L_{N_1+i-1}(q^{-1})$ , we obtain the following characteristic polynomial of the closed-loop system:

$$D(q^{-1}) = D_0(q^{-1})C(q^{-1}), \quad (19)$$

with  $D_0(q^{-1}) = \tilde{D}(q^{-1}) + g^*B(q^{-1})$ ,  $\tilde{D}(q^{-1}) = \hat{A}(q^{-1}) + q^{-1}L(q^{-1})$ , and

$$g^* = g - \sum_{i=1}^{N_0} k_i = \sum_{i=1}^{N_0} k_i (r_{N_1+i-1} - 1). \quad (20)$$

As  $C(q^{-1})$  is assumed to be stable, the closed loop system is stable if  $D_0(q^{-1})$  is stable. A classical root-locus technique applied to  $\tilde{D}(q^{-1}) + g^*B(q^{-1})$  immediately shows (Kowalczyk *et al.*, 1996) that for a given  $B(q^{-1})$  and with a properly shaped  $\tilde{D}(q^{-1})$ , preferably of a low degree, there is an open interval for real  $g^*$  such that the resulting zeros of  $\tilde{D}(q^{-1}) + g^*B(q^{-1})$  lie in the unit open circle. Yet another observer structure of the AF-GPC system is shown in Fig. 3, where the feedforward ( $g$ ) and feedback ( $g^*$ ) gains are distinguished and an altered observer filter is applied to the plant output.

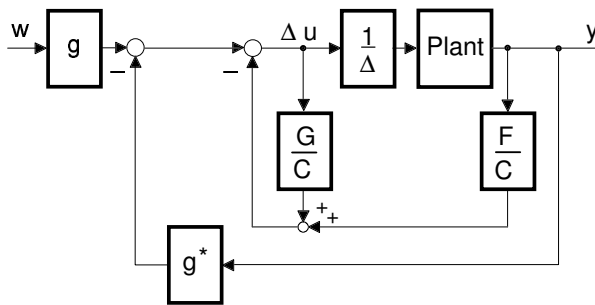


Fig. 3. Alternative observer structure of the AF-GPC system.

Note that switching off the AF filter ( $r_i = 1$ ,  $i = N_1, \dots, N_2$ ) makes  $g^*$  zero. Otherwise, the closed-loop transfer functions from the command  $w(t)$  to the plant output  $y(t)$  can be shown as

$$G_{wy}(q^{-1}) = \frac{gB(q^{-1})}{\tilde{D}(q^{-1}) + g^*B(q^{-1})}. \quad (21)$$

If there is no ‘differentiation’ in the incremental-control channel of the plant model (i.e.,  $B(1) \neq 0$ ), it

can be easily found out that the closed-loop system has a unit DC gain (i.e.,  $G_{wy}(1) = 1$ ). Note that in the case of coprime  $\hat{A}(q^{-1})$  and  $B(q^{-1})$ , the condition  $B(1) \neq 0$  must always be fulfilled. Moreover, from the first Diophantine equation (6) it follows that  $F_i(1) = C(1)$  for  $i \geq 1$ . This means that the derivative action appears in the output observer filter  $\tilde{F}(q^{-1})/C(q^{-1})$  shown in Fig. 2.

### 3. Solvability of the GPC Problem

We shall now consider zeroing the coefficient  $\lambda$  in the cost function (12) which implies that the control effort is not included in the optimal design procedure. Such a choice is principally motivated by the presumption of looking for a fully analytical design algorithm. In such a case the nominal control signal can still be moderately shaped by suitably selecting the gain coefficient (17), which is related to the analyzed design parameters (the time horizons and the anticipation filter parameters). This is one of the design tools which allows the designer to find a balance between the control effort and the speed of the controlled process.

With such a simplifying assumption ( $\lambda = 0$ ), a necessary and sufficient condition for the existence of  $\mathbf{K}$  can be entirely based on the full column rank of  $\mathbf{H}_{N_1, N_2}^{N_u}$ . Consequently, a useful characterization of solvability conditions in terms of preliminary GPC parameters and precise conditions for low-order closed-loop GPC systems are given in the following subsections.

#### 3.1. Basic Characteristics of Solvability

For  $N_2 \geq N_1 + N_u - 1$  the following two conditions assuring that  $\text{rank} \mathbf{H}_{N_1, N_2}^{N_u} = N_u$  were stated in (Peng and Hanus, 1991):

$$(N_1 = N_B, N_u \geq 1)$$

and

$$(N_1 > N_B, N_u = N_A + 1).$$

Alternatively, based on the assumption that  $N_B = N_A$ ,  $n_B = 1$  and  $N_2 \geq N_1 + N_u - 1$ , the following two (partial) conditions for the solvability of the GPC problem were derived in (Zhang, 1996):

$$(N_1 \geq 1, N_u = N_A + 1)$$

and

$$(N_1 = N_A = N_B, N_u \geq N_A + 2).$$

Our complete assertions related to the solvability problem are given in Lemma 1 and 2 (cf. Kowalczyk and Suchomski, 2001).

**Lemma 1.** (Basic solvability regions). *For coprime  $\hat{A}(q^{-1})$  and  $B(q^{-1})$ , the matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  has a full column rank ( $\text{rank} \mathbf{H}_{N_1, N_2}^{N_u} = N_u$ ) if one of the following two triple conditions is satisfied:*

$$\begin{aligned} (1') \quad N_u &= N_A + 1, & (1'') \quad N_u &> N_A + 1, \\ (2') \quad N_1 &\geq N_B, & (2'') \quad N_1 &= N_B, \\ (3') \quad N_2 &\geq N_1 + N_A, & (3'') \quad N_2 &\geq N_1 + N_u - 1. \end{aligned}$$

*Proof.* An elementary but tedious proof is given in Appendix E. ■

**Remark 1.** Note that in the above lemma in  $(1'')$   $N_u \geq N_A + 1$  is also admissible. Moreover, for the first set  $(1')$ – $(3')$  the other necessary condition  $(3'')$  is satisfied, which can be stated in terms of the effective observation horizon as  $N_0 \geq N_u$ .

**Lemma 2.** (Admitted and denied regions). *Let  $\hat{A}(q^{-1})$  and  $B(q^{-1})$  be coprime.*

(a) *The matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  has a full column rank ( $\text{rank} \mathbf{H}_{N_1, N_2}^{N_u} = N_u$ ) if one of the following four triple conditions is satisfied:*

$$\begin{aligned} (1^0) \quad N_u &\geq 1, & (1'') \quad N_u &\leq N_A + 1, \\ (2^0) \quad N_1 &\leq n_B, & (2'') \quad n_B &\leq N_1 \leq N_B \\ (3^0) \quad N_2 &\geq n_B + N_u - 1, & (3'') \quad N_2 &\geq N_A + N_B, \\ (1'_+) \quad N_u &\geq N_A + 1, & (1'_-) \quad N_u &\leq N_A + 1, \\ (2'_+) \quad n_B &\leq N_1 \leq N_B, & (2'_-) \quad N_1 &\geq N_B, \\ (3'_+) \quad N_2 &\geq N_B + N_u - 1, & (3'_-) \quad N_2 &\geq N_1 + N_A. \end{aligned}$$

(b) *The matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  has a degenerated (column) rank ( $\text{rank} \mathbf{H}_{N_1, N_2}^{N_u} = N_A + 1 < N_u$ ) if the following condition is fulfilled:*

$$\begin{aligned} (1'_+) \quad N_u &> N_A + 1, \\ (2'_+) \quad N_1 &> N_B, \\ (3'_+) \quad N_2 &\geq N_1 + N_u - 1 \\ &\text{(and if } N_2 < N_1 + N_u - 1\text{)}. \end{aligned}$$

*Proof.* **Claims  $(1^0)$ – $(3^0)$ :** For  $N_u \geq 1$ ,  $N_1 \leq n_B \leq N_B$  and  $N_2 \geq n_B + N_u - 1$  the matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  has a square submatrix  $\mathbf{H}_{n_B, n_B + N_u - 1}^{N_u}$  of the lower-triangular Toeplitz structure with a non-zero main diagonal composed of  $h_{n_B - 1} = h_{n_b} = b_{n_B}$ . This implies

$$\begin{aligned} \text{rank} \mathbf{H}_{N_1, N_2}^{N_u} \Big|_{\substack{N_u \geq 1 \\ N_1 \leq n_B \leq N_B \\ N_2 \geq n_B + N_u - 1}} & \\ &= \text{rank} \mathbf{H}_{n_B, n_B + N_u - 1}^{N_u} = N_u. \end{aligned}$$

**Claims  $(1'')$ – $(3'')$ :** Taking into account the claims  $(1')$ – $(3')$  of Lemma 1, we can observe that if  $N_u \leq N_A + 1$ ,  $N_1 = N_B$  and  $N_2 \geq N_A + N_B$ , the matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  is of a full column rank. Thus, for the selected horizons  $N_u$  and  $N_2$ , the matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  with an increased number of rows maintains its column rank

$$\text{rank} \mathbf{H}_{N_1, N_2}^{N_u} \Big|_{\substack{N_u \leq N_A + 1 \\ N_1 \leq N_B \\ N_2 \geq N_A + N_B}} = N_u.$$

Note that for  $N_1 \leq n_B < N_B$  the restriction on  $N_2$  established by the claim  $(3^0)$  is weaker than the one represented by the claim  $(3'')$ . In such a case, using the set  $(1^0)$ – $(3^0)$  is recommended.

**Claim  $(1'_+)$ – $(3'_+)$ :** The conditions  $(1'')$ – $(3'')$  of Lemma 1 can be expanded as

$$\begin{aligned} (1'') \quad N_u &\geq N_A + 1, \\ (2'') \quad N_1 &= N_B, \\ (3'') \quad N_2 &\geq N_B + N_u - 1. \end{aligned}$$

Hence, for  $N_u$  and  $N_2$  chosen such that  $N_u \geq N_A + 1$  and  $N_2 \geq N_B + N_u - 1$ , increasing the number of rows of the analysed matrix by diminishing  $N_1 \leq N_B$  does not affect the matrix rank

$$\text{rank} \mathbf{H}_{N_1, N_2}^{N_u} \Big|_{\substack{N_u \geq N_A + 1 \\ N_1 \leq N_B \\ N_2 \geq N_B + N_u - 1}} = N_u.$$

Note that the case  $N_1 \leq n_B \leq N_B$  has a wider interpretation in the claims  $(1^0)$ – $(3^0)$ . Therefore, if  $N_1 = n_B < N_B$ , this set of conditions is recommended.

**Claim  $(1'_-)$ – $(3'_-)$ :** From the claims  $(1')$ – $(3')$  of Lemma 1 it follows that with  $N_1 \geq N_B$  and  $N_2 \geq N_1 + N_A$  the matrix  $\mathbf{H}_{N_1, N_2}^{N_A + 1}$  has a full column rank. Hence, it can be deduced that the matrix having a decreased number of columns is also of a full column rank:

$$\text{rank} \mathbf{H}_{N_1, N_2}^{N_u} \Big|_{\substack{N_u \leq N_A + 1 \\ N_1 \geq N_B \\ N_2 \geq N_1 + N_A}} = N_u.$$

**Claim  $(1'_+)$ – $(3'_+)$ :** Let  $N_u \geq N_A + 2$ ,  $N_1 > N_B$  and  $N_2 \geq N_1 + N_u - 1$ . From (D.5) and (D.7) in Appendix D it follows that  $\mathbf{H}_{N_1, N_2}^{N_u} \hat{\mathbf{a}}_{N_u}^{N_A + 1} = \mathbf{0}_{N_0}$ . As  $\hat{\mathbf{a}}_{N_u}^{N_A + 1}$  is non-zero, it is evident that the columns of  $\mathbf{H}_{N_1, N_2}^{N_u}$  are not linearly independent. Clearly, the necessary condition for  $\mathbf{H}_{N_1, N_2}^{N_u}$  to be of a full column rank takes the form of the basic inequality  $N_2 \geq N_1 + N_u - 1$ . ■

The basic solvability regions have their plane (angular) interpretations in the three dimensional space of the design parameters  $(N_1, N_2, N_u)$  as shown in Fig. 4. This area can be used as an outset for determining other regions pertinent to the design of the GPC controller that are defined in Lemma 2 and illustrated in Figs. 5 through 7.

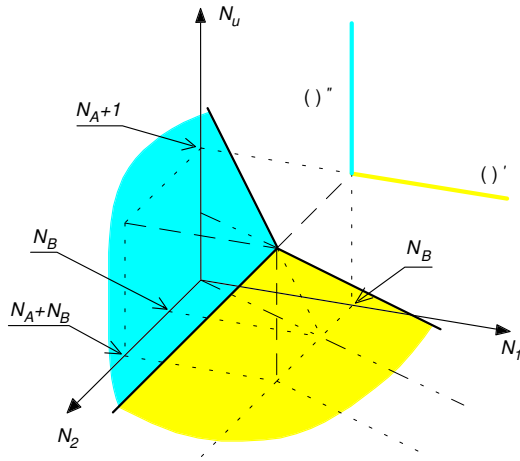


Fig. 4. Basic GPC solvability regions.

The restrictive condition (b) can be interpreted as a region that has a complementary projection with respect to orthogonal projections of all the regions of (a) on a quarter-plane  $(N_1, N_u)$ , as is shown in Fig. 7.

A useful reformulation of the solvability conditions is given in Lemmas 3 and 4 solely from the viewpoint of the prediction/observation horizon parameters  $N_1$  and  $N_2$ .

**Lemma 3.** (Solvability regions with respect to  $N_1$ ). *The GPC-design solvability area can be entered in the following ways:*

- $(\circ)^0$  if  $1 \leq N_1 \leq n_B$ , then  $N_u \geq 1$   
and  $N_2 \geq n_B + N_u - 1$ ;
- $(\circ)''$  with  $n_B \leq N_1 \leq N_B$ , if
  - $(\circ)''_-$   $1 \leq N_u \leq N_A + 1$ , then  $N_2 \geq N_B + N_A$  or
  - $(\circ)''_+$   $N_u \geq N_A + 1$ , then  $N_2 \geq N_B + N_u - 1$ ;

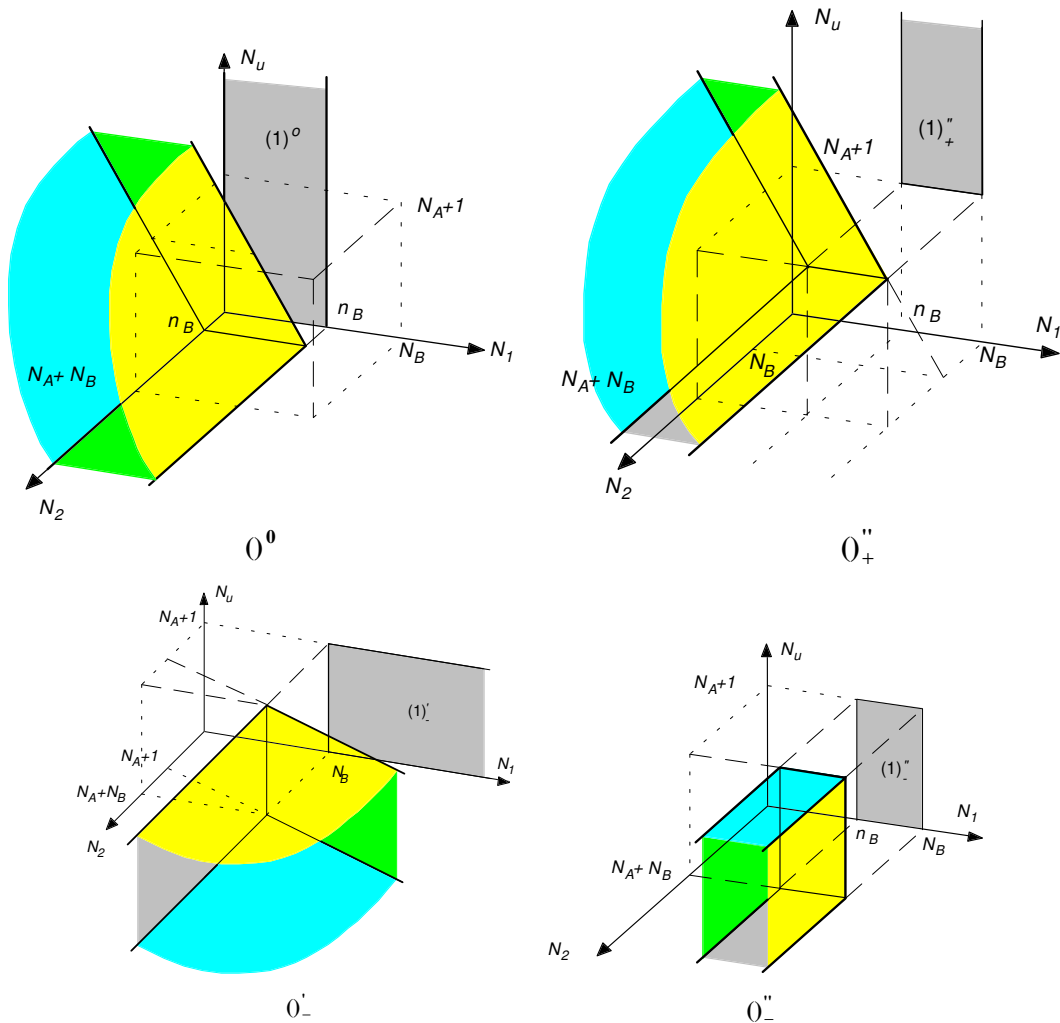


Fig. 5. Four regions admitted for GPC solutions.

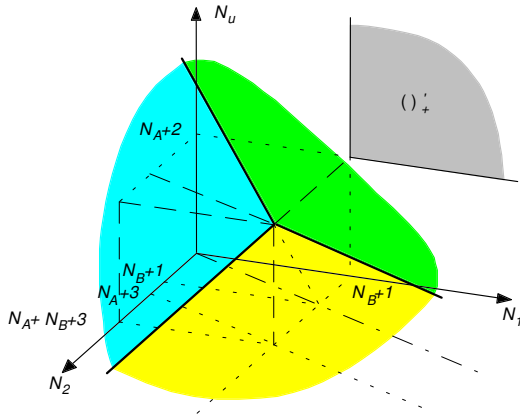


Fig. 6. Denied region from the GPC solvability viewpoint.

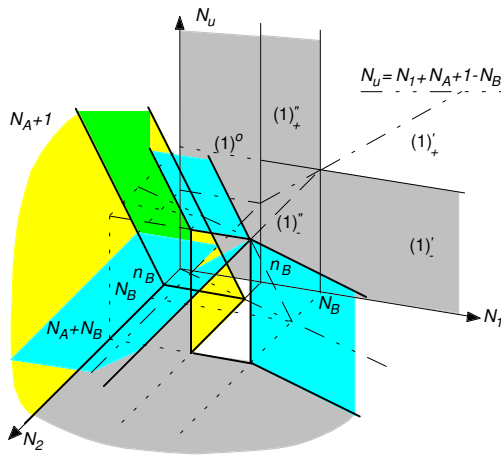


Fig. 7. Entire area of solvability and its orthogonal projections on  $(N_1, N_u)$ .

$(1)'_-$  for  $N_1 \geq N_B$  it is necessary that  $1 \leq N_u \leq N_A + 1$  and  $N_2 \geq N_1 + N_A$ .

**Lemma 4.** (Restricted certainty of solvability with respect to  $N_1$ ). For any pair  $(N_1, N_u)$  such that  $N_1 \leq N_B$  or  $N_u \leq N_A + 1$ , there exists an  $N_2$  such that the matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  associated with the coprime  $\hat{A}(q^{-1})$  and  $B(q^{-1})$  has a full column rank. Alternatively, for the pairs  $(N_1, N_u)$  with  $N_1 > N_B$  and  $N_u > N_A + 1$ , the matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  does not maintain its column rank.

**Remark 2.** The conditions  $(1)^0$ ,  $(1)'_-$  and  $(1)''$  may seem too conservative as compared to the ‘strictly’ necessary condition  $N_2 = N_1 + N_u - 1$ . However, it is worth noticing that, in general, this condition is not a sufficient one.

**Example 1.** Let us consider  $A(q^{-1}) = 1 + q^{-1} + 0.75q^{-2} + 0.75q^{-3}$  and  $B(q^{-1}) = q^{-1} + 0.5q^{-2}$ . It can be easily verified that for the case of the condition  $(1)'_-$  we have  $\text{rank } \mathbf{H}_{2,3}^2 = \text{rank } \mathbf{H}_{2,4}^2 = 1$  but  $\text{rank } \mathbf{H}_{2,5}^2 = 2$ , and  $\text{rank } \mathbf{H}_{3,4}^2 = 1$ , whereas  $\text{rank } \mathbf{H}_{2,6}^2 = 2$ . As another example, let us take  $A(q^{-1}) = 1 - 0.25q^{-1} - 0.5q^{-2} + 0.25q^{-3} + 0.25q^{-4} - 0.1q^{-5}$  and  $B(q^{-1}) = q^{-1} - 0.75q^{-2} - 0.675q^{-3} + 0.45q^{-4}$ , where in the case  $(1)''$ :  $\text{rank } \mathbf{H}_{2,4}^3 = 2$ , while  $\text{rank } \mathbf{H}_{2,9}^3 = 3$ . For yet another set of polynomials ( $A(q^{-1}) = 1 + 0.5q^{-1} + 3.125q^{-2} - 0.5q^{-3}$ ,  $B(q^{-1}) = q^{-1} + 0.5q^{-2} + 1.125q^{-3}$ ) and the case  $(1)''_+$  we have  $\text{rank } \mathbf{H}_{2,5}^4 = 3$  and  $\text{rank } \mathbf{H}_{2,6}^4 = 4$ .

The completeness of the above lemmas should be confronted with the results obtained previously. Note that an extensive discussion of the relevant results is given in (Kowalczyk and Suchomski, 2001; Suchomski and Kowalczyk, 2002a). Some generic guidelines for the selection of the tuning parameters  $(N_1, N_2, N_u, \lambda)$  of the basic GPC algorithm can be found in (Clarke *et al.*, 1987; Clarke, 1988; Clarke and Mohtadi, 1989; Mohtadi and Clarke, 1986). Simple methods, such as model algorithmic control rules (Rouhani and Mehra, 1982), require  $N_1 = 1$  and  $N_u = N_2$  with  $\lambda = 0$ , which, however, do not guarantee the existence of  $\mathbf{K}$  (Clarke and Mohtadi, 1989). The dead-beat (DB) GPC controller can also be derived via setting  $N_1 = N_A + 1$ ,  $N_2 \geq 2N_A + 1$ ,  $N_u = N_A + 1$  and  $\lambda = 0$  (Clarke, 1988; Clarke and Mohtadi, 1989). Moreover, it has been claimed that in most practical cases of minimal plant models the following set of GPC parameters should be effective:  $N_1 = N_A + 1$ ,  $N_2 \geq 2N_A + 1$ ,  $N_u \leq N_A + 1$  and  $\lambda = 0$ . The choice  $N_u = 1$  can be satisfactory, whilst a ‘difficult’ plant requires  $N_u$  equal to the number of unstable/underdamped poles (Clarke, 1988; Clarke and Mohtadi, 1989). Since we assume that  $N_2 \geq N_1 + N_u - 1$  and  $\lambda = 0$ , the following two conditions for the DB control law can be proposed (Peng and Hanus, 1991):  $(N_1 > N_B, N_u = N_A + 1)$  and  $(N_1 = N_B, N_u \geq N_A + 1)$ . These settings can be used for a suitably augmented system to make the GPC strategy equivalent to a typical pole-placement design. An incomplete theorem on reducing the GPC closed-loop order by properly choosing the tuning parameters was suggested in (Zhang, 1996), where, under the assumption that  $N_B = N_A$ ,  $n_B = 1$ ,  $N_2 \geq N_1 + N_u - 1$  and  $\lambda = 0$ , the following two conditions were derived:  $(N_1 \geq N_A = N_B, N_u = N_A + 1)$  and  $(N_1 = N_A = N_B, N_u \geq N_A + 2)$ , which lead to the DB property of the GPC system. On the other hand, the following experimentally confirmed strategies for selecting the GPC design parameters were recommended in (McIntosh *et al.*, 1991):  $(N_1 = 1, N_2 \geq n_b + 1, N_u = 1)$ ,  $(N_1 = 1, N_2 > n_b + N_u, N_u = N_A + 1)$  and  $(N_1 = N_B + 1, N_2 \geq N_1 + N_u - 1, N_u = N_A + 1)$  with  $\lambda = 0$ . An in-



interesting discussion concerning various methods of tuning GPC parameters can also be found, e.g., in (Maciejowski, 2002; Rossiter, 2003).

Ending the basic analysis of the GPC tuning problem, let us give a short comment on the consequences of using the zeroed lambda. On the one hand, within our methodology the coefficient  $\lambda$  does not play the role of a contributing safety factor in assuring that the inversion in the formula (14) will exist. Note that such a naive regularization of the solvability of the GPC problem can be helpful in the lack of knowledge about the full column rank of the celebrated Markov matrix  $H_{N_1, N_2}^{N_u}$ , but here this trick is not necessary because we utilize the demonstrated beauty of  $H$ . On the other hand, after pre-selecting design parameters according to the proposed methodology, a non-zero lambda can always be used as an additional re-tuning instrument in the GPC control design. What is more, even the analytical nature of the design procedure can then be partially preserved. Namely, as the coefficient  $\lambda$  contributes to the prediction control gain matrix and to the closed-loop characteristic polynomial, a suitable analysis can be performed with the use of the standard root-locus technique (Kowalczyk *et al.*, 1996; Kowalczyk and Suchomski, 2002; Suchomski and Kowalczyk, 2002b). Thus the  $\lambda$ -optimization objective simply represents yet another design space, which can be explored (in the case of concrete unfulfilled control signal constraints, for instance).

### 3.2. Conditions for Low-Order Polynomials $\tilde{D}(q^{-1})$

Principal conditions for the existence of the optimal solution vector  $\mathbf{k}|_{\lambda=0}$  in terms of pertinent GPC design parameters were analyzed in (Suchomski and Kowalczyk, 2002a), where an explicit parameterization of the polynomials  $\tilde{D}(q^{-1})$  of an arbitrary degree was presented. Clearly, by employing  $\tilde{D}(q^{-1})$  of a sufficiently low degree, the designer can substantially facilitate the issue of stabilizing the closed-loop system. The following summarizing theorem deals with the conditions for  $\tilde{D}(q^{-1})$  of the zero, one and second degrees as the most relevant

cases of practical importance, including stability. In order to save space, we use a concise table form to present the details of the theorem. Note that this set can be derived on the basis given in (Kowalczyk and Suchomski, 2001; 2002; Suchomski and Kowalczyk, 2002a).

**Theorem 1.** (Conditions for  $\tilde{D}(q^{-1})$  of a low degree). *With a minimal plant model applied in the GPC design, the polynomial  $\tilde{D}(q^{-1})$  has a prescribed low degree if the corresponding conditions listed in Tables 1–3 are fulfilled (Table 4 details the components of the parameters of these polynomials).*

Table 1. Zero degree conditions:  $\tilde{D}(q^{-1}) = 1$ .

		$N_B$	$N_u$	$N_1$	$N_2$
$()^0$	<b>a</b>	$n_B$	$\geq 1$	$\leq n_B$	$n_B + N_u - 1$
	<b>b</b>	$n_B$	$\geq N_A + 1$	$\leq n_B$	$\geq n_B + N_u$
$()''_+$			$\geq N_A + 1$	$N_B$	$\geq N_B + N_u - 1$
$()'_-$			$N_A + 1$	$\geq N_B$	$\geq N_1 + N_A$

**Remark 3.** The closed-loop control system with  $\tilde{D}(q^{-1})$  of the first degree is stable iff  $|\tilde{d}_1| < 1$ . The closed-loop control system with  $\tilde{D}(q^{-1})$  of the second degree is stable iff  $1 + \tilde{d}_1 + \tilde{d}_2 > 0$ ,  $1 - \tilde{d}_1 + \tilde{d}_2 > 0$ , and  $\tilde{d}_2 < 1$ .

**Remark 4.** Assuming that  $N_u \geq N_A + 1$ ,  $1 \leq N_1 \leq n_B$ , and  $N_2 \geq n_B + N_u - 1$ , we can rewrite the conditions  $()^0b$  by taking  $N_B = n_B$ ,  $N_B = n_B + 1$ , and  $N_B = n_B + 2$  for  $\tilde{D}(q^{-1})$  of the zero, first and second degrees, respectively.

Concluding this section, let us emphasize that from the viewpoint of design completeness our results are pioneering (see Lemma 2 and the denied and admitted regions in Figs. 5 and 6). The designer obtains new design perspectives, although the optimal parameter choice apparently depends on both the desirable control goals (stability, quality, complexity) and other parameters being fixed. For example, the full range of the values of  $N_1$  (see Lemma 3) can be utilized if necessary (from

Table 2. First degree conditions:  $\tilde{D}(q^{-1}) = 1 + \tilde{d}_1 q^{-1}$ .

		$N_B$	$N_u$	$N_1$	$N_2$	$\tilde{d}_1$
$()^0$	<b>a</b>	$n_B + 1$	$\geq 1$	$\leq n_B$	$n_B + N_u - 1$	$\alpha_1$
	<b>b</b>	$n_B + 1$	$\geq N_A + 1$	$\leq n_B$	$\geq n_B + N_u$	$\alpha_1$
	<b>c1</b>	$n_B + 1$	$N_A$	$\leq n_B$	$\geq n_B + N_u$	$\alpha_1 + \alpha_2$
	<b>c2</b>	$n_B$	$N_A$	$\leq n_B$	$\geq n_B + N_u$	$\alpha_3$
$()''_-$			$N_A$	$N_B - 1$	$\geq N_A + N_B$	$\alpha_4 + \alpha_5$
$()''_+$			$\geq N_A + 1$	$N_B - 1$	$\geq N_B + N_u - 1$	$\alpha_4$
$()'_-$			$N_A$	$\geq N_B$	$\geq N_1 + N_A$	$\alpha_6$

Table 3. Second degree conditions:  $\tilde{D}(q^{-1}) = 1 + \tilde{d}_1 q^{-1} + \tilde{d}_2 q^{-2}$ .

		$N_B$	$N_u$	$N_1$	$N_2$	$\tilde{d}_1$	$\tilde{d}_2$
$()^0$	<b>a</b>	$n_B + 2$	$\geq 1$	$\leq n_B$	$n_B + N_u - 1$	$\gamma_1$	$\gamma_2$
	<b>b</b>	$n_B + 2$	$\geq N_A + 1$	$\leq n_B$	$\geq n_B + N_u$	$\gamma_1 + \gamma_3$	$\gamma_2$
	<b>c1</b>	$n_B$	$N_A - 1$	$\leq n_B$	$\geq N_A + n_B - 1$	$\beta_1 + \beta_2$	$\beta_3$
	<b>c2</b>	$n_B + 1$	$N_A - 1$	$\leq n_B$	$\geq N_A + n_B - 1$	$\beta_1 + \beta_2 + \gamma_3$	$\beta_3$
	<b>c3</b>	$n_B + 2$	$N_A - 1$	$\leq n_B$	$\geq N_A + n_B - 1$	$\beta_1 + \beta_2 + \gamma_1 + \gamma_3$	$\beta_3 + \gamma_2$
	<b>c4</b>	$n_B + 2$	$N_A$	$\leq n_B$	$\geq N_A + n_B$	$\beta_2 + \gamma_1 + \gamma_3$	$\gamma_2$
$()''$	<b>a</b>		$N_A - 1$	$N_B - 2$	$\geq N_A + N_B$	$\beta_4 + \beta_5 + \gamma_1 + \gamma_3$	$\beta_6 + \gamma_2$
	<b>b</b>		$N_A - 1$	$N_B - 1$	$\geq N_A + N_B$	$\beta_4 + \beta_5 + \gamma_3$	$\beta_6$
	<b>c</b>		$N_A$	$N_B - 2$	$\geq N_A + N_B$	$\beta_5 + \gamma_1 + \gamma_3$	$\gamma_2$
$()'_+$			$\geq N_A + 1$	$N_B - 2$	$\geq N_B + N_u - 1$	$\gamma_1 + \gamma_3$	$\gamma_2$
$()'_-$			$N_A - 1$	$\geq N_B$	$\geq N_1 + N_A$	$\beta_4 + \beta_5$	$\beta_6$

Table 4. Constituent parameters of the parameters of the characteristic polynomials described in Tables 2 and 3.

$i$	$\alpha_i$	$\beta_i$	$\gamma_i$
1	$k_{N_B - N_1} b_{N_B}$	$(a_{N_A - 1} - a_{N_A}) \sum_{i=1}^{N_2 - n_B + 1} k_{i+n_B - N_1} h_{i+n_B - N_A - 1}$	$k_{N_B - N_1 - 1} b_{N_B - 1}$
2	$a_{N_A} \sum_{i=1}^{N_2 - n_B + 1} k_{i+n_B - N_1} h_{i+n_B - N_A - 3}$	$a_{N_A} \sum_{i=1}^{N_2 - n_B + 1} k_{i+n_B - N_1} h_{i+n_B - N_A - 2}$	$k_{N_B - N_1 - 1} b_{N_B}$
3	$a_{N_A} \sum_{i=1}^{N_2 - n_B + 1} k_{i+n_B - N_1} h_{i+n_B - N_A - 2}$	$a_{N_A} \sum_{i=1}^{N_2 - n_B + 1} k_{i+n_B - N_1} h_{i+n_B - N_A - 1}$	$k_{N_B - N_1} b_{N_B}$
4	$k_1 b_{N_B}$	$(a_{N_A - 1} - a_{N_A}) \sum_{i=1}^{N_0} k_i h_{i+N_1 - N_A - 1}$	
5	$a_{N_A} \sum_{i=1}^{N_0} k_i h_{i+n_B - N_A - 3}$	$a_{N_A} \sum_{i=1}^{N_0} k_i h_{i+N_1 - N_A - 2}$	
6	$a_{N_A} \sum_{i=1}^{N_0} k_i h_{i+N_1 - N_A - 2}$	$a_{N_A} \sum_{i=1}^{N_0} k_i h_{i+N_1 - N_A - 1}$	

some technical standpoint). Greater  $N_1$  can be used, for instance, to comply with some control signal restrictions. Note that, at the same time, all the low order solutions determined by Theorem 1 are available to the designer.

A discussion concerning the closed-loop performance resulting from the overrated value of the bottom horizon parameter  $N_1$  can be found in (Kowalczyk and Suchomski, 2002; Suchomski and Kowalczyk, 2002a). Overestimating the plant model order leads to enlarging both the horizon bottom  $N_1$  (related to the structure of the numerator of the plant model,  $n_B$ ,  $N_B$ ) and the effective observation horizon  $N_0 = N_2 - N_1 + 1$  (related usually to the degree of the plant denominator,  $N_0 \geq N_A + 1$ ). Consequently, this design imperative can cause degrada-

tion in the speed of the output, which, at the same time, can alleviate the control effort (such an effect of inaccurate modeling can be amended by appropriately tuning the top horizon  $N_2$ , (Kowalczyk and Suchomski, 2002; Suchomski and Kowalczyk, 2002a)).

It should also be mentioned here that, in general, with the non-linear characteristics of design, one could not expect a monotonous progress of the control indices in terms of a given design tuning parameter (see also the results presented by Suchomski and Kowalczyk (2002a)). Though, in most cases (surely, not always), such an effect is possible to be obtained as long as one keeps the applied conditioning of the GPC design (defined in Tables 1 through 3).

#### 4. GPC Design for Non-minimal Models

In systems theory, it is often required to determine (without resorting to factorization) whether two given polynomials are relatively prime. For example, the knowledge of an ‘unstable’ pole-zero cancellation in a real rational system model is essential for the analysis of most crucial system characteristics, such as stability and sensitivity to parameter variations. A cardinal algebraic test for detecting a common factor in two polynomials requires checking the rank deficiency of a resultant Sylvester matrix associated (in terms of coefficients) with these polynomials (Cullen and Hall, 1971; Middleton and Goodwin, 1990; Ogata, 1995, Landau *et al.*, 1998). This, however, implies numerically inept handling of a matrix of the order equal to the sum of the degrees of the two polynomials. A simple necessary and sufficient condition for two polynomials to have a common factor was given in (Vogt and Bose, 1970), and it is based on a test for the zeroing of the determinant of one (matrix) polynomial for an argument being the row companion (regulator) form of a system model, having the other polynomial as its characteristic polynomial. A comparison of the computational effort required for the above approaches can be found in (Cullen and Hall, 1971). The determination of the model order based on tests for common factors in models estimated via direct least-squares and maximum-likelihood methods was discussed in (Unbehauen and Göhring, 1974; van den Boom and van den Enden, 1974). A systematic way of performing tests for possible pole-zero cancellations in uncertain models obtained in least-squares identification was presented in (Söderström, 1975). A non-iterative maximum-likelihood approach for the detection and estimation of the greatest common factor (divisor) of two given polynomials, whose coefficients are assumed to have a normal distribution with a consistently estimable covariance, was introduced in (Stoica and Söderström, 1996).

In order to facilitate a further discussion, let us introduce the following notation associated with a given polynomial  $P(q^{-1}) = \sum_{i=0}^{N_P} p_i q^{-i}$  of  $\deg P(q^{-1}) = N_P$ :

$$\mathbf{p} = [p_0 \quad \cdots \quad p_{N_P}]^T, \quad (22)$$

$$\mathbf{T}_m^P = \begin{bmatrix} \mathbf{p} & \mathbf{0}_{N_P+1} & \vdots & \mathbf{0}_{N_P+1} \\ \mathbf{0}_{N_P+1} & \mathbf{p} & \vdots & \mathbf{0}_{N_P+1} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{0}_{N_P+1} & \mathbf{0}_{N_P+1} & \vdots & \mathbf{p} \end{bmatrix}, \quad (23)$$

$m \geq 1, \quad \mathbf{T}_m^P \in \mathbb{R}^{(N_P+m) \times m}.$

The Diophantine equations D1–D4 can be written in the following compact forms of linear equations:

$$D1 : \begin{bmatrix} \mathbf{T}_i^{\hat{A}} & \mathbf{0}_{i, N_A+1} \\ & \mathbf{I}_{N_A+1} \end{bmatrix} \begin{bmatrix} \mathbf{e}_i \\ \mathbf{f}_i \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{0}_{N_A-N_C+i} \end{bmatrix}, \quad (24)$$

$$D2 : \begin{bmatrix} \mathbf{T}_i^C & \mathbf{0}_{i, N_G+1} \\ \mathbf{0}_{N_G-N_C+1, i} & \mathbf{I}_{N_G+1} \end{bmatrix} \begin{bmatrix} \mathbf{h}_i \\ \mathbf{g}_i \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{e}}_i \\ \mathbf{0}_{N_G-N_B+2} \end{bmatrix}, \quad (25)$$

$$D3 : \begin{bmatrix} \mathbf{T}_i^{\hat{A}} & \mathbf{0}_{i, N_A+1} \\ & \mathbf{I}_{N_A+1} \end{bmatrix} \begin{bmatrix} \mathbf{h}_i \\ \mathbf{l}_i \end{bmatrix} = \begin{bmatrix} \bar{\mathbf{b}} \\ \mathbf{0}_{N_A-N_B+i+1} \end{bmatrix}, \quad (26)$$

$$D4 : \begin{bmatrix} \mathbf{T}_{N_G+1}^{\hat{A}} & \mathbf{T}_{N_A+1}^{\bar{B}} \\ & \end{bmatrix} \begin{bmatrix} \mathbf{g}_i \\ \mathbf{f}_i \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{l}}_i \\ \mathbf{0}_{N_G-N_C+1} \end{bmatrix}, \quad (27)$$

where  $\tilde{\mathbf{e}}_i \in \mathbb{R}^{N_B+i-1}$  and  $\tilde{\mathbf{l}}_i \in \mathbb{R}^{N_A+N_A+1}$  are associated with polynomials  $\tilde{E}_i(q^{-1}) = \bar{B}(q^{-1})E_i(q^{-1})$  and  $\tilde{L}_i(q^{-1}) = C(q^{-1})L_i(q^{-1})$ , respectively. The matrices of (24)–(26) are of a lower triangular structure with unit diagonal elements. Solutions to these equations can thus be easily (i.e., recursively) obtained by performing a standard low-cost and numerically stable procedure of ‘forward substitution’ (Golub and van Loan, 1996; Higham, 1996). The matrix (27) has a Sylvester-like structure that, for any coprime pair  $(\hat{A}(q^{-1}), B(q^{-1}))$ , achieves guaranteed non-singularity (Higham, 1996). However, the solution of (27) requires higher computational efforts when compared to (24)–(26) and can even become numerically ill-conditioned when an approximate (near-) cancellation appears in  $(\hat{A}(q^{-1}), B(q^{-1}))$ .

Assume that the model equation (1) describes an overparameterized model of the controlled plant with  $A(q^{-1}) = A'(q^{-1})\Lambda(q^{-1})$  and  $B(q^{-1}) = B'(q^{-1})\Lambda(q^{-1})$ , where  $A'(q^{-1})$  and  $B'(q^{-1})$  denote the factual (true) plant polynomials, while  $\Lambda(q^{-1})$  of  $\deg \Lambda(q^{-1}) = N_\Lambda > 0$  is a monic polynomial, representing the greatest common factor (GCF) of  $A(q^{-1})$  and  $B(q^{-1})$ .  $N_\Lambda$  will be referred to as the cancellation order. Likewise, all ‘primed’ items (polynomials and parameters) will be henceforth associated with the minimal model.

Three crucial problems can be formulated:

**Problem 1.** (Principal GPC-design problems resulting from overparameterization).

1. How does the lack of knowledge about the cancellation order influence the GPC design?
2. How is the resulting GPC control law affected by the presence of common factors?
3. Are there any difficulties the controller calculations that are imposed by the common factors?

#### 4.1. Regional Behavior (No Knowledge on $N_\Lambda$ )

Let us consider the effect of uncertainty about the system order, which in the case of its overestimation and in virtue of Lemmas 1 and 2 clearly results in the overestimation of  $N_u$ ,  $N_1$  and  $N_2$ . Taking into account the distinguished regions for  $\text{rank } \mathbf{H}_{N_1, N_2}^{N_u} = N_u$ , the following corollary can be proposed:

**Corollary 1.** (Sensitivity to overparameterization). *In the specific regions of invertability of (14) for  $\lambda = 0$  model overparameterization has the following effects:*

- (j) The claim  $(\ )'_+$ :
  - for larger  $N_1$  the loss of solvability is possible,
  - letting  $N_1 = n_B$  guarantees solvability.
- (jj) The claim  $(\ )''_-$ :
  - for larger  $N_1$  solutions of type  $(\ )'_-$  are probable,
  - for larger  $N_u$  solutions can be of type  $(\ )''_+$ ,
  - the loss of solvability is possible (for both  $N_1$  and  $N_u$  overestimated),
  - letting  $N_1 = n_B$  or  $N_u = 1$  guarantees solvability.
- (jjj) The claim  $(\ )'_-$ :
  - too big  $N_u$  may lead to the loss of solvability,
  - letting  $N_u = 1$  guarantees solvability.

*In general, in all the above cases of overparameterization, there is no possibility of obtaining  $\hat{D}(q^{-1})$  of an assumed degree. Thus, in particular, the DB control is also impossible.*

An important inference follows from Corollary 1, i.e., the existence of  $\mathbf{K}$  for  $\lambda = 0$  can always be enforced. With the lack of knowledge about the cancellation order  $N_\Lambda$ , however, there are no means of establishing  $\hat{D}(q^{-1})$  of a desired low degree. And this practically implies difficulties in guaranteeing the stability of GPC closed-loop systems. For example, as we can see from (Suchomski and Kowalczyk, 2002a), taking  $N_u = 1$  always gives  $\hat{D}(q^{-1})$  with a ‘high’ degree of  $N_A$ . Moreover, it is worth noticing that the lack of knowledge about the exact value of the plant transportation delay may lead to the loss of solvability solely if both  $N_A$  and  $N_B$  are overestimated.

#### 4.2. Closed Loop Analysis

As the system’s Markov parameters  $h_i$ ,  $i \geq 1$ , are independent of the common factor  $\Lambda(q^{-1})$ , for properly chosen design parameters  $(N_1, N_2, N_u)$  the gain  $g$  is also unaffected by overparameterization. At the same time, however, the polynomials  $G(q^{-1})$ ,  $F(q^{-1})$ ,  $\tilde{F}(q^{-1})$  and  $L(q^{-1}) = L'(q^{-1})\Lambda(q^{-1})$  have different forms for different  $\Lambda(q^{-1})$ .

Since there holds  $\hat{A}'(q^{-1})G(q^{-1}) + B'(q^{-1})F(q^{-1}) = q^{-1}C(q^{-1})L'(q^{-1})$ , we conclude that by employing the control law with the observer filters  $G(q^{-1})/C(q^{-1})$  and  $\tilde{F}(q^{-1})/C(q^{-1})$  (or  $F(q^{-1})/C(q^{-1})$ ) to the actual plant described with the aid of  $A'(q^{-1})$  and  $B'(q^{-1})$ , the following closed-loop characteristic polynomial results:

$$D(q^{-1}) = D'_0(q^{-1})C(q^{-1}), \quad (28)$$

where  $D'_0(q^{-1}) = \hat{A}'(q^{-1}) + q^{-1}L'(q^{-1}) + g^*B'(q^{-1})$  does not depend on  $\Lambda(q^{-1})$ .

**Remark 5.** Note that from the viewpoint of identification (used in adaptive GPC schemes), it is of great significance that in this case there is no need for  $\Lambda(q^{-1})$  to be stable. On the other hand, if  $C(q^{-1}) = C'(q^{-1})\Lambda(q^{-1})$ , we have  $G(q^{-1}) = G'(q^{-1})\Lambda(q^{-1})$  and  $F(q^{-1}) = F'(q^{-1})\Lambda(q^{-1})$ . Consequently, in such cases, as well as when the control channel of the plant model itself is non-minimal with a common factor  $\Lambda(q^{-1})$ , the internal stability condition requires  $\Lambda(q^{-1})$  to be stable.

#### 4.3. Cancellation Order Detection and Design Perspectives

From the third Diophantine equation (8) it follows that for given  $\hat{A}(q^{-1})$  and  $\bar{B}(q^{-1})$  a quite ‘natural’ way of determining the cancellation order  $N_\Lambda$  can be based on the examination of the column rank of subsequent left column submatrices of the matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  with properly chosen parameters  $N_1$  and  $N_2$ .

**Lemma 5.** (Rank deficiency of  $\mathbf{H}_{N_1, N_2}^{N_u}$  for a reducible/non-coprime pair  $(\hat{A}(q^{-1}), \bar{B}(q^{-1}))$ ). *Let  $N_1 \geq N_B$  and  $N_2 \geq N_1 + N_A$ . For polynomials  $\hat{A}(q^{-1})$  of  $\deg \hat{A}(q^{-1}) = N_A + 1$  and  $\bar{B}(q^{-1})$  of  $\deg \bar{B}(q^{-1}) = N_B - 1$  having a greatest common factor  $\Lambda(q^{-1})$  of  $\deg \Lambda(q^{-1}) = N_\Lambda > 0$ , we have*

$$\text{rank } \mathbf{H}_{N_1, N_2}^{N_u} = \begin{cases} N_u & \text{if } N_u \leq N_{A'} + 1, \\ N_{A'} + 1 & \text{if } N_u > N_{A'} + 1, \end{cases} \quad (29)$$

*and the range space of  $\mathbf{H}_{N_1, N_2}^{N_u}$  can be found as  $\mathbf{R}[\mathbf{H}_{N_1, N_2}^{N_u}] = \mathbf{R}[\mathbf{H}_{N_1, N_2}^{\min\{N_u, N_{A'} + 1\}}]$ .*

*Proof.* Taking into account the fact that  $N_1 \geq N_B > N_{B'}$  and  $N_2 \geq N_1 + N_A > N_1 + N_{A'}$ , we observe that the claim  $(\ )'_-$  of Lemma 2 produces

$$\text{rank } \mathbf{H}_{N_1, N_2}^{N_u} \left| \begin{array}{l} N_u \leq N_{A'} + 1 \\ N_1 \geq N_{B'} \\ N_2 \geq N_1 + N_{A'} \end{array} \right. = N_u,$$

while the claim  $(\ )'_+$  of the same lemma provides

$$\text{rank } \mathbf{H}_{N_1, N_2}^{N_u} \left| \begin{array}{l} N_u > N_{A'} + 1 \\ N_1 > N_{B'} \\ N_2 \geq N_1 + N_u - 1 \end{array} \right. < N_u.$$

■

A recursive algorithm developed for examining the rank deficiency of  $\mathbf{H}_{N_1, N_2}^{N_u}$ , being a version of a recursive procedure (Boullion and Odell, 1971; Bunch and Nielsen, 1978) for computing the Moore-Penrose pseudoinverse of a matrix, can be found in (Kowalczyk *et al.*, 1996; Kowalczyk and Suchomski, 1999; 2002).

Consider an increasing sequence of  $N_u = 1, 2, \dots$ , and let  $\bar{N}_u$  denote the lowest  $N_u$  for which the deficiency of the column rank of  $\mathbf{H}_{N_1, N_2}^{N_u}$  is detected. Then we infer that the cancellation order  $N_\Lambda$  can be established as

$$N_\Lambda = N_A - \bar{N}_u + 2. \quad (30)$$

In the process of detecting the column rank deficiency of  $\mathbf{H}_{N_1, N_2}^{N_u}$ , any ‘thresholding’ mechanism of decision can be employed. The basic criterion can be obtained by examining singular values of subsequent left submatrices of  $\mathbf{H}_{N_1, N_2}^{N_u}$ : it is well known that a deficiency in the rank of a matrix may be signaled by a ‘gap’ in its singular values (Golub and van Loan, 1996; Stewart, 1998). A numerically cheap, ‘SVD-free’ algorithm can also be considered that is based on a recursively computed measure of an angular distance  $\kappa_{N_1, N_2}^i$ ,  $i \geq 1$ , between a left submatrix  $\mathbf{H}_{N_1, N_2}^i$  of  $\mathbf{H}_{N_1, N_2}^{N_u}$  and its subsequent column  $\mathbf{h}_{N_1, N_2}^{i+1} = [h_{N_1-i-1} \ \dots \ h_{N_2-i-1}]^T$ :

$$\kappa_{N_1, N_2}^i = \sin \angle \left\{ \mathbf{h}_{N_1, N_2}^{i+1}, \mathbf{P}_{R[\mathbf{H}_{N_1, N_2}^i]} \mathbf{h}_{N_1, N_2}^{i+1} \right\}, \quad (31)$$

where  $\mathbf{P}_{R[\mathbf{H}_{N_1, N_2}^i]}$  denotes the orthogonal projector onto  $R[\mathbf{H}_{N_1, N_2}^i]$ . It is worth noticing that while completing the above-mentioned  $N_\Lambda$ -estimation algorithm for chosen  $N_1 \geq N_B$  and  $N_2 \geq N_1 + N_A$ , a sequence of the gain vectors  $\mathbf{k}$  for consecutive  $N_u = 1, \dots, N_{A'} + 1$  is established as a by-product (at the same time).

**Remark 6.** A critical point of the above approaches is a right answer to the thresholding question: How small should the tested value (a singular value or an angular distance) be to be classified as ‘zero’? In other words: At

what level should a decision threshold be placed? This question can also be posed as a classical problem of determining a numerical rank of a given matrix (Higham, 1996; Stewart, 1998). When numerically computed (perturbed) singular values of  $\mathbf{H}_{N_1, N_2}^{N_u}$  are small and differ very slightly, the standard algorithms for solving this problem may have inferior sensitivity (Bjorck, 1996; Golub and van Loan, 1996). It is also worth emphasizing that the observed ‘gap’ between the singular values, which serves as a basis for the decision on rank deficiency, must be reasonably large as compared to an actual estimate ( $\varepsilon$ ) of computation errors. On the other hand, overly large  $\varepsilon$  can easily result in missing the gap (Stewart, 1998). The same drawback is characteristic if algorithms are based on checking the angular distance between the subsequent column of  $\mathbf{H}_{N_1, N_2}^{N_u}$  and the range subspace of its corresponding left submatrix (Kowalczyk *et al.*, 1996; Kowalczyk and Suchomski, 1999; 2002).

Now, let  $N_\Lambda$  be known and  $C(q^{-1})$  of  $\deg C(q^{-1}) = N_C \leq N_A - N_\Lambda + 1$  represent an arbitrarily chosen stable observer polynomial. From the fourth Diophantine (D4) of the equation (9) it follows that, for  $i \geq 1$ ,

$$\hat{A}(q^{-1})G'_i(q^{-1}) + \bar{B}(q^{-1})F'_i(q^{-1}) = C(q^{-1})L_i(q^{-1}), \quad (32)$$

where the polynomials  $G'_i(q^{-1})$  of  $\deg G'_i(q^{-1}) = N_{G'} = \max\{N_B - N_\Lambda - 2, N_C - 1\} \geq 0$  and  $F'_i(q^{-1})$  of  $\deg F'_i(q^{-1}) = N_A - N_\Lambda$  denote the reduced-in-degree numerators of the input and output observer filters, respectively, while  $L_i(q^{-1})$  is appropriately derived by using the non-minimal model of the control channel. The polynomials  $G'_i(q^{-1})$  and  $F'_i(q^{-1})$  can thus be computed based on the triplet  $(A(q^{-1}), B(q^{-1}), C(q^{-1}))$  without performing any reduction in the control channel model.

By considering the Diophantine equations (D1)–(D3) of (6)–(8), we derive the formula

$$(G'_i(q^{-1}) + H_i(q^{-1})F'_i(q^{-1}))\Lambda(q^{-1}) = L_i(q^{-1})E'_i(q^{-1}), \quad (33)$$

in which the polynomials of the plant model of (5) do not appear. Having obtained  $G'_i(q^{-1})$  and  $F'_i(q^{-1})$ , we can regard this formula as a Diophantine equation in  $\Lambda(q^{-1})$  and  $E'_i(q^{-1})$  of  $\deg E'_i(q^{-1}) = i - 1$ . On the other hand, the coefficients of  $E'_i(q^{-1})$  and  $E_i(q^{-1})$  are obviously related via the following relation:

$$\sum_{j=0}^{\infty} e'_j q^{-j} = \Lambda(q^{-1}) \sum_{j=0}^{\infty} e_j q^{-j}. \quad (34)$$

Note that solving the (Diophantine) equation (33) should give the same  $\Lambda(q^{-1})$  for all  $i \geq 1$ .

The above observations establish a suitable basis for the following procedure for estimating  $N_\Lambda$ : At this stage, since  $N_\Lambda$  is unknown, it is convenient to assume the simplest case  $C(q^{-1}) = 1$ . Considering a hypothesis  $N_\Lambda = m$ , where  $m \in \{1, \dots, N_B - n_B\}$ , we obtain the following coupled Diophantine equations:

$$\hat{A}(q^{-1})G_i^m(q^{-1}) + \bar{B}(q^{-1})F_i^m(q^{-1}) = L_i(q^{-1}), \quad (35)$$

$$\begin{aligned} (G_i^m(q^{-1}) + H_i(q^{-1})F_i^m(q^{-1})) \Lambda_i^m(q^{-1}) \\ = L_i(q^{-1})E_i^m(q^{-1}), \quad (36) \end{aligned}$$

where for  $i \geq 1$  the following degree-constraints are assumed:  $\deg \Lambda_i^m(q^{-1}) = m$ ,  $\deg G_i^m(q^{-1}) = N_B - m - 2$  (with  $G_i^m(q^{-1}) = 0$  if  $m = N_B - n_B$  and  $n_B = 1$ ),  $\deg F_i^m(q^{-1}) = N_A - m$ , and  $\deg E_i^m(q^{-1}) = i - 1$ . The equation (35) can be transformed into the following sets of linear equations for the coefficients of the polynomials  $G_i^m(q^{-1})$  and  $F_i^m(q^{-1})$ :

$$\begin{cases} \begin{bmatrix} \mathbf{T}_{N_B-m+1}^{\hat{A}} & \mathbf{T}_{N_A-m+1}^{\bar{B}} \end{bmatrix} \begin{bmatrix} \mathbf{g}_i^m \\ \mathbf{f}_i^m \end{bmatrix} = \begin{bmatrix} \mathbf{l}_i \\ \mathbf{0}_{N_B-m-1} \end{bmatrix}, \\ \mathbf{T}_{N_A-N_B+2}^{\bar{B}} \mathbf{f}_i^m = \mathbf{l}_i \\ \text{if } m = N_B - n_B \text{ and } n_B = 1. \end{cases} \quad (37)$$

In order to transform (36) into an equation linear solely in  $\Lambda_i^m(q^{-1})$ , we should consider an estimate of  $E_i^m(q^{-1})$ . By virtue of (34) the following recursive formula can be written for  $i > 1$ :

$$\sum_{j=0}^{\infty} e_{i,j}^m q^{-j} = \Lambda_{i-1}^m(q^{-1}) \sum_{j=0}^{\infty} e_j q^{-j}, \quad (38)$$

where  $\Lambda_i^m(q^{-1}) = \sum_{j=0}^m \lambda_{i,j}^m q^{-j}$  with  $\lambda_{i,0}^m = 1$ . Now, taking into account the fact that  $E_i^m(q^{-1}) = \sum_{j=0}^{i-1} e_{i,j}^m q^{-j}$  with  $e_{i,0}^m = 1$ , we obtain

$$e_{i,j}^m = \sum_{k=0}^{\min\{j,m\}} \lambda_{i,k}^m e_{j-k} \text{ for } j = 0, \dots, i-1. \quad (39)$$

Hence the required set of linear equations in  $\tilde{\lambda}_i^m$  takes the subsequent aggregate form:

$$\begin{bmatrix} \mathbf{0}_m^T \\ \mathbf{T}_m^{M_i^m} \end{bmatrix} \tilde{\lambda}_i^m = \mathbf{T}_i^{L_i} e_i^m - \begin{bmatrix} \mathbf{m}_i^m \\ \mathbf{0}_m \end{bmatrix}, \quad (40)$$

where  $\tilde{\lambda}_i^m = [\lambda_{i,1}^m \ \lambda_{i,2}^m \ \dots \ \lambda_{i,m}^m]^T$  is a vector of the coefficients of the cancellation polynomial  $\Lambda_i^m(q^{-1})$ , and

$$M_i^m(q^{-1}) = G_i^m(q^{-1}) + H_i(q^{-1})F_i^m(q^{-1}), \quad (41)$$

$$\lambda_i^m = \begin{bmatrix} 1 \\ \tilde{\lambda}_i^m \end{bmatrix}. \quad (42)$$

For each  $m$ , a sequence of least-squares solutions  $\Lambda_i^m(q^{-1})$  can be evaluated for  $i = 1, \dots, N_q$ , where  $N_q \geq 2$  is a free parameter. In general, three cases should be discerned while examining the solutions of the design equations (36)–(39):

- (i)  $m < N_\Lambda$  – (37) with the matrix of a column-rank deficiency has a non-unique least-squares solution with a zero residue, – (40) with the matrix of a full column rank has a unique least-squares solution with a non-zero residue,
- (ii)  $m = N_\Lambda$  – (37) and (40) with the matrices of a full column rank have unique least-squares solutions of zero residue,
- (iii)  $m > N_\Lambda$  – (37) and (40) with the matrices of a full column rank have unique least-squares solutions with non-zero residues.

Comparing the consecutive solutions  $\lambda_i^m$  permits the detection of  $N_\Lambda$  with the use of a direct  $\Lambda$ -estimation index (43) utilized in the following lemma:

**Lemma 6.** (Robust detection of the cancellation order  $N_\Lambda$ ). *Let*

$$J_{N_q}^m = \sum_{i=1}^{N_q-1} \frac{\|\lambda_i^m - \lambda_{i+1}^m\|}{\|\lambda_i^m\|}. \quad (43)$$

*Starting from  $m = N_B - n_B$ , a specific value  $\bar{m}$  of  $m$  is sought, for which  $J_{N_q}^{\bar{m}} = 0$ . Consequently,  $N_\Lambda = \bar{m}$ . In the case of a non-zero  $J_{N_q}^m$  for all  $m \in \{1, \dots, N_B - n_B\}$ , we conclude that  $(\hat{A}(q^{-1}), \bar{B}(q^{-1}))$  are coprime. It is important that during an effective stage of searching for  $\bar{m}$ , i.e., for  $m \geq N_\Lambda$ , the corresponding least-squares problems described by the system (37) are numerically well-conditioned by having the matrices of full column-ranks.*

With a particular ‘design relevant’  $C(q^{-1})$  and after having determined  $N_\Lambda$ , the reduced-in-degree polynomials  $G_i'(q^{-1})$  and  $F_i'(q^{-1})$ ,  $i \geq 1$ , can be easily obtained in a numerically robust way by solving (32).

**Remark 7.** The parameterization of  $\mathbf{H}_{N_1, N_2}^{N_u}$  given in Lemma 5 guides the designer to the region  $(\cdot)'_-$  of the GPC design solvability. In this case, having three degrees of freedom, i.e.,  $N_1 > N_B$ ,  $N_2 \geq N_1 + N_A$  and  $N_u \leq N_{A'} + 1$ , we observe that, as far as closed-loop stability is concerned, the parameter  $N_u$  makes a basic design ‘knob.’ It can then be easily shown (Suchomski and Kowalczyk, 2002a) that  $\deg \bar{D}(q^{-1}) = N_{A'} + 1 - N_u$  (cf. also the results of Theorem 1). Thus, by choosing sufficiently large  $N_u$ , we can simply obtain a suitable polynomial  $\bar{D}(q^{-1})$  of a low degree. Note, however, that in this

design case increasing  $N_u$  leads to more intense control actions. On the other hand, choosing larger  $N_1$  and  $N_2$  means further trading-off between the speed of the closed-loop system transient reaction and the control-signal energy (Suchomski and Kowalczyk, 2002a).

From the above discussion it follows that, having estimated  $N_\Lambda$ , the designer has three ways of designing GPC controllers, which are given in the following proposition:

**Proposition 1.** (GPC design perspectives).

1. A low-order controller can be computed based on the minimal model  $(A'(q^{-1}), B'(q^{-1}), C(q^{-1}))$  with  $\deg C(q^{-1}) \leq N_{A'} + 1$  obtained by virtue of Appendix F. In order to derive the polynomials  $G'_i(q^{-1})$  and  $F'_i(q^{-1})$ ,  $i \geq 1$ , it is recommended that Eqns. (D1) and (D2) be solved according to (24) and (25).
2. A high-order controller can be computed with the use of the non-minimal model  $(A(q^{-1}), B(q^{-1}), C(q^{-1}))$  with  $\deg C(q^{-1}) \leq N_A + 1$ . Solving (24) and (25) is recommended in order to acquire the Diophantine products  $G_i(q^{-1})$  and  $F_i(q^{-1})$ ,  $i \geq 1$ .
3. A low-order controller can be computed based on the non-minimal model  $(A(q^{-1}), B(q^{-1}), C(q^{-1}))$  with  $\deg C(q^{-1}) \leq N_{A'} + 1$ . This time, the (D3) and (D4)-related polynomials  $G'_i(q^{-1})$  and  $F'_i(q^{-1})$ ,  $i \geq 1$ , should be obtained by solving (26) and (37).

Proposition 1 represents a concise summary of known facts and design perspectives resulting from the presented analysis of the GPC problem in the context of overparameterized plant models. At this point it may be worth emphasizing that in this contribution we are not concerned with the consequences of wrong modeling (results of such a discussion are given by Kowalczyk and Suchomski (2002), and Suchomski and Kowalczyk (2002a)), but instead we focus our attention on proper utilization of any product of modeling and identification in terms of retrieving the necessary information about the ‘exact’ plant model in a numerically robust way.

Several numerical examples of GPC designs based on all the admitted solvability regions specified in Lemma 2 can be found in (Kowalczyk and Suchomski, 2001; 2002; Suchomski and Kowalczyk, 2002a).

## 5. Illustrative Design Examples

Let us illustrate the above discussion by considering two design examples. The first example illustrates some basic properties of the analysed method of GPC tuning.

Namely, the AF mechanism is applied to solutions from the region  $()'_-$  of the GPC feasibility area, and a simple case of overparameterization is given. The second example illustrates some more complicated issues of the detection of the cancellation order.

### 5.1. Example 1: Unstable Minimum-Phase Plant

Assume that an unstable minimum-phase plant is characterized by the following minimal model (Filali and Wertz, 2001):

$$A'(q^{-1}) = 1 - 2.4428q^{-1} + 1.4918q^{-2}, \quad (44)$$

$$B'(q^{-1}) = 0.2672q^{-1} + 0.2181q^{-2}. \quad (45)$$

Hence we set  $N_{A'} = 2$ ,  $N_{B'} = 2$ , and  $n_B = 1$ . Moreover, let us assume that  $C(q^{-1}) = 1 - 0.7q^{-1} + 0.12q^{-2}$ .

#### 5.1.1. Design Based on the Region $()'_-$

As has been shown in the previous section, the proposed design methodology for overparameterized models can be founded on exploring the region  $()'_-$  of the GPC solvability area. Let us illustrate some properties of this approach by starting from the above minimal model  $(A'(q^{-1}), B'(q^{-1}))$ . The DB solution results from the design settings  $N_1 = 2$ ,  $N_2 = 4$  and  $N_u = 3$ .

First of all, let us consider the effect of the anticipation filter (AF) by varying the value of its first parameter  $r_2 \in [0.5, 1.0]$ , while keeping fixed  $r_3 = r_4 = 1$ . Figure 8(a) outlines the resulting GPC controller gains. The plots presented in Fig. 8(b) illustrate closed-loop step responses. The norms of the control error and incremental control signals are given in Figs. 8(c) and 8(d), respectively. It is clear that employing the AF mechanism offers a simple and convenient tool for making a trade-off between the speed of the system response and the control effort.

Secondly, let us reflect on the possibility of control based on  $\tilde{D}(q^{-1}) = 1 + \tilde{d}_1 q^{-1}$ . From Theorem 2 it follows that such an effect can be obtained by setting  $N_1 = 2$ ,  $N_2 \geq 4$  and  $N_u = 2$  with the AF mechanism switched off ( $g^* = 0$ ). Let  $N_2$  serve as a free design parameter that allows for shaping the coefficient  $\tilde{d}_1$  of the factor  $D_0(q^{-1}) = \tilde{D}(q^{-1})$  of the closed-loop characteristic polynomial (19). The plot given in Fig. 9(a) illustrates how  $\tilde{d}_1$  depends on  $N_2$ . On this basis we conclude that the control system can be stabilized and that via increasing  $N_2$  a similar-as-before effect can be expected: The system response can be made slower and the control signal can be made less active. The corresponding plots depicted in Figs. 9(b)–(e) show the controller gain and the achievable control system performance, respectively, and confirm the above conjecture.

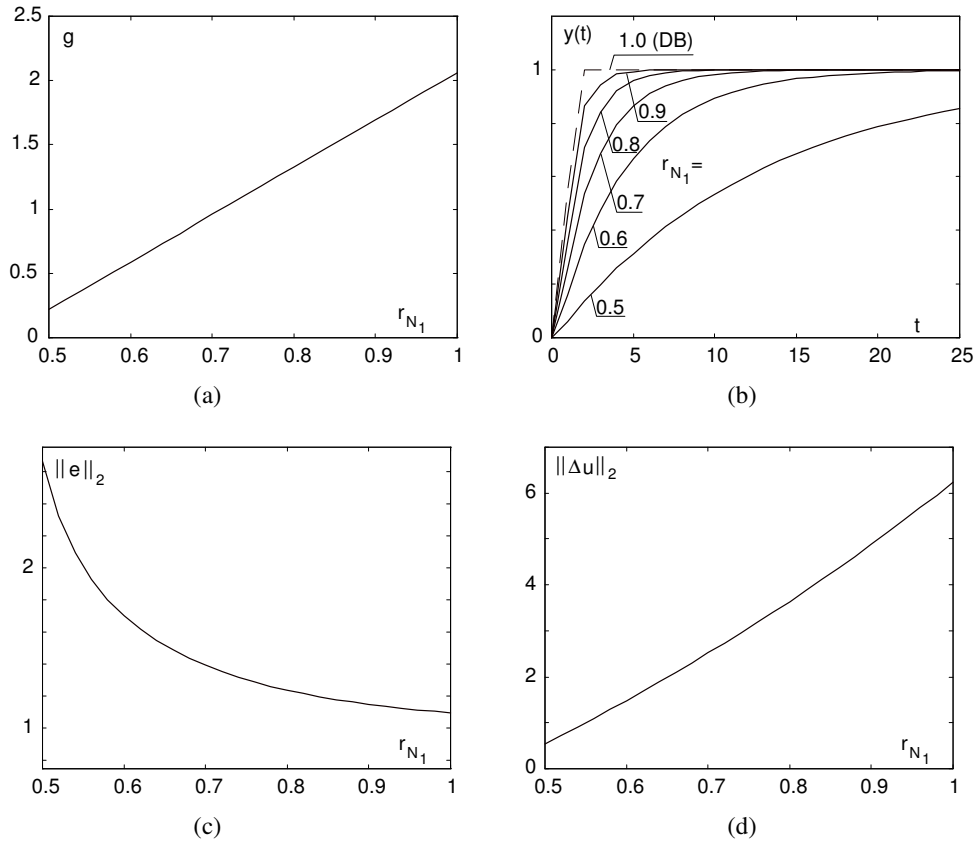


Fig. 8. Example 1: Effects of anticipative filtration.

### 5.1.2. Detection of the Cancellation Order

Let

$$\Lambda(q^{-1}) = (1 + 0.3q^{-1})(1 - 0.7q^{-1})(1 + 1.2q^{-1}). \quad (46)$$

The cancellation order is thus  $N_\Lambda = 3$ . Three methods of the detection of  $N_\Lambda$  will be considered. As  $N_A = 5$  and  $N_B = 5$ , Lemma 5 shows that the matrix  $\mathbf{H}_{5,10}^6$  with reasonably big  $N_u = N_A + 1 = 6$  can be appropriate in this case. Figure 10(a) illustrates  $\kappa_{5,10}^i$  of (31) being the angular distance between the left submatrix  $\mathbf{H}_{5,10}^i$  of  $\mathbf{H}_{5,10}^6$  and its subsequent column  $\mathbf{h}_{5,10}^{i+1}$ ,  $i \geq 1$ . The draft given in Fig. 10(b) shows the quantity of a *relative gap in the singular values*

$$\rho_{5,10}^i = \frac{\sigma_{i+1}(\mathbf{H}_{5,10}^6)}{\sigma_i(\mathbf{H}_{5,10}^6)}, \quad (47)$$

which is a standard measure representing a relative gap in the singular values  $\sigma_i$  of  $\mathbf{H}_{5,10}^6$ ,  $i \geq 1$ . Ultimately, the direct  $\Lambda$ -estimation index  $J_3^n$  of (43) for  $N_q = 3$  is depicted in Fig. 10(c).

It can be easily shown that all the discussed methods of the detection of the cancellation order give the same correct solution. Unfortunately, this is not always the case,

as will be learned from the next, a bit more complicated example of the GPC tuning procedure.

### 5.2. Example 2: Unstable Non-minimum-Phase Plants

Consider two unstable and non-minimum-phase plants numerically represented by the following overparameterized models with  $N_{A'} = 6$ ,  $N_{B'} = 7$ ,  $n_B = 2$  and  $N_\Lambda = 3$ :

$$A'(q^{-1}) = (1 - 1.9q^{-1} + 0.965q^{-2})(1 - q^{-1}) \times (1 - 0.8q^{-1})(1 - 0.5q^{-1})(1 + 1.5q^{-1}), \quad (48)$$

$$\Lambda(q^{-1}) = (1 - 0.3q^{-1})(1 - 0.2q^{-1})(1 + 6.2q^{-1}), \quad (49)$$

and

$$B'_I(q^{-1}) = -0.2q^{-2}(1 - 3.5q^{-1})(1 - 0.2q^{-1}) \times (1 - 0.1q^{-1})(1 + 0.2q^{-1})(1 + 2.5q^{-1}), \quad (50)$$

$$B'_{II}(q^{-1}) = -0.2q^{-2}(1 - 3.5q^{-1})(1 - 0.2q^{-1}) \times (1 - 0.51q^{-1})(1 + 0.2q^{-1})(1 + 2.5q^{-1}). \quad (51)$$

Observe that an additional near-cancellation occurs in the second pair  $(A'(q^{-1}), B'_{II}(q^{-1}))$ , which can have



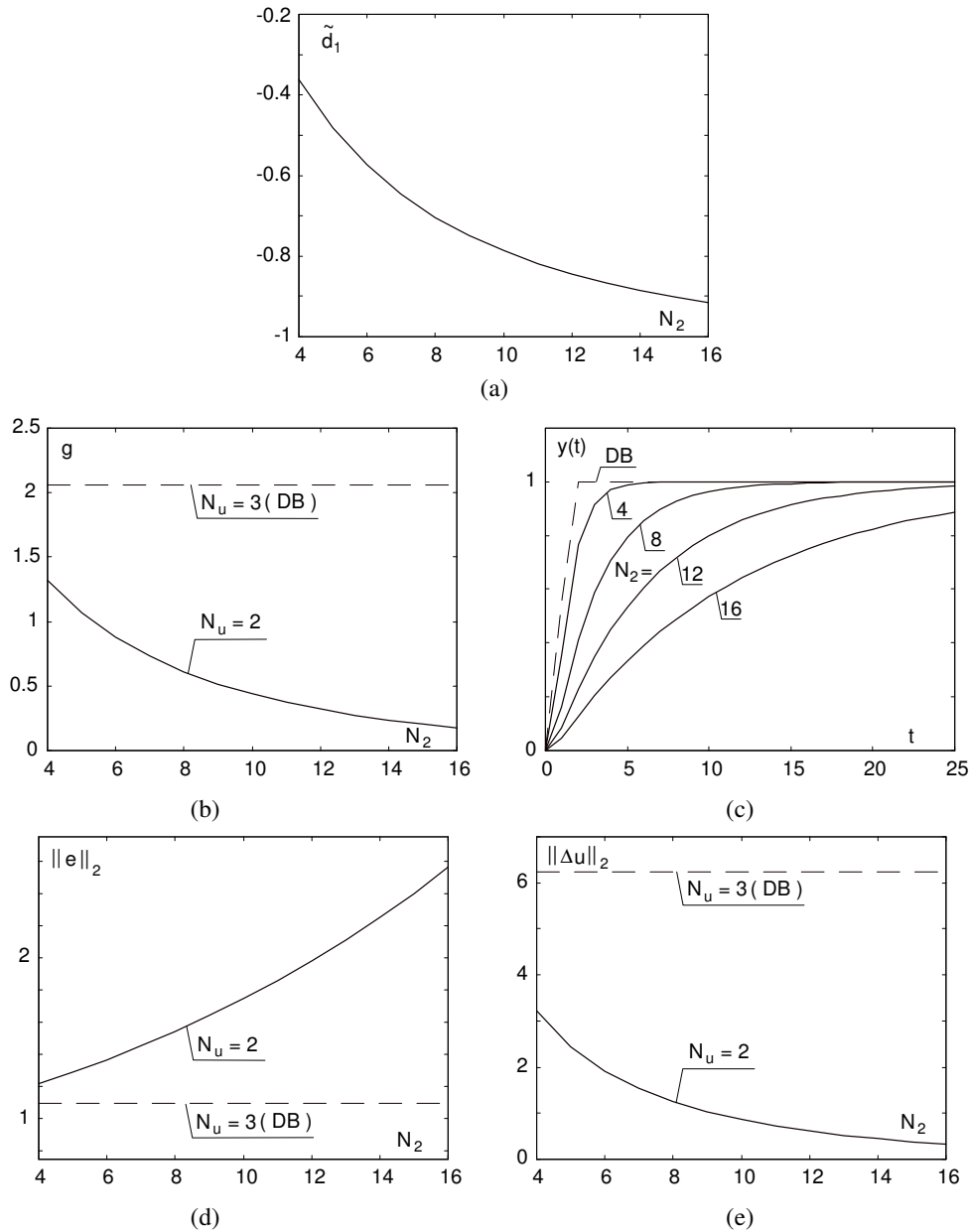


Fig. 9. Example 1: GPC control based on  $\tilde{D}(q^{-1})$  of the first degree.

a critical impact on the design process, including the  $N_\Lambda$ -detection procedure. At the same time, it is perceptible that the appearance of a double zero at  $q=0.2$ , which is partially cancelled in the entire numerical model  $B(q^{-1})/A(q^{-1})$  of the plant, has no effect on our GPC design.

### 5.2.1. Detection of the Cancellation Order

With  $N_A = 9$  and  $N_B = 10$ , similarly as before (by virtue of Lemma 5), we infer that it is the matrix  $\mathbf{H}_{10,19}^{10}$  (with  $N_u = 10$ ) that should be the subject of our exam-

ination. Moreover, let  $N_q = 4$ . The indices of the angular distance (31), the relative gap (47) and the direct  $\Lambda$ -estimation index (43), denoted by  $\kappa_{10,19}^i$ ,  $\rho_{10,19}^i$  and  $J_4^m$ , respectively, were computed for the above two models and depicted in Figs. 11 and 12, respectively.

As can be seen from the above, only the utilization of  $J_4^m$  leads to the right solution. The other methods based on the indices  $\kappa_{10,19}^i$  and  $\rho_{10,19}^i$  generally fail by roughly advising that the cancellation order is  $N_\Lambda = 2$ , or, alternatively, based on  $\rho_{10,19}^i$  from Fig. 12(b), that  $N_\Lambda = 4$  (which could have resulted from the additional near cancellation).

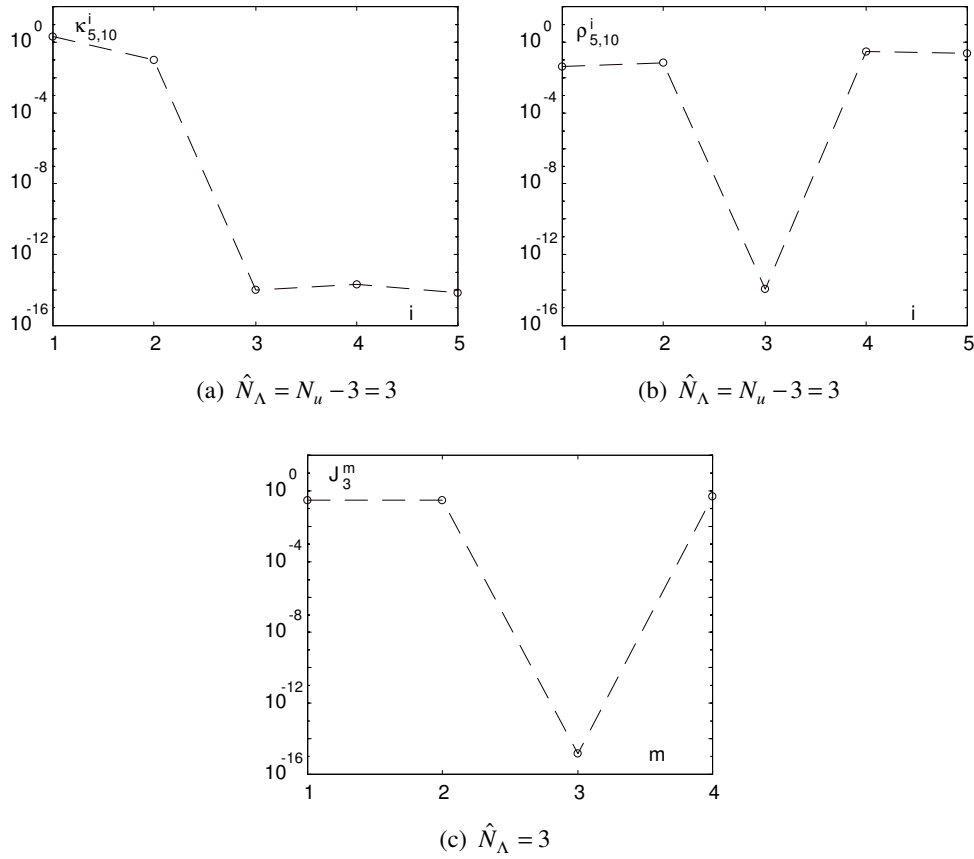


Fig. 10. Example 1: Indices of the detection of the cancellation order.

### 5.2.2. Design Based on the Region $(\cdot)'_-$

Once we have obtained the correct cancellation order and assumed the design simplifying terms (making also the DB-setting possible):  $\lambda = 0$  and  $g^* = 0$ , three GPC controllers can be completely designed based on the tuning specifications concerning the region  $(\cdot)'_-$  of GPC solvability and given in Theorem 1 (for  $\tilde{D}(q^{-1})$  of the three lowest degrees, respectively). Moreover, let us assume  $C(q^{-1}) = (1 - 0.3q^{-1})(1 - 0.4q^{-1})(1 - 0.5q^{-1})$ .

Considering the model with  $B'_I(q^{-1})$  and taking  $N_1 = 7$  together with  $N_2 = 13$ , we achieve GPC solutions of the following characteristics:

- (i)  $N_u = 7$ :  $\deg \tilde{D}(q^{-1}) = 0$  (DB):  $g = 0.6614$ ,  
 $\|e\|_2 = 2.0698$ ,  $\|\Delta u\|_2 = 5.9979$  ;
- (ii)  $N_u = 6$ :  $\deg \tilde{D}(q^{-1}) = 1$ :  $g = 0.4205$ ,  
 $\|e\|_2 = 2.0733$ ,  $\|\Delta u\|_2 = 3.0428$ ,  
 $\tilde{d}_1 = -0.3641$ ;
- (iii)  $N_u = 5$ :  $\deg \tilde{D}(q^{-1}) = 2$ :  $g = 0.2235$ ,  
 $\|e\|_2 = 2.1648$ ,  $\|\Delta u\|_2 = 1.2152$ ,  
 $\tilde{d}_1 = -0.8600$ ,  $\tilde{d}_2 = 0.1980$ .

The plots given in Fig. 13 describe the step responses and control signals for the above systems. The results confirm that enlarging  $N_u$  intensifies the control action. A further improvement in closed-loop performance is possible by utilizing the AF mechanism and the  $N_2$ -tuning (Kowalczyk *et al.*, 1996; Kowalczyk and Suchomski, 2001; 2002; Suchomski and Kowalczyk, 2002a).

## 6. Concluding Remarks

The overparameterization of the identified models used in adaptive systems can be considered as a suitable treatment of the uncertainty or variability of identified and controlled plant structures. What is more, in the case of system non-stationarity, noise, disturbances, and other systemic properties, as well as due to limited numerical (finite wordlength) mechanization, approximate cancellations can always occur in the system model being processed. In a usual course of the control system design such models give rise to severe difficulties (unfeasibility of design goals, ill-conditioning of design solutions, rank deficiency of design matrices, large-valued controller coefficients, degraded closed-loop performance, etc.).

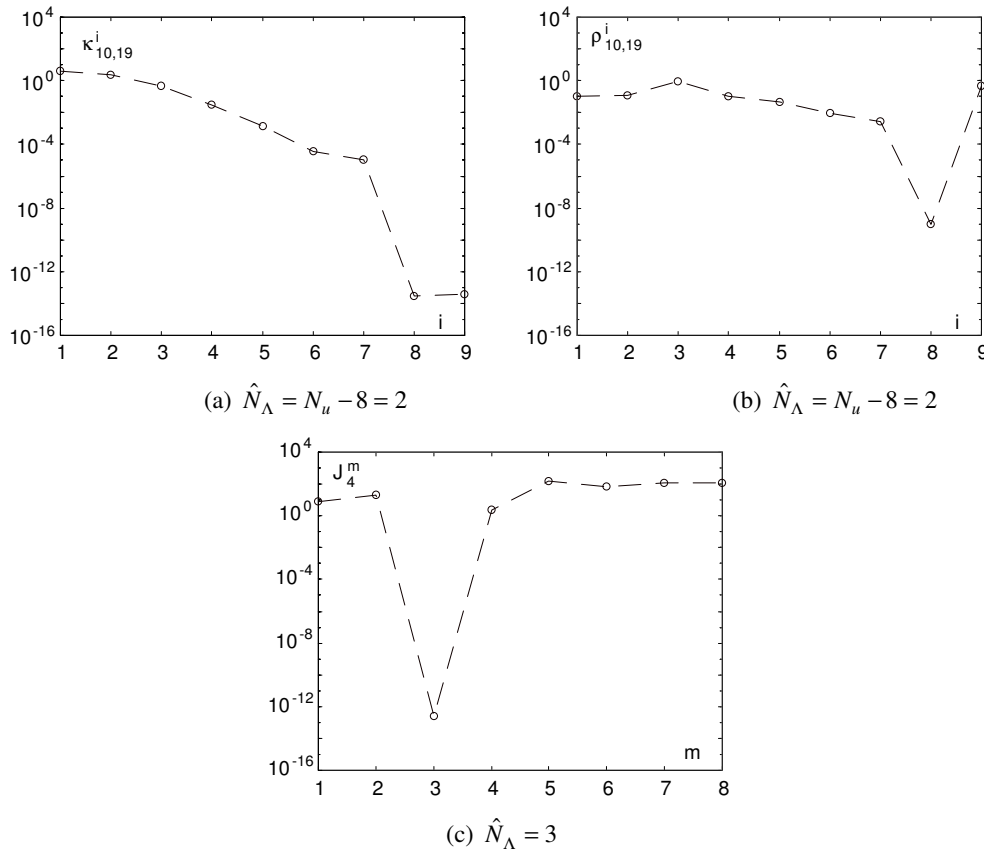


Fig. 11. Example 2: Detecting the cancellation order for the model  $(A'(q^{-1})\Lambda(q^{-1}), B'_l(q^{-1})\Lambda(q^{-1}))$ .

A discrete-time AF-GPC controller design procedure appropriate for plants with a non-zero transportation delay and robust to an overparameterization of the plant model was provided along with conditions for the existence and stability of solutions of the AF-GPC design in terms of relevant GPC design parameters. In this context, four basic GPC-solvability regions were distinguished based on a rigorous proof. Moreover, explicit analytical forms of GPC closed-loop characteristic polynomials were supplied.

From the systems theory viewpoint, our GPC synthesis conception based on the Toeplitz matrix of system Markov parameters can be considered as dual (complementary) to the classical Kalman-Ho identification-based methodology using Hankel matrices.

With the approach to the estimation of the cancellation order suggested in this paper and, in particular, with the algorithm based on the direct  $\Lambda$ -estimation index, we are able to identify models of an overestimated order, evaluate the model cancellation order, recalculate the effective orders of system polynomials, and design a suitable controller of a reasonably low order. Thus the set of tuning rules acquired from the derived solvability conditions can be safely used in adaptive control. The necessary design calculations can be performed with the aid

of suitable GPC-design algorithms. These issues are discussed, for instance, in (Kowalczyk *et al.*, 1996; Kowalczyk and Suchomski, 2001; 2002; Suchomski and Kowalczyk, 2002a).

Consequently, we hope to fill up the exiting gap connected with the lack of explicit tuning rules in the classical literature on predictive control that can impair using the stable GPC procedure in practice. Moreover, apart from the challenge concerning overparameterization, solvability, system structure and complexity, the proposed methodology can be considered as a functional contribution to adaptive GPC control system applications, basically, in the framework of GPC without constraints. Though, in view of the tuning results obtained, one can take into consideration the possibility of controlling both system stability and the trajectories of plant input and output signals by means of the defined set of design parameters.

The illustrating numerical material highlights certain general design problems and the properties of GPC design methodology for overparameterized plant models, as well as confirms the applicability of the proposed approach. The presented numerical examples also portray the way in which our approach insures both the stability of the closed-loop system and a limitation of the control sig-

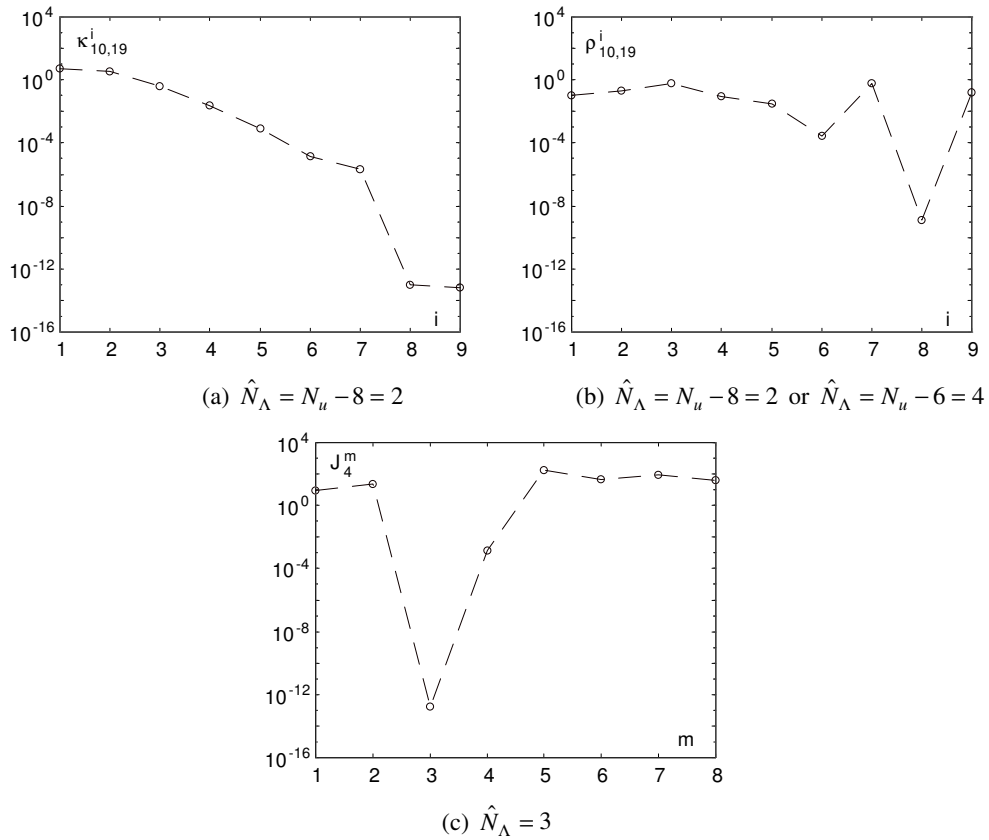


Fig. 12. Example 2: Detecting the cancellation order for the model  $(A'(q^{-1})\Lambda(q^{-1}), B'_{II}(q^{-1})\Lambda(q^{-1}))$ .

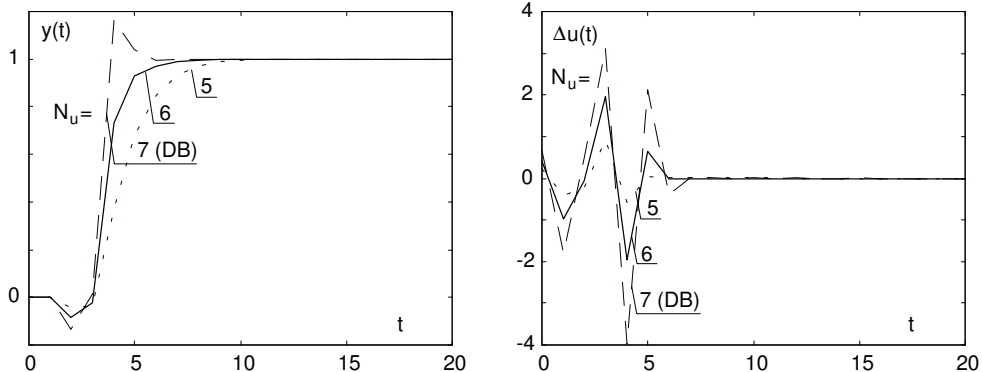


Fig. 13. Example 2: GPC system performance for the model  $(A'(q^{-1})\Lambda(q^{-1}), B'_I(q^{-1})\Lambda(q^{-1}))$ .

nal, as opposed to the regular final constrained receding-horizon control approach, which is inappropriate in the presence of common factors.

With overparameterization inducing the well-known problem of pole-zero cancellations, the estimation of the cancellation order is necessary to obtain the characteristic polynomial of a desired degree. To do so, we consider two ways: the classic SVD-based approach, the effectiveness of which depends on an *a priori*-set threshold, and a new one based on a distance measure between subsequent so-

lutions of a properly defined set of Diophantine equations. The merit of the latter method, based on a numerically robust algorithm, has also been explained by two complementary design computation examples given in the text.

The fact that the obtained practical tuning rules have been analytically derived due to the simplifying assumption that  $\lambda$  is equal to zero does not imply any design restriction. Indeed a non-zero  $\lambda$  can always be used in the GPC design after the proposed arrangement of pre-design tuning parameters.

In this context, it is also worth noticing that discrete-time GPC designs, embedded in the pole-placement or dead-beat paradigms, often lead to an excessive control action that can easily result in a very limited practicable range of  $\lambda$ , which implies numerical difficulties (note that, in general, non-zero  $\lambda$  does not necessarily assure the desired matrix regularization) and some other related sensitivity problems (the value of  $\lambda$  used in minimizing the cost function can appear to be too small to be feasible in a practical context).

In conclusion, let us also annex some remarks on anticipative filtering (AF) applied to the GPC design strategy in order to fictitiously abate the command signal. As has been shown in (Kowalczyk and Suchomski, 1999; 2002; Suchomski and Kowalczyk, 2002a), AF can have a desired effect on both the closed-loop system behavior and the control effort (thus, to some extent, the effect of AF is similar to that of non-zero  $\lambda$ ). This leads to an improved balance within the combined cost function, what makes the  $\lambda$ -tuning feasible and permits an alleviation of the control effort. At the same time, as results from the mathematical characteristics of the AF-GPC design that consist in reducing the effect of the AF filter to a scalar outer-loop gain coefficient ( $g$ ), the simplest AF mechanization with a single knob  $r = r_{N_1}$  is sufficient.

Furthermore, even though we have considered the unconstrained design case, the results can be used in more general settings. Namely, it is possible to use the proposed analytical methodology by suitably handling GPC design parameters within their tuning scopes defined by the lemmas and theorems recommended in this paper, with the purpose of shaping the nominal output and control signals (balancing the control effort and the regulation speed, for instance). Moreover, on a similar basis, the control weighting factor  $\lambda$  can be used to additionally re-tune the GPC control system in order, for instance, to comply with some control signal restrictions (at the same time, the standard root-locus technique based on the prediction control gain can allow the designer to preserve some analytical character of the design). Such a  $\lambda$ -optimization objective, representing yet another design space not considered in this paper, can also be explored in the constrained control case (cf. also Maciejowski, 2002; Rossiter, 2003). Certainly, this could be an interesting subject of another study.

Another interesting search direction both from the practical and theoretical points of view is associated with the problem of a control system's robustness to modeling uncertainty (Rossiter, 2003). Within this subject, a simple contribution to robust GPC was presented in (Kowalczyk and Suchomski, 1998), whereas an effective Youla-Kučera approach to the robustification of continuous-time C-GPC systems was proposed in (Kowalczyk and Suchomski, 2004).

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## Appendix A

### MBPC–GPC Odyssey

The original MBPC strategies are mostly based on input/output models of the controlled plant (Camacho and Bordons, 1999; Maciejowski, 2002; Rossiter, 2003). Let some of these strategies be listed below:

- MPHPC – model predictive heuristic control derived from the impulse response representation of the controlled plant (Richalet *et al.*, 1978),
- MAC – model algorithmic control also using the plant impulse response (Rouhani and Mehra, 1982),
- DMC – dynamic matrix control based on the step response of the plant (Cutler and Ramaker, 1980; Georgiou *et al.*, 1988; Hinde and Cooper, 1994),
- GDMC – generalized dynamic matrix control utilizing the step response model of the plant (Xi, 1989),
- EHAC – extended horizon adaptive control applying the variable forgetting factor (VFF) method for the identification of a polynomial (ARMA) model of the plant (Ydstie, 1984; Ydstie *et al.*, 1985),



- EPSAC – extended prediction self-adaptive control built on parallel CARMA models (de Keyser and van Cauwenberghe, 1981),
- GPC – generalized predictive control (Clarke *et al.*, 1987; Clarke and Mohtadi, 1989) considered in this paper.

Having in mind the advantageous properties of state equations, effective extensions of the MBPC paradigm to multivariable plants are also essential. Some key observations concerning the state-space reformulation of MBPC can be found in (Li *et al.*, 1989; Ricker, 1990; Berlin and Frank, 1992; Kwon *et al.*, 1992b; Ordys and Clarke, 1993; Lee *et al.*, 1994; Elshafei *et al.*, 1995). A variety of multivariable algorithms recently developed, i.e.,

- RHTC – receding horizon tracking control (Kwon and Byun, 1989; Kwon *et al.*, 1992a),
- CRHPC – constrained receding horizon predictive control (Clarke and Scattolini, 1991; Scokaert and Clarke, 1994),
- SIORHC – stabilizing I/O receding horizon control (Mosca and Zhang, 1992),
- MGPC – multivariable generalized predictive control (Shah *et al.*, 1987),
- MSGPC – multivariable stable generalized predictive control (Kouvaritakis and Rossiter, 1993; Kouvaritakis *et al.*, 1997), and
- LQGPC – linear quadratic generalized predictive control (Grimble, 1990; 1993; Taube and Lampe, 1992; Hangstrup *et al.*, 1997),

well illustrate the broad applicability of the predictive control paradigm.

There are stability outcomes concerning the receding horizon LQ control both in the discrete- and continuous-time domains (Kleinman, 1974; Kwon and Pearson, 1975, 1978; Longchamp, 1983). These results were employed in a redesign of GPC controllers (Clarke and Scattolini, 1991; Mosca and Zhang, 1992), with a finite end-point weighting (i.e., equality constraints on the system output) added to the usual GPC cost. On the other hand, additional weighting on a predicted ultimate system-state was utilized in (Demircioglu and Clarke, 1993; Jolly and Bentsman, 1993), where it was demonstrated that closed-loop stability can be guaranteed by choosing sufficiently robust weighting. A similar technique of receding horizon predictive control (Yoon and Clarke, 1993) utilizes exponentially increasing weights on the predicted tracking errors. Another application of the end-point constraint principle along with a stabilizing inner feedback

loop called SGPC (stable generalized predictive control) is presented in (Kouvaritakis *et al.*, 1992), where the minimization of a control error norm leads to a monotonically decreasing cost, which guarantees closed-loop stability and asymptotic tracking. The results of SGPC from the input-output framework are extended to state-space models taking into account numerical conditioning issues (Kouvaritakis and Rossiter, 1993; Kouvaritakis *et al.*, 1997; Rossiter and Kouvaritakis, 1994; Rossiter *et al.*, 1998; Rossiter, 2003). Moreover, as was shown in (Rossiter, 1997), the constrained receding horizon predictive control CRHPC (Clarke and Scattolini, 1991), the stabilizing I/O receding horizon control SIORHC (Mosca and Zhang, 1992) and the SGPC (Kouvaritakis *et al.*, 1992) approaches implement the same control law.

A modified GPC routine combined with a pole-placement technique is presented in (Peng *et al.*, 1992; 1993), where a set of controller parameters is fixed by solving a set of non-linear algebraic equations, and the control law is implemented in a receding horizon manner. In turn, in (Lim *et al.*, 1998), a GPC design method is reported where the closed-loop poles are located in a restricted region determined by the required settling time and overshoot of the control system step response.

Several simple rules for tuning MBPC controllers can be found in (Scattolini and Bittanti, 1990; Taube, 1991; Banerjee and Shah, 1992). Some generic guidelines for the selection of tuning parameters (time horizons)  $N_1$ ,  $N_2$  and  $N_u$  of the basic GPC algorithm are given in (Mohtadi and Clarke, 1986; Clarke *et al.*, 1987; Clarke, 1988; Clarke and Mohtadi, 1989). A comprehensive presentation of studies on tuning GPC can be found in (Kowalczyk and Suchomski, 2002; Suchomski and Kowalczyk, 2002a). Specifically, in (McIntosh *et al.*, 1991) it is illustrated how the GPC control law can be expressed by an equivalent linear transfer function, and certain simple tuning strategies are recommended as well. The frequently considered case of GPC systems with a one-step control horizon was examined in (Elshafei *et al.*, 1991). Sufficient conditions for dead-beat settings of the GPC controller given in (Peng and Hanus, 1991) can be used to make the GPC strategy (suitably augmented) equivalent to the common pole-placement design. Several existing tuning guidelines for predictive controllers, including the GPC algorithm, were investigated in (Rani and Unbehauen, 1997), where an auto-tuning procedure based on the speed and shape of the closed-loop system response and on its integral squared errors was also proposed. An incomplete theorem about reducing the GPC closed-loop order by properly choosing tuning parameters along with some results concerning the dead-beat GPC control was proposed in (Zhang, 1996). Workable design guidelines for plants with monotone or convex step responses were given in (Zhang and Xi, 1998) along with a conjecture

that the minimal dynamic matrix controller (DMC) can be regarded as a specific type of the GPC control law. Extensive simulations (Krämer and Unbehauen, 1992; Rani and Unbehauen, 1996, 1997) illustrate the usefulness of the MBPC methods under study. Genetic approaches applied to the optimization of design parameters of GPC controllers can also be found in the literature (Filali and Wertz, 2001).

## Appendix B State Space Models

**B1.** The order  $n$  of the irreducible model of the incremental control channel  $B(q^{-1})/\hat{A}(q^{-1})$  is equal to the dimension of a minimal state space representation associated with this input-output model. The order  $n$  can also be established by expressing the transfer function model  $B(q^{-1})/\hat{A}(q^{-1})$  in the forward shift operator  $q$ . If  $N_B > N_A + 1$ , the model has  $N_B - N_A - 1$  poles at the origin of the complex plane ( $z = 0$ ). Assuming that  $n_B \leq N_B \leq N_A + 1$ , we can constrain our deliberations to the models  $B(q^{-1})/\hat{A}(q^{-1})$  having no zero poles:

$$\begin{aligned} n &= \text{ord}(B(q^{-1})/\hat{A}(q^{-1})) = \max\{N_A + 1, N_B\} \\ &= N_A + 1. \end{aligned}$$

**B2.** In order to clarify the number of the properties of such models let us recall their controllable canonical state-space form  $(\hat{A}, \hat{b}, \hat{c})$  with  $\hat{A} \in \mathbb{R}^{n \times n}$ ,  $\hat{b} \in \mathbb{R}^n$  and  $\hat{c} \in \mathbb{R}^{1 \times n}$ :

$$\hat{A} = \begin{bmatrix} \mathbf{0} & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \\ -\hat{a}_{N_A+1} & -\hat{a}_{N_A} & \cdots & -\hat{a}_1 \end{bmatrix}, \quad (\text{B.1})$$

$$\hat{b} = \begin{bmatrix} \mathbf{0}_{N_A} \\ 1 \end{bmatrix}, \quad (\text{B.2})$$

$$\hat{c} = \begin{cases} \begin{bmatrix} b_{N_B} & \cdots & b_{n_B} \end{bmatrix} & \text{if } N_B = N_A + 1, n_B = 1, \\ \begin{bmatrix} b_{N_B} & \cdots & b_{n_B} & \mathbf{0}_{n_B-1}^T \end{bmatrix} & \text{if } N_B = N_A + 1, n_B > 1, \\ \begin{bmatrix} \mathbf{0}_{N_A-N_B+1}^T & b_{N_B} & \cdots & b_{n_B} \end{bmatrix} & \text{if } N_B < N_A + 1, n_B = 1, \\ \begin{bmatrix} \mathbf{0}_{N_A+1-N_B}^T & b_{N_B} & \cdots & b_{n_B} & \mathbf{0}_{n_B-1}^T \end{bmatrix} & \text{if } N_B < N_A + 1, n_B > 1. \end{cases} \quad (\text{B.3})$$

As there are no zero poles, the state matrix  $\hat{A}$  has a full rank:  $\text{rank} \hat{A} = N_A + 1$ .

**B3.** Considering the minimal state-space model  $(\hat{A}, \hat{b}, \hat{c})$  associated with  $\bar{B}(q^{-1})/\hat{A}(q^{-1})$ , we immediately find that  $h_i = \hat{c} \hat{A}^i \hat{b}$ ,  $i = 0, 1, \dots$ .

## Appendix C Classical Kalman-Ho System Theory using Hankel Matrices

Recall that Markov parameters of linear time-invariant (LTI) systems play an important role in deterministic realization theory. The first solution to the determination of a minimal state-space representation from impulse response data was given in a seminal work by Ho and Kalman (1966). The problem consists of two sub-problems: (i) how to determine the McMillan degree of an input/output rational model from a potentially infinite set of data, and (ii) how to compute the matrix parameters  $(A, B, C, D)$  of the corresponding state-space model.

The generic solution (Ho and Kalman, 1966) is based on the factorization of a Hankel matrix composed of system Markov parameters into the product of an indefinite observability matrix and an indefinite controllability matrix (similar results were independently gained by other researchers, cf. e.g., (Silverman, 1971; Tether, 1970)). Many improvements of the basic Kalman-Ho approach were proposed. Specifically, it is clear that in real conditions, e.g., when data are contaminated by noise, the Hankel matrix tends to have a full rank. This makes the problem of determining a minimum-order state-space model non-trivial. Consequently, some approximate finite-dimensional techniques have to be applied (seemingly, the most reliable approaches to the problem of guessing the ‘true’ order of the system are based on the SVD technique (Zeiger and McEwen, 1974; Kung, 1978)). An effective recursive solution to the minimal partial state-space realization problem based on a decomposition of submatrices of the Hankel matrix was given in (Rissanen, 1971). Contemporary versions of such algorithms can be found in (van Overschee *et al.*, 1997).

## Appendix D Properties of Markov Matrices

**D1.** The relationship between the pair of polynomials  $(\hat{A}(q^{-1}), B(q^{-1}))$  and the Markov parameters  $h_i$ ,  $i = 0, 1, \dots$ , of the model  $\bar{B}(q^{-1})/\hat{A}(q^{-1})$  can be expressed by applying the following doubly infinite lower-triangular

Toeplitz-like matrix:

$$n_b \left\{ \begin{array}{cccc} 0 & 0 & 0 & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots \\ h_{n_b} & 0 & 0 & \cdots \\ h_{n_b+1} & h_{n_b} & 0 & \cdots \\ h_{n_b+2} & h_{n_b+1} & h_{n_b} & \cdots \\ \cdots & \cdots & \cdots & \ddots \end{array} \right\} \begin{bmatrix} \hat{a}_0 \\ \hat{a}_1 \\ \vdots \\ \hat{a}_{N_A+1} \\ 0 \\ \vdots \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \\ b_{n_B} \\ \vdots \\ b_{N_B} \\ 0 \\ \vdots \end{bmatrix} \Bigg\} n_b \quad (D.1)$$

**D2.** Let an auxiliary vector  $\mathbf{h}_{N_1, N_2}^k \in \mathbb{R}^{N_0+k}$  with  $k \geq N_1 - N_2$  be defined as

$$\mathbf{h}_{N_1, N_2}^k = \begin{cases} \begin{bmatrix} h_{N_1-1} & \cdots & h_{N_2-1} \end{bmatrix}^T & \text{if } N_1 - N_2 \leq k \leq 0, \\ \begin{bmatrix} \mathbf{0}_k^T & h_{N_1-1} & \cdots & h_{N_2-1} \end{bmatrix}^T & \text{if } k > 0. \end{cases} \quad (D.2)$$

Note that for  $k \leq 0$ , we have  $\mathbf{h}_{N_1, N_2}^k = \mathbf{h}_{N_1-k, N_2}^0$ . For  $N_u > 1$ , the following two partitions of the Markov matrix  $\mathbf{H}_{N_1, N_2}^{N_u}$  will play an important role in the further development:

$$\mathbf{H}_{N_1, N_2}^{N_u} = \begin{bmatrix} \mathbf{h}_{N_1, N_2}^0 & \mathbf{H}_{N_1-1, N_2-1}^{N_u-1} \end{bmatrix}, \quad (D.3)$$

$$\mathbf{H}_{N_1, N_2}^{N_u} = \begin{bmatrix} \mathbf{H}_{N_1, N_2}^{N_u-1} & \mathbf{h}_{1, N_2-N_u+1}^{N_u-N_1} \end{bmatrix}. \quad (D.4)$$

**D3.** Assuming that  $N_u > N_A + 1$  and employing the equation (D.1), we obtain

$$\mathbf{H}_{N_1, N_2}^{N_u} \hat{\mathbf{a}}_{N_u}^{N_A+1} = \mathbf{b}_{N_1, N_2}^{N_B}, \quad (D.5)$$

where  $\hat{\mathbf{a}}_{N_u}^{N_A+1} \in \mathbb{R}^{N_u}$  is given by

$$\hat{\mathbf{a}}_{N_u}^{N_A+1} = \begin{cases} \hat{\mathbf{a}} & \text{if } N_u = N_A + 2 \\ \begin{bmatrix} \hat{\mathbf{a}}^T & \mathbf{0}_{N_u-N_A-2}^T \end{bmatrix}^T & \text{if } N_u > N_A + 2 \end{cases} \quad (D.6)$$

with the vector  $\hat{\mathbf{a}} \in \mathbb{R}^{N_A+2}$  defined as  $\hat{\mathbf{a}} = [\hat{a}_0 \ \hat{a}_1 \ \cdots \ \hat{a}_{N_A+1}]^T = [\hat{a}_0 \ \tilde{\mathbf{a}}^T]^T$ , and likewise,  $\tilde{\mathbf{a}} = [\hat{a}_1 \ \hat{a}_2 \ \cdots \ \hat{a}_{N_A+1}]^T$ ,  $\tilde{\mathbf{a}} \in \mathbb{R}^{N_A+1}$ , while  $\mathbf{b}_{N_1, N_2}^{N_B} \in \mathbb{R}^{N_2-N_1+1}$  takes the following form:

$$\mathbf{b}_{N_1, N_2}^{N_B} = \begin{cases} \begin{bmatrix} \mathbf{0}_{N_2-N_1+1} & \cdots & b_{N_B} & \mathbf{0}_{N_2-N_B}^T \end{bmatrix}^T & \text{if } N_1 > N_B \vee N_2 < n_B, \\ \begin{bmatrix} b_{N_1} & \cdots & b_{N_2} \end{bmatrix}^T & \text{if } N_1 \geq n_B \wedge N_2 > N_B, \\ \begin{bmatrix} b_{N_1} & \cdots & b_{N_2} \end{bmatrix}^T & \text{if } N_1 \geq n_B \wedge N_2 \leq N_B, \\ \begin{bmatrix} \mathbf{0}_{n_B-N_1}^T & b_{n_B} & \cdots & b_{N_B} & \mathbf{0}_{N_2-N_B}^T \end{bmatrix}^T & \text{if } N_1 < n_B \wedge N_2 > N_B, \\ \begin{bmatrix} \mathbf{0}_{n_B-N_1}^T & b_{n_B} & \cdots & b_{N_2} \end{bmatrix}^T & \text{if } N_1 < n_B \wedge N_2 \leq N_B. \end{cases} \quad (D.7)$$

If  $N_u = N_A + 2$ , then (D.5) becomes  $\mathbf{H}_{N_1, N_2}^{N_A+2} \hat{\mathbf{a}}_{N_A+2}^{N_A+1} = \mathbf{b}_{N_1, N_2}^{N_B}$ . Thus (D.3) and (D.6) result in

$$\mathbf{h}_{N_1, N_2}^0 = \mathbf{b}_{N_1, N_2}^{N_B} - \mathbf{H}_{N_1-1, N_2-1}^{N_A+1} \tilde{\mathbf{a}}. \quad (D.8)$$

**D4.** Consider  $\mathbf{H}_{N_1, N_2}^{N_u}$  with  $N_1 \geq N_u$ . In this case, the  $(i, j)$ -th entry of  $\mathbf{H}_{N_1, N_2}^{N_u}$ , i.e., the Markov parameter  $h_{N_1+i-j-1}$ ,  $1 \leq i \leq N_2 - N_1 + 1$  and  $1 \leq j \leq N_u$ , can be represented as  $h_{N_1+i-j-1} = \hat{\mathbf{c}} \hat{\mathbf{A}}^{N_1+i-j-1} \hat{\mathbf{b}}$ . Thus

$$\mathbf{H}_{N_1, N_2}^{N_u} = \begin{bmatrix} \hat{\mathbf{c}} \\ \hat{\mathbf{c}} \hat{\mathbf{A}} \\ \vdots \\ \hat{\mathbf{c}} \hat{\mathbf{A}}^{N_2-N_1} \end{bmatrix} \times \hat{\mathbf{A}}^{N_1-N_u} \begin{bmatrix} \hat{\mathbf{A}}^{N_u-1} \hat{\mathbf{b}} & \cdots & \hat{\mathbf{A}} \hat{\mathbf{b}} & \hat{\mathbf{b}} \end{bmatrix}. \quad (D.9)$$

## Appendix E

### Proof of Lemma 1

**E1. Sufficiency of the conditions (1')–(3').** Assuming that the claims (1') and (3'), i.e.,  $N_u = N_A + 1$  and  $N_2 \geq N_1 + N_A$ , respectively, provide a suitable means of distinguishing between the following two sub-conditions related to the basic region border (2') of Lemma 1:

- (i)  $N_1 \geq N_A + 1$  (in this case we have  $N_1 \geq N_B$ ),
- (ii)  $N_1 \leq N_A$  (valid only if  $N_B \leq N_1 \leq N_A$ ).

Let us start from the inequality (i). If  $N_1 \geq N_A + 1$  and  $N_u = N_A + 1$ , from (D.9) we have

$$\mathbf{H}_{N_1, N_2}^{N_A+1} = \begin{bmatrix} \hat{c} \\ \hat{c}\hat{\mathbf{A}} \\ \vdots \\ \hat{c}\hat{\mathbf{A}}^{N_2-N_1} \end{bmatrix} \times \hat{\mathbf{A}}^{N_1-N_A-1} \begin{bmatrix} \hat{\mathbf{A}}^{N_A} \hat{\mathbf{b}} & \dots & \hat{\mathbf{A}} \hat{\mathbf{b}} & \hat{\mathbf{b}} \end{bmatrix}. \quad (\text{E.1})$$

The coprimeness of  $\hat{\mathbf{A}}(q^{-1})$  and  $B(q^{-1})$  implies that the two sets of (column and row) vectors  $\{\hat{\mathbf{A}}^i \hat{\mathbf{b}}\}_{i=0}^{N_A}$  and  $\{\hat{c}\hat{\mathbf{A}}^i\}_{i=0}^{N_2-N_1}$ , respectively, are linearly independent. Since  $N_2 - N_1 + 1 \geq N_A + 1$ , we have

$$\text{rank} \begin{bmatrix} \hat{\mathbf{A}}^{N_A} \hat{\mathbf{b}} & \dots & \hat{\mathbf{A}} \hat{\mathbf{b}} & \hat{\mathbf{b}} \end{bmatrix} = \text{rank} \begin{bmatrix} \hat{c} \\ \hat{c}\hat{\mathbf{A}} \\ \vdots \\ \hat{c}\hat{\mathbf{A}}^{N_2-N_1} \end{bmatrix} = N_A + 1. \quad (\text{E.2})$$

Combining the above with the result that  $\text{rank } \hat{\mathbf{A}} = N_A + 1$  yields

$$\text{rank } \mathbf{H}_{N_1, N_2}^{N_A+1} \Big|_{\substack{N_1 \geq N_A + 1 \\ N_2 \geq N_1 + N_A}} = N_A + 1. \quad (\text{E.3})$$

Considering the inequality (ii), we conclude that (D.7) together with the assumption that  $N_1 > N_B$  enforce the zeroing of  $\mathbf{b}_{N_1, N_2}^{N_B}$ :  $\mathbf{b}_{N_1, N_2}^{N_B} = \mathbf{0}_{N_2-N_1+1}$ . Consequently, (D.8) generates

$$\mathbf{h}_{N_1, N_2}^0 = -\mathbf{H}_{N_1-1, N_2-1}^{N_A+1} \tilde{\mathbf{a}} \quad \text{if } N_1 > N_B, \quad (\text{E.4})$$

$$\mathbf{h}_{N_1+1, N_2+1}^0 = -\mathbf{H}_{N_1, N_2}^{N_A+1} \tilde{\mathbf{a}} \quad \text{if } N_1 \geq N_B. \quad (\text{E.5})$$

Now we shall proceed with the following scheme by induction:

- (iii)  $N_1 = N_A$ ,
- (iv)  $N_1 = N_A - 1$ ,
- and so on.

Considering the setting (iii) and taking  $N_2 \geq N_1 + N_A$  (see also the basic region limitation (3')), by virtue of (E.5) we have  $\mathbf{h}_{N_A+1, N_2+1}^0 = -\mathbf{H}_{N_A, N_2}^{N_A+1} \tilde{\mathbf{a}}$ . This means that  $\mathbf{h}_{N_A+1, N_2+1}^0$  is a linear combination of the columns of  $\mathbf{H}_{N_A, N_2}^{N_A+1}$ . As  $\hat{a}_{N_A+1} \neq 0$ , the last column of this matrix, i.e.,  $\mathbf{h}_{1, N_2-N_A}^1$ , can be substituted by  $\mathbf{h}_{N_A+1, N_2+1}^0$  without affecting the rank of  $\mathbf{H}_{N_A, N_2}^{N_A+1}$ .

$$\begin{aligned} & \text{rank } \mathbf{H}_{N_1, N_2}^{N_A+1} \Big|_{N_1=N_A} \\ &= \text{rank} \begin{bmatrix} \mathbf{H}_{N_A, N_2}^{N_A} & \mathbf{h}_{N_A+1, N_2+1}^0 \end{bmatrix}. \quad (\text{E.6}) \end{aligned}$$

By interchanging the first and the last column of the matrix  $\begin{bmatrix} \mathbf{H}_{N_A, N_2}^{N_A} & \mathbf{h}_{N_A+1, N_2+1}^0 \end{bmatrix}$  the rank is not affected. Thus

$$\begin{aligned} & \text{rank } \mathbf{H}_{N_1, N_2}^{N_A+1} \Big|_{N_1=N_A} \\ &= \text{rank} \begin{bmatrix} \mathbf{h}_{N_A+1, N_2+1}^0 & \mathbf{H}_{N_A, N_2}^{N_A} \end{bmatrix}. \quad (\text{E.7}) \end{aligned}$$

Now, taking into account (D.3), we obtain

$$\text{rank } \mathbf{H}_{N_1, N_2}^{N_A+1} \Big|_{N_1=N_A} = \text{rank } \mathbf{H}_{N_A+1, N_2+1}^{N_A+1}. \quad (\text{E.8})$$

In order to apply the effect of (E.3) to the resulting matrix

$$\mathbf{H}_{N_A+1, N_2+1}^{N_A+1} = \mathbf{H}_{N_1', N_2'}^{N_A+1} \Big|_{\substack{N_1' = N_A + 1 \\ N_2' = N_2 + 1}}, \quad (\text{E.9})$$

we should check the following two conditions:  $N_1' \geq N_A + 1$  and  $N_2' \geq N_1' + N_A$ . The first one is apparently fulfilled, as  $N_1' = N_A + 1$ . In order to deal with the other one, let observe that  $N_2' = N_2 + 1 \geq 2N_A + 1 = N_1' + N_A$ . Consequently,  $\text{rank } \mathbf{H}_{N_A+1, N_2+1}^{N_A+1} = N_A + 1$ , and

$$\text{rank } \mathbf{H}_{N_1, N_2}^{N_A+1} \Big|_{\substack{N_1 = N_A \\ N_2 \geq N_1 + N_A}} = N_A + 1. \quad (\text{E.10})$$

To perform the next induction Step (iv), let us consider the matrix  $\mathbf{H}_{N_A-1, N_2}^{N_A+1}$ . If  $N_1 = N_A - 1 \geq N_B$  and  $N_2 \geq N_1 + N_A$  hold, then from (E.5) we have  $\mathbf{h}_{N_A, N_2+1}^0 = -\mathbf{H}_{N_A-1, N_2}^{N_A+1} \tilde{\mathbf{a}}$ . Proceeding similarly to the above case (iii), we get

$$\begin{aligned} & \text{rank } \mathbf{H}_{N_1, N_2}^{N_A+1} \Big|_{N_1=N_A-1} \\ &= \text{rank} \begin{bmatrix} \mathbf{H}_{N_A-1, N_2}^{N_A} & \mathbf{h}_{N_A, N_2+1}^0 \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} \mathbf{h}_{N_A, N_2+1}^0 & \mathbf{H}_{N_A-1, N_2}^{N_A} \end{bmatrix} \\ &= \text{rank } \mathbf{H}_{N_A, N_2+1}^{N_A+1}. \quad (\text{E.11}) \end{aligned}$$

In order to employ the previously shown result (E.10) to the matrix

$$\mathbf{H}_{N_A, N_2+1}^{N_A+1} = \mathbf{H}_{N_1', N_2'}^{N_A+1} \Big|_{\substack{N_1' = N_A \\ N_2' = N_2 + 1}}, \quad (\text{E.12})$$

it should be guaranteed that  $N_2' \geq N_1' + N_A$ . Clearly, this is the case, because  $N_2' = N_2 + 1 \geq N_1 + N_A + 1 = N_1' + N_A$  and

$$\text{rank } \mathbf{H}_{N_1, N_2}^{N_A+1} \Big|_{\substack{N_1 = N_A - 1 \\ N_2 \geq N_1 + N_A}} = N_A + 1. \quad (\text{E.13})$$

The above procedure can be carried until

$$\text{rank } \mathbf{H}_{N_1, N_2}^{N_A+1} \Big|_{\substack{N_1 = N_B \\ N_2 \geq N_1 + N_A}} = N_A + 1. \quad (\text{E.14})$$

**E2. Sufficiency of the conditions (1'')–(3'').** Combining (2'') and (3'') provides  $N_2 \geq N_B + N_u - 1$ . From (D.7) it follows that for  $N_1 = N_B$  and  $N_2 > N_B$  we have

$$\mathbf{b}_{N_1, N_2}^{N_B} \Big|_{\substack{N_1 = N_B \\ N_2 > N_B}} = \begin{bmatrix} b_{N_B} \\ \mathbf{0}_{N_2 - N_B} \end{bmatrix}. \quad (\text{E.15})$$

The proof starts from the lowest admissible  $N_u$ :

- (i)  $N_u = N_A + 2$ ,
- (ii)  $N_u = N_A + 3$ ,
- ⋮

(i) In this case, (D.5) can be rewritten as  $\mathbf{H}_{N_B, N_2}^{N_A+2} \hat{\mathbf{a}} = \mathbf{b}_{N_B, N_2}^{N_B}$ . It follows that  $\mathbf{b}_{N_B, N_2}^{N_B}$  acts as a linear combination of the columns of  $\mathbf{H}_{N_B, N_2}^{N_A+2}$ . From (D.8) we conclude that  $\mathbf{b}_{N_B, N_2}^{N_B} = \mathbf{h}_{N_B, N_2}^0 + \mathbf{H}_{N_B-1, N_2-1}^{N_A+1} \tilde{\mathbf{a}}$ . It follows that the first column of  $\mathbf{H}_{N_B, N_2}^{N_A+2}$  (i.e., the vector  $\mathbf{h}_{N_B, N_2}^0$ ) can be replaced with  $\mathbf{b}_{N_B, N_2}^{N_B}$ . And the last conclusion, according to (E.15), results in

$$\begin{aligned} & \text{rank } \mathbf{H}_{N_B, N_2}^{N_u} \Big|_{N_u = N_A + 2} \\ &= \text{rank} \begin{bmatrix} \mathbf{b}_{N_B, N_2}^{N_B} & \mathbf{H}_{N_B-1, N_2-1}^{N_A+1} \end{bmatrix} \\ &= \text{rank} \begin{bmatrix} b_{N_B} & h_{N_B-2} & \cdots & h_{N_B - N_A - 2} \\ \mathbf{0}_{N_2 - N_B} & \mathbf{H}_{N_B, N_2-1}^{N_A+1} \end{bmatrix}. \end{aligned} \quad (\text{E.16})$$

Therefore

$$\text{rank } \mathbf{H}_{N_B, N_2}^{N_u} \Big|_{N_u = N_A + 2} = 1 + \text{rank } \mathbf{H}_{N_B, N_2-1}^{N_A+1}. \quad (\text{E.17})$$

Applying the first basic region (1')–(3') in order to evaluate the rank of  $\mathbf{H}_{N_B, N_2-1}^{N_A+1}$ , we have

$$\text{rank } \mathbf{H}_{N_1', N_2'}^{N_A+1} \Big|_{\substack{N_1' = N_B \\ N_2' = N_2 - 1}} = N_A + 1. \quad (\text{E.18})$$

It is a consequence of the fact that  $N_2' = N_2 - 1 \geq N_1 + N_u - 2 = N_A + N_B = N_1' + N_A$ . From this we deduce that

$$\text{rank } \mathbf{H}_{N_B, N_2}^{N_u} \Big|_{\substack{N_u = N_A + 2 \\ N_2 \geq N_B + N_u - 1}} = N_A + 2. \quad (\text{E.19})$$

(ii) Now  $N_u = N_A + 3$  and  $N_2 \geq N_A + N_B + 2$ . With reference to the matrix equation (D.1), we obtain

$$\mathbf{H}_{N_B, N_2}^{N_A+3} \begin{bmatrix} \hat{\mathbf{a}} \\ 0 \end{bmatrix} = \mathbf{b}_{N_B, N_2}^{N_B} = \begin{bmatrix} b_{N_B} \\ \mathbf{0}_{N_2 - N_B} \end{bmatrix}. \quad (\text{E.20})$$

A similar-as-before judgment applied to this case yields

$$\text{rank } \mathbf{H}_{N_B, N_2}^{N_u} \Big|_{N_u = N_A + 3} = 1 + \text{rank } \mathbf{H}_{N_B, N_2-1}^{N_A+2}, \quad (\text{E.21})$$

and, consequently,

$$\text{rank } \mathbf{H}_{N_B, N_2}^{N_u} \Big|_{\substack{N_u = N_A + 3 \\ N_2 \geq N_B + N_u - 1}} = N_A + 3. \quad (\text{E.22})$$

Almost the same development can be repeated for any  $N_u \geq N_A + 2$ , and can be shown to yield the final assertion as

$$\text{rank } \mathbf{H}_{N_B, N_2}^{N_u} \Big|_{\substack{N_u \geq N_A + 2 \\ N_2 \geq N_B + N_u - 1}} = N_u. \quad (\text{E.23})$$

## Appendix F Reconstruction of the Minimal Model ( $A'(q^{-1}), B'(q^{-1})$ )

Let  $A(q^{-1}) = A'(q^{-1})\Lambda(q^{-1})$  of  $\deg A(q^{-1}) = N_A$  and  $B(q^{-1}) = B'(q^{-1})\Lambda(q^{-1})$  of  $\deg B(q^{-1}) = N_B$  be reducible with a greatest common factor  $\Lambda(q^{-1})$  of the known degree  $\deg \Lambda(q^{-1}) = N_\Lambda > 0$ . The coefficients of the minimal polynomials  $A'(q^{-1}) = \sum_{i=0}^{N_A - N_\Lambda} a'_i q^{-i}$ ,  $a'_0 = 1$ , and  $B'(q^{-1}) = \sum_{i=n_B}^{N_B - N_\Lambda} b'_i q^{-i}$  satisfy  $a'_0 = 1$ ,  $b'_{n_B} = b_{n_B}$  and

$$\begin{bmatrix} \mathbf{T}_{N_A - N_\Lambda}^{B^0} & -\mathbf{T}_{N_B - n_B - N_\Lambda}^A \end{bmatrix} \begin{bmatrix} \mathbf{a}' \\ \mathbf{b}' \end{bmatrix} = \mathbf{t}_{ab} \quad \text{if } N_\Lambda < N_B - n_B, \quad (\text{F.1})$$

$$\mathbf{T}_{N_A - N_\Lambda}^{B^0} \mathbf{a}' = \bar{\mathbf{t}}_{ab} \quad \text{if } N_\Lambda = N_B - n_B, \quad (\text{F.2})$$

where

$$\mathbf{a}' = \begin{bmatrix} a'_1 & \cdots & a'_{N_A - N_\Lambda} \end{bmatrix}^T, \quad \mathbf{a}' \in \mathbb{R}^{N_A - N_\Lambda}, \quad (\text{F.3})$$

$$\mathbf{b}' = \begin{bmatrix} b'_{n_B+1} & \cdots & b'_{N_B-N_\Lambda} \end{bmatrix}^T, \\ \mathbf{b}' \in \mathbb{R}^{N_B-n_B-N_\Lambda} \text{ if } N_\Lambda < N_B - n_B, \quad (\text{F.4})$$

$$\mathbf{t}_{ab} = - \begin{bmatrix} b_{n_B+1} \\ \vdots \\ b_{N_B} \\ \mathbf{0}_{N_A-N_\Lambda} \end{bmatrix} + b_{n_B} \begin{bmatrix} a_1 \\ \vdots \\ a_{N_A} \\ \mathbf{0}_{N_B-n_B-N_\Lambda} \end{bmatrix}, \\ \mathbf{t}_{ab} \in \mathbb{R}^{N_A+N_B-n_B-N_\Lambda} \text{ if } N_\Lambda < N_B - n_B, \quad (\text{F.5})$$

$$\bar{\mathbf{t}}_{ab} = - \begin{bmatrix} b_{n_B+1} \\ \vdots \\ b_{N_B} \\ \mathbf{0}_{N_A-N_\Lambda} \end{bmatrix} + b_{n_B} \begin{bmatrix} a_1 \\ \vdots \\ a_{N_A} \end{bmatrix}, \\ \bar{\mathbf{t}}_{ab} \in \mathbb{R}^{N_A} \text{ if } N_\Lambda = N_B. \quad (\text{F.6})$$

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