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LMI–BASED ADAPTIVE FUZZY INTEGRAL SLIDING MODE CONTROL OF MISMATCHED UNCERTAIN SYSTEMS

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Integral sliding mode design is considered for a class of uncertain systems in the presence of mismatched uncertainties in both state and input matrices, as well as norm-bounded nonlinearities and external disturbances. A sufficient condition for the robust stability of the sliding manifold is derived by means of linear matrix inequalities. The initial existence of the sliding mode is guaranteed by the proposed control law. The improvement of the proposed control scheme performances, such as chattering elimination and estimation of norm bounds of uncertainties, is then considered with the application of an adaptive fuzzy integral sliding mode control law. The validity and efficiency of the proposed approaches are investigated through a sixth order uncertain mechanical system.

Keywords: adaptive fuzzy control, integral sliding mode, LMI, mismatched uncertainties.

1. Introduction

The relationship between models and the reality they represent is subtle and complex. The differences or errors between them define the term uncertainty. Thus, it is necessary to analyze dynamic systems subject to uncertainties. In other words, for control design purposes we need to handle simple models. However, the obtained controller must work when connected to a real system. Control strategies based on this philosophy have attracted the attention of researchers and have been widely considered. Variable Structure Control (VSC) with Sliding Modes (SMs) has been regarded as a robust technique for its insensitivity to external disturbances and model uncertainties satisfying the matching condition, i.e., perturbations that affect the system model through the input channel (Decarlo *et al*., 1988; Hung *et al*., 1993). In addition, the use of SMC can offer fast response, good transient performance and order reduction. These advantages make the SM technique very widespread in robust control design (Ha *et al*., 1999; Utkin, 1992).

The existence of the reaching phase in the resulting closed loop system yields the sensitivity of the dynamics to perturbations in an initial period of time in which the system has not yet reached the sliding manifold (Ackermann and Utkin, 1998). Hence, the new approach known as Integral SMC (ISMC) has as the main contribution the elimination of the reaching phase. The basic idea of this concept is the addition of an integral term in the sliding surface allowing immediate sliding mode emergence (Utkin and Shi, 1996). ISMC is used in the works of Mnasri and Gasmi (2007a; 2007b) and good results are obtained for large scale systems with matched uncertainties.

Application of the same method in the control of matched uncertain MIMO systems produced good performance as for robustness and tracking (Mnasri and Gasmi, 2008). However, in many cases, the uncertainties and external disturbances do not always satisfy the matching conditions. Contrary to the matched case, any mismatched uncertainty affects the behavior of the sliding mode directly, even when the ISMC approach is used. To solve this problem, the main idea is the combination of SMC with other robust techniques.

A majority of the existing methods are based on Classic SMC (CSMC) (Kim *et al*., 2000; Choi, 2001; Xia and Jia, 2003). These methods are affected by the aforementioned insufficiency of SMC with the reaching phase. Recently, much research has been focused on the advantages of ISMC in the control of systems with mismatched uncertainties. This method is considered by Cao and Xu (2004) for the case of systems with mismatched uncertainties in the state matrix, but it is limited to matched uncertainties in the input matrix and to external disturbances. A few recent studies have included the case of mismatched uncertainties in the input matrix (Shaocheng and Yongji, 2006; Choi, 2007). However, neither of them is applicable in the presence of mismatched disturbances not related to the input channel.

In this paper, we propose a new approach to design ISMC for a class of uncertain systems. This class regroups mismatched uncertainties in both state and input matrices, as well as mismatched norm-bounded nonlinearities and external disturbances. Our approach, based on Linear Matrix Inequalities (LMIs), combines the advantages of ISMC with H_∞ control. This gives a sufficient condition for the robust stability of the system in the sliding mode. The chattering phenomenon, the main drawback of SMC, is also considered in this paper. An adaptive fuzzy ISMC law is proposed to improve the performance of the control scheme, by allowing elimination of chattering and the estimation of the norm bounds of uncertainties. The efficiency of the proposed control laws is investigated through a sixth-order mechanical system example.

2. Problem formulation

2.1. System description. Consider the following uncertain system:

$$\begin{cases} \dot{x} = [A + \Delta A]\, x \\ \qquad + [B + \Delta B]\, u(t) + f(x,t) + H\omega(t), \\ y = Cx, \end{cases} \tag{1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^m$ is the input control, $y \in \mathbb{R}^q$ is the controlled output, $f(x,t) \in \mathbb{R}^n$ is the vector of nonlinearities and unmodelled dynamics. Here $\omega \in \mathbb{R}^p$ is a square-integrable external disturbance. $A \in \mathbb{R}^{n\times n}$ is the system characteristic matrix, $B \in \mathbb{R}^{n\times m}$ is the input matrix, $H \in \mathbb{R}^{n\times p}$ is the matrix of external disturbances, $C \in \mathbb{R}^{q\times n}$ is the output matrix, ΔA and ΔB represent system and input matrix uncertainties, respectively. We make the following assumptions:

(i) The pair (A, B) is stabilizable.

(ii) The input matrix B has full rank.

(iii) There exist known positive constants a, b, g, and ω_0 such that $\|\Delta A\| \le a$, $\|\Delta B\| \le b$, $\|f(x,t)\| \le g\,\|x\|$ and $\|\omega(t)\| \le \omega_0$, for all $t \in \mathbb{R}^+$.

(iv) $\|B^+\Delta B\| \le b_m < 1$, where b_m is a positive known scalar and $B^+ \equiv \left(B^T B\right)^{-1} B^T$.

2.2. Preliminary results. In this section, we give some preliminary results that will be helpful to obtain our main results.

Lemma 1. (Boyd *et al.*, 1994) *Consider the following unforced system:*

$$\begin{cases} \dot{x} = Ax + H\omega, \\ y = Cx. \end{cases} \tag{2}$$

This system is quadratically stable and satisfies the H_∞ constraint $\|T_{y\omega}\|_\infty < \gamma$ if there exists a quadratic Lyapunov function $V(x) = x^T P x, P > 0$ such that, for all $t > 0$,

$$\dot{V} + y^T y - \gamma^2 \omega^T \omega < 0. \tag{3}$$

Lemma 2. (Choi, 2007) *For any vectors x and y with appropriate dimensions, the following inequality holds:*

$$2x^T y \le \epsilon x^T x + \epsilon^{-1} y^T y, \quad \forall \epsilon > 0. \tag{4}$$

Lemma 3. (Boyd *et al.*, 1994) *Consider a block symmetric matrix*

$$\begin{bmatrix} A & B^T \\ B & C \end{bmatrix}, \tag{5}$$

where A and C are square matrices, with C being negative definite. This matrix is negative definite if and only if $A - B^T C^{-1} B$ is negative semi-definite.

3. Sliding mode stability

3.1. Sliding surface choice. In this work, we choose the switching function as follows:

$$S(t) = B^+ x + z, \tag{6}$$

where $z \in \mathbb{R}^m$ is the solution of the following dynamic equation:

$$\dot{z} = -(B^+ A + K)x, z(0) = -B^+ x(0), \tag{7}$$

where $K \in \mathbb{R}^{m\times n}$ is a state feedback gain which should be designed to lead the closed loop system to the desired performances in the sliding mode. The sliding surface considered allows the elimination of the reaching phase characterizing CSMC, because the initial value $S(0) = 0$ for any initial conditions. The time derivative of the switching function is derived using (1) and (6) as follows:

$$\begin{aligned} \dot{S}(t) &= B^+ \left[[A + \Delta A]\, x + [B + \Delta B]\, u + f + H\omega \right] \\ &\quad - B^+ A x - K x \\ &= B^+ \left[\Delta A x + [B + \Delta B]\, u + f + H\omega \right] - Kx. \end{aligned} \tag{8}$$

Let us suppose that

$$\Gamma = I_n - BB^+, \tag{9}$$

where $I_n \in \mathbb{R}^{n\times n}$ is the $n \times n$ identity matrix. Accordingly, it is easy to deduce that

$$B^+\Gamma = B^+ - B^+BB^+ = B^+ - B^+ = 0. \tag{10}$$

In addition, we can rewrite the uncertainty terms as follows:

$$\Delta A(t) = B\Delta A_m(t) + \Delta A_u(t), \tag{11a}$$
$$\Delta B(t) = B\Delta B_m(t) + \Delta B_u(t), \tag{11b}$$
$$f(x,t) = Bf_m(x,t) + f_u(x,t), \tag{11c}$$
$$H = BH_m + H_u, \tag{11d}$$

where

$$\begin{aligned} \Delta A_m(t) &= B^+\Delta A(t), & \Delta B_m(t) &= B^+\Delta B(t),\\ f_m(x,t) &= B^+f(x,t), & H_m &= B^+H,\\ \Delta A_u(t) &= \Gamma\Delta A(t), & \Delta B_u(t) &= \Gamma\Delta B(t),\\ f_u(x,t) &= \Gamma f(x,t), & H_u &= \Gamma H. \end{aligned} \tag{12}$$

Furthermore, there exist known positive constants a_m, a_u, b_u, g_m, and g_u such that $\|\Delta A_m\| \le a_m$, $\|\Delta A_u\| \le a_u$, $\|\Delta B_u\| \le b_u$, $\|f_m(x,t)\| \le g_m\|x\|$, and $\|f_u(x,t)\| \le g_u\|x\|$.

Therefore, using (9)–(12), Eqn. (8) can be transformed into

$$\begin{aligned} \dot{S} = \Delta A_m x + (I_m + \Delta B_m)\,u\\ + f_m(x,t) + H_m\omega - Kx. \end{aligned} \tag{13}$$

The intrinsic condition of the sliding mode emergence is

$$S(t) = 0, \quad \dot{S}(t) = 0. \tag{14}$$

This condition allows deriving the expression for the equivalent control as follows:

$$\begin{aligned} u_{eq} &= -(I_m + \Delta B_m)^{-1}\\ &\times \Big[\Delta A_m x + f_m(x,t) + H_m\omega - Kx\Big]. \end{aligned} \tag{15}$$

Remark 1. Equation (15) requires that the matrix $(I_m + \Delta B_m)$ be nonsingular. This requirement is guaranteed by Assumption (iv).

3.2. Stability of the sliding motion. By substituting (15) in (1), the sliding mode dynamics can be described by

$$\begin{aligned} \dot{x}(t) = Ax + BKx + \tilde{B}Kx + \Delta A_u x - \tilde{B}\Delta A_m x\\ + f_u - \tilde{B}f_m + H_u\omega - \tilde{B}H_m\omega, \end{aligned} \tag{16}$$

where

$$\tilde{B} = \Delta B_u(I_m + \Delta B_m)^{-1}. \tag{17}$$

It is clear from (16) that the system dynamics in the sliding mode are affected by the existence of uncertainties and disturbances. Our objective is the design of a state feedback gain K. This gain guarantees the stability of the closed loop system while satisfying the H_∞ constraint $\|T_{y\omega}\|_\infty < \gamma$. In order to reach this goal, we proceed by means of the LMI method.

Theorem 1. *The uncertain system (1) with the assumptions (i)–(iv) is quadratically stable on the sliding surface described by (6) and satisfies the H_∞ constraint $\|T_{y\omega}\|_\infty < \gamma$ if there exist a symmetric positive-definite matrix X, a matrix R and positives scalars ϵ_i, $i = 1,\dots,6$, such that the following LMI holds:*

$$\begin{bmatrix} \Xi & \Theta \\ * & \Psi \end{bmatrix} < 0, \tag{18}$$

where

$$\Xi = \begin{bmatrix} \Sigma & XC^T & a_uX & b_ua_mX & g_uX\\ * & -I & 0 & 0 & 0\\ * & * & -\epsilon_1 I & 0 & 0\\ * & * & * & -\epsilon_2^* I & 0\\ * & * & * & * & -\epsilon_3 I \end{bmatrix}, \tag{19}$$

$$\Sigma = AX + XA^T + BR + R^TB^T + \sum_{i=1}^{6}\epsilon_i I, \tag{20}$$

$$R = KX, \qquad \epsilon_i^* = (1-b_m)^2\epsilon_i, \tag{21}$$

$$\Theta = \begin{bmatrix} b_ug_mX & b_uR^T & H_u & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{22}$$

$$\Psi = \begin{bmatrix} -\epsilon_4^* I & 0 & 0 & 0\\ * & -\epsilon_5^* I & 0 & 0\\ * & * & -\gamma^2 I & b_uH_m^T\\ * & * & * & -\epsilon_6^* I \end{bmatrix}. \tag{23}$$

Here the asterisk denotes the transpose of the corresponding block above the main diagonal, and I denotes the identity matrix of appropriate dimension.

Proof. Consider a symmetric positive-definite matrix P and choose a candidate Lyapunov function,

$$V(x) = x^TPx. \tag{24}$$

In order to complete the proof, we proceed by verification of Lemma 1:

$$\begin{aligned} &\dot{V} + y^Ty - \gamma^2\omega^T\omega\\ &\quad = x^T\left[PA + A^TP + PBK + K^TB^TP\right]x\\ &\qquad + 2x^TP\Delta A_u x - 2x^TP\tilde{B}\Delta A_m x + 2x^TPf_u\\ &\qquad - 2x^TP\tilde{B}f_m + 2x^TP\tilde{B}Kx + 2x^TPH_u\omega\\ &\qquad - 2x^TP\tilde{B}H_m\omega + x^TC^TCx - \gamma^2\omega^T\omega. \end{aligned}$$

Using Lemma 2, we get

$$\begin{aligned}
2x^T P\Delta A_u x &\leq \epsilon_1 x^T P^2 x + \epsilon_1^{-1} x^T \Delta A_u x \\
&\leq x^T \left[\epsilon_1 P^2 + a_u^2 \epsilon_1^{-1} I\right] x, \\
-2x^T P\tilde{B}\Delta A_m x &\leq \epsilon_2 x^T P^2 x \\
&\quad + \epsilon_2^{-1} x^T \Delta A_m^T \tilde{B}^T \tilde{B}\Delta A_m x \\
&\leq x^T \left[\epsilon_2 P^2 + \frac{b_u^2 a_m^2}{(1-b_m)^2}\epsilon_2^{-1} I\right] x, \\
2x^T P f_u &\leq \epsilon_3 x^T P^2 x + \epsilon_3^{-1} f_u^T f_u \\
&\leq x^T \left[\epsilon_3 P^2 + g_u^2 \epsilon_3^{-1} I\right] x, \\
-2x^T P\tilde{B} f_m &\leq \epsilon_4 x^T P^2 x + \epsilon_4^{-1} f_m^T \tilde{B}^T \tilde{B} f_m \\
&\leq x^T \left[\epsilon_4 P^2 + \frac{b_u^2 g_m^2}{(1-b_m)^2}\epsilon_4^{-1} I\right] x, \\
2x^T P\tilde{B}Kx &\leq \epsilon_5 x^T P^2 x + \epsilon_5^{-1} x^T K^T \tilde{B}^T \tilde{B}Kx \\
&\leq x^T \big[\epsilon_5 P^2 \\
&\quad + \frac{b_u^2}{(1-b_m)^2}\epsilon_5^{-1} K^T K\big] x, \\
-2x^T P\tilde{B}H_m\omega &\leq \epsilon_6 x^T P^2 x + \epsilon_6^{-1}\omega^T H_m^T \tilde{B}^T \tilde{B} H_m \omega \\
&\leq \epsilon_6 x^T P^2 x \\
&\quad + \frac{b_u^2}{(1-b_m)^2}\epsilon_6^{-1}\omega^T H_m^T H_m \omega,
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
&\dot{V} + y^T y - \gamma^2 \omega^T \omega \\
&\quad \leq x^T \Omega x + x^T P H_u \omega + \omega^T H_u^T P x \\
&\quad\quad + \omega^T \left[-\gamma^2 I + \frac{b_u^2}{(1-b_m)^2}\epsilon_6^{-1} H_m^T H_m\right]\omega,
\end{aligned} \tag{25}$$

where

$$\begin{aligned}
\Omega &= PA + A^T P + PBK + K^T B^T P + C^T C \\
&\quad + \sum_{i=1}^{6} \epsilon_i P^2 + \big(a_u^2 \epsilon_1^{-1} + \frac{b_u^2 a_m^2}{(1-b_m)^2}\epsilon_2^{-1} + g_u^2 \epsilon_3^{-1} \\
&\quad + \frac{b_u^2 g_m^2}{(1-b_m)^2}\epsilon_4^{-1}\big) I + \frac{b_u^2}{(1-b_m)^2}\epsilon_5^{-1} K^T K.
\end{aligned}$$

The inequality (25) can be reformulated as follows:

$$\begin{aligned}
&\dot{V} + y^T y - \gamma^2 \omega^T \omega \\
&\quad \leq \begin{bmatrix} x^T & \omega^T \end{bmatrix} \\
&\quad \times \begin{bmatrix} \Omega & PH_u \\ H_u^T P & \frac{b_u^2}{(1-b_m)^2}\epsilon_6^{-1} H_m^T H_m - \gamma^2 I \end{bmatrix} \begin{bmatrix} x \\ \omega \end{bmatrix}
\end{aligned}$$

Thus Lemma 1 is satisfied if

$$\begin{bmatrix} \Omega & PH_u \\ H_u^T P & \frac{b_u^2}{(1-b_m)^2}\epsilon_6^{-1} H_m^T H_m - \gamma^2 I \end{bmatrix} < 0. \tag{26}$$

The next LMI can be derived from (26), by using Lemma 3:

$$\begin{bmatrix} \Omega & PH_u & 0 \\ H_u^T P & -\gamma^2 I & b_u H_m^T \\ 0 & b_u H_m & -(1-b_m)^2 \epsilon_6 I \end{bmatrix} < 0. \tag{27}$$

After pre-multiplying and post-multiplying (27) by $\operatorname{diag}\left[P^{-1}, I, I\right]$, considering $X = P^{-1}$ and $R = KX$, the LMI (18) is obtained by the successive use of Lemma 3. Therefore, the proof is complete. ■

4. Reachability analysis

In the preceding section a sufficient condition was derived for the quadratic stability of the uncertain system on the sliding manifold $S(t) = 0$. Now, we proceed with the second task, which is the design of an SMC law, such that the reachability of the specified sliding surface is guaranteed.

4.1. Integral sliding mode control law. The proposed ISMC law is specified through the following result.

Theorem 2. *Consider the uncertain system (1) with the assumptions (i)–(iv). Suppose that the switching surface is given by (6) with $K = RX^{-1}$, where X and R are solutions of the LMI (18). Suppose also that the SMC law is*

$$u = Kx - \rho\frac{S}{\|S\|}, \tag{28}$$

where

$$\begin{aligned}
\rho &= \frac{1}{1-b_m}\rho_1, \\
\rho_1 &= q + (a_m + b_m\|K\| + g_m)\|x\| + \|H_m\|\omega_0,
\end{aligned} \tag{29}$$

with q being a small positive scalar. Then a stable sliding mode exists from the initial time.

Proof. Consider the Lyapunov function

$$V = \frac{1}{2}S^T S. \tag{30}$$

Using (13), the derivative of this function with respect to time is given as follows:

$$\begin{aligned}
S^T\dot{S} &= S^T\big[\Delta A_m x + (I_m + \Delta B_m)u \\
&\quad + H_m\omega + f_m - Kx\big] \\
&= S^T(\Delta A_m x + \Delta B_m Kx + H_m\omega + f_m \\
&\quad - (I_m + \Delta B_m)\rho\frac{S}{\|S\|}) \\
&= S^T(\Delta A_m x + \Delta B_m Kx + H_m\omega + f_m) \\
&\quad - \rho\|S\| - \rho S^T\Delta B_m\frac{S}{\|S\|} \\
&= S^T(\Delta A_m x + \Delta B_m Kx + H_m\omega + f_m) \\
&\quad - \rho_1\|S\| - \frac{b_m}{1-b_m}\rho_1\|S| - \rho S^T\Delta B_m\frac{S}{\|S\|}
\end{aligned}$$

$$\leq \|S\| \left[\|\Delta A_m x\| + \|\Delta B_m K x\| + \|H_m \omega\| + \|f_m\| \right] - \rho_1 \|S\| - b_m \rho \|S\| \leq -q \|S\| < 0.$$

Then, the SMC law considered guarantees the reachability of the switching surface. In addition, the initial value of $S(t)$ is given by $S(0) = 0$. Thus, the proof is complete. ∎

Remark 2. The switching gain (29) of the proposed controller is a function of the norm-bounds of the matched components of uncertainties and disturbances a_m, b_m, g_m and $\|H_m\|$. Moreover, $a_m \leq a$, $b_m \leq b$, $g_m \leq g$, $\|H_m\| \leq \|H\|$, which allows the optimization of the discontinuous controller magnitude. This argument justifies the procedure proposed in (11)–(12).

4.2. Adaptive fuzzy ISMC law. Two major problems may affect the applicability of the proposed SMC law. The first is the difficulty to obtain the exact values of uncertainties and disturbance bounds. The second is the phenomenon of chattering, a major disadvantage of SMC, which is induced by the switching nature of the controller. Hence, to overcome these problems, an Adaptive Fuzzy ISMC (AFISMC) law is presented in this section. The proposed AFISMC is based on the introduction of a Fuzzy Logic (FL) inference mechanism which replaces the switching control law. The switching function (6) can be written as

$$S = [s_1 \cdots s_i \cdots s_m]^T.$$

Let s_i be the input linguistic variable of FL, and $u_{F,i}$ be the output linguistic variable. The associated fuzzy sets are expressed as follows:

- for the antecedent proposition (s_i): P (Positive), N (Negative), and Z (Zero);
- for the consequent proposition ($u_{F,i}$): PE (Positive Effort), NE (Negative Effort), and ZE (Zero Effort).

In order to make the sliding surface attractive, the fuzzy linguistic rule base can be given as follows:

1. Rule 1: If s_i is P, then $u_{F,i}$ is PE.
2. Rule 2: If s_i is Z, then $u_{F,i}$ is ZE.
3. Rule 3: If s_i is N, then $u_{F,i}$ is NE.

The membership functions of the input fuzzy sets are of the triangle type, and those of the output fuzzy sets are of the singleton type. The singleton defuzzification method is used in this work. Then the fuzzy controller (output of the defuzzification module) can be written as

$$u_{F,i} = \frac{\sum_{k=1}^{3} \mu_{jk} \delta_{jk}}{\sum_{k=1}^{3} \mu_{jk}}, \tag{31}$$

where $0 \leq \mu_{jk} \leq 1$ is the firing strength of rule k, $k = 1, \dots, 3$, $\delta_{j1} = \delta_j, \delta_{j2} = 0$, and $\delta_{j3} = -\delta_j$ stand for the centres of the membership functions PE, ZE, and NE, respectively. Owing to the special choice of triangular membership functions, we get

$$\sum_{k=1}^{3} \mu_{jk} = 1. \tag{32}$$

As a result, (31) can be reduced to the following:

$$u_{F,j} = (\mu_{j1} - \mu_{j3})\, \delta_j. \tag{33}$$

According to the aforementioned fuzzy rule base, it is easy to observe that

$$u_{F,j} = \begin{cases} \mu_{j1}\delta_j, & \text{if} \quad s_j > 0, \\ -\mu_{j3}\delta_j, & \text{if} \quad s_j < 0. \end{cases} \tag{34}$$

Then we can conclude that

$$s_j\,(\mu_{j1} - \mu_{j3})\, \delta_j \geq 0. \tag{35}$$

Consider again the Lyapunov candidate function (30). As was mentioned at the beginning of the proof of Theorem 2, its derivative with respect to time is given by

$$\dot{V} = S^T \Big[\Delta A_m x + (I_m + \Delta B_m)\, u + H_m \omega + f_m - Kx \Big]. \tag{36}$$

Consequently, if the controller is selected as follows:

$$u = Kx + u_F, \tag{37}$$

where u_F is the fuzzy controller specified by

$$u_F = -\frac{1}{1 - b_m} \left[u_{F,1} \cdots u_{F,m} \right]^T, \tag{38}$$

then (36) can be written as

$$\dot{V} = S^T \Big[\varphi(x, \omega, t) + (I_m + \Delta B_m) u_F \Big], \tag{39}$$

where

$$\begin{aligned} \varphi(x, \omega, t) &= \Delta A_m x + \Delta B_m K x + H_m \omega + f_m \\ &= [\varphi_1 \cdots \varphi_m]^T. \end{aligned} \tag{40}$$

Therefore,

$$\begin{aligned} \dot{V} &= \sum_{j=1}^{m} s_j \varphi_j - \frac{1}{1 - b_m} \sum_{j=1}^{m} s_j u_{F,j} - S^T \Delta B_m u_F \\ &\leq \sum_{j=1}^{m} |s_j \varphi_j| - \frac{1}{1 - b_m} \sum_{j=1}^{m} s_j (\mu_{j1} - \mu_{j3}) \delta_j \\ &\quad + \|\Delta B_m\| \left\| S^T u_F \right\| \\ &\leq \sum_{j=1}^{m} |s_j \varphi_j| - \frac{1}{1 - b_m} \sum_{j=1}^{m} s_j (\mu_{j1} - \mu_{j3}) \delta_j \\ &\quad + \frac{b_m}{1 - b_m} \sum_{j=1}^{m} s_j (\mu_{j1} - \mu_{j3}) \delta_j. \end{aligned}$$

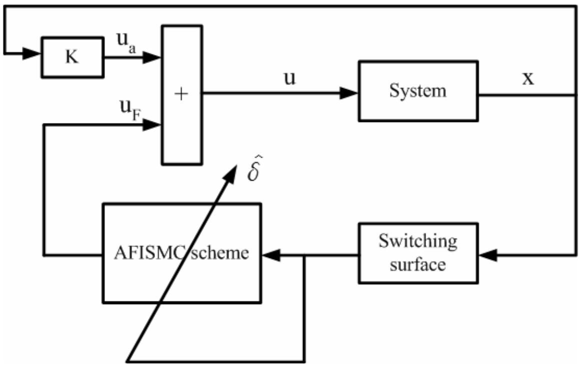

Fig. 1. Structure of the proposed AFISMC scheme.

Hence we get

$$\dot{V} \leq \sum_{j=1}^{m} |s_j| \left[|\varphi_j| - |\mu_{j1} - \mu_{j2}| \, \delta_j \right]. \tag{41}$$

As a result, $\dot{V} < 0$ if the following inequality holds:

$$\delta_j > \frac{|\varphi_j|}{|\mu_{j1} - \mu_{j3}|}, \quad j = 1, \dots, m. \tag{42}$$

According to Wang's theorem (Wang, 1997), there exists an optimal value $\overline{\delta_j}$ which satisfies the preceding inequality. However, this value cannot be accurately determined because the uncertainties bounds cannot be easily extracted. Then, δ_j is chosen as the parameter to be updated. The structure of the proposed scheme is indicated in Fig. 1. The following theorem describes this control law.

Theorem 3. *Consider the uncertain system (1) with the assumptions (i)–(iv). Suppose that the switching surface is given by (6), where X and R are solutions to be LMI (18). If the control law is given by (37), where u_F is the fuzzy controller (38), and δ_j is replaced by the adaptive parameter $\hat{\delta}_j$ described as follows:*

$$\dot{\hat{\delta}}_j = \beta_j s_j (\mu_{j1} - \mu_{j3}), \tag{43}$$

with β_j being a nonnegative scalar, then a stable sliding mode exists from the initial time.

Proof. The estimated error between the adaptive parameter $\hat{\delta}_j$ and the optimal value $\overline{\delta_j}$ is defined as

$$\tilde{\delta}_j = \hat{\delta}_j - \overline{\delta_j}. \tag{44}$$

Thereafter, we choose the following Lyapunov candidate:

$$V_1 = V + \frac{1}{2} \sum_{j=1}^{m} \beta_j^{-1} \tilde{\delta}_j^2. \tag{45}$$

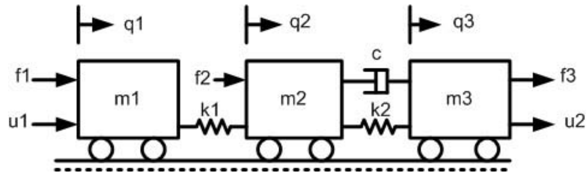

Fig. 2. Mechanical system example.

Thus we obtain

$$\dot{V}_1 = \dot{V} + \sum_{j=1}^{m} \beta_j^{-1} \tilde{\delta}_j \dot{\tilde{\delta}}_j. \tag{46}$$

By using (37), (39), (43) and (44), (46) can be rewritten as

$$\begin{aligned}
\dot{V}_1 &= \sum_{j=1}^{m} s_j \varphi_j - \frac{1}{1 - b_m} \sum_{j=1}^{m} s_j u_{F,j} - S^T \Delta B_m u_F \\
&\quad + \sum_{j=1}^{m} (\tilde{\delta}_j - \overline{\delta_j}) s_j (\mu_{j1} - \mu_{j3}) \\
&= \sum_{j=1}^{m} \left[s_j \phi_j - s_j \overline{\delta_j} (\mu_{j1} - \mu_{j3}) \right] - S^T \Delta B_m u_F \\
&\quad - \frac{1}{1 - b_m} \sum_{j=1}^{m} s_j u_{F,j} + \sum_{j=1}^{m} \hat{\delta}_j s_j (\mu_{j1} - \mu_{j2}) \\
&\leq - \frac{1}{1 - b_m} \sum_{j=1}^{m} s_j \hat{\delta}_j (\mu_{j1} - \mu_{j3}) \\
&\quad + \sum_{j=1}^{m} s_j \hat{\delta}_j (\mu_{j1} - \mu_{j3}) \\
&\quad + \sum_{j=1}^{m} \left[s_j \phi_j - s_j \overline{\delta_j} (\mu_{j1} - \mu_{j3}) \right] \\
&\quad + \|\Delta B_m\| \left\| S^T u_F \right\| \\
&\leq \sum_{j=1}^{m} |s_j| \, |\phi_j| - \overline{\delta_j} \, |\mu_{j1} - \mu_{j3}| \\
&\quad - \sum_{j=1}^{m} s_j \hat{\delta}_j (\mu_{j1} - \mu_{j3}) \\
&\quad + \sum_{j=1}^{m} s_j \hat{\delta}_j (\mu_{j1} - \mu_{j3}) \\
&\leq \sum_{j=1}^{m} \left| s_j \left[| \, |\phi_j| - \overline{\delta_j} \, |\mu_{j1} - \mu_{j3}| \right] \right| < 0.
\end{aligned}$$

Thus, the time derivative of the Lyapunov candidate function is negative. This completes the proof. ■

5. Example

In this section we shall evaluate the proposed control laws through the application to a sixth-order mechanical sys-

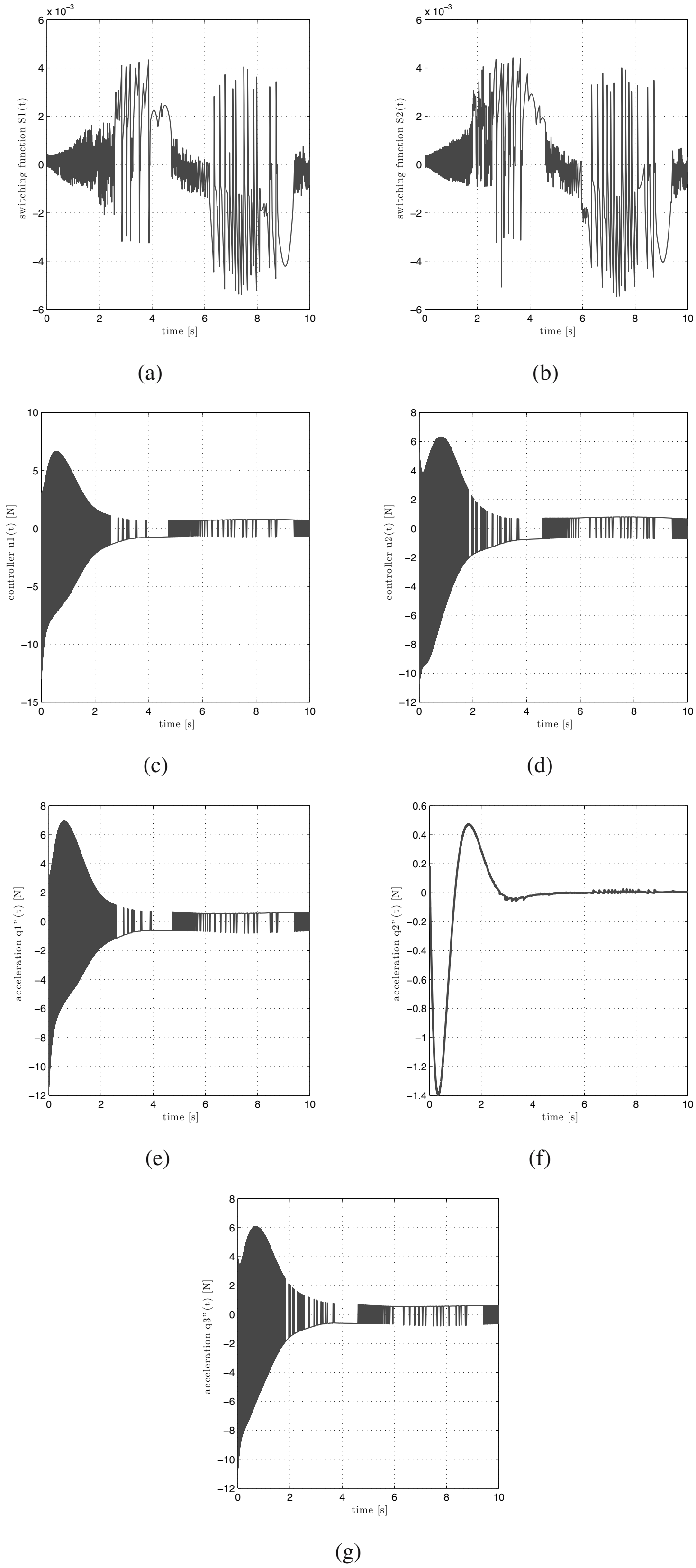

Fig. 3. Simulation results using ISMC: switching function $S_1(t)$ (a), switching function $S_2(t)$ (b), controller $u_1(t)$ (c), controller $u_2(t)$ (d), acceleration $\ddot{q}_1(t)$ (e), acceleration $\ddot{q}_2(t)$ (f), acceleration $\ddot{q}_3(t)$ (g).

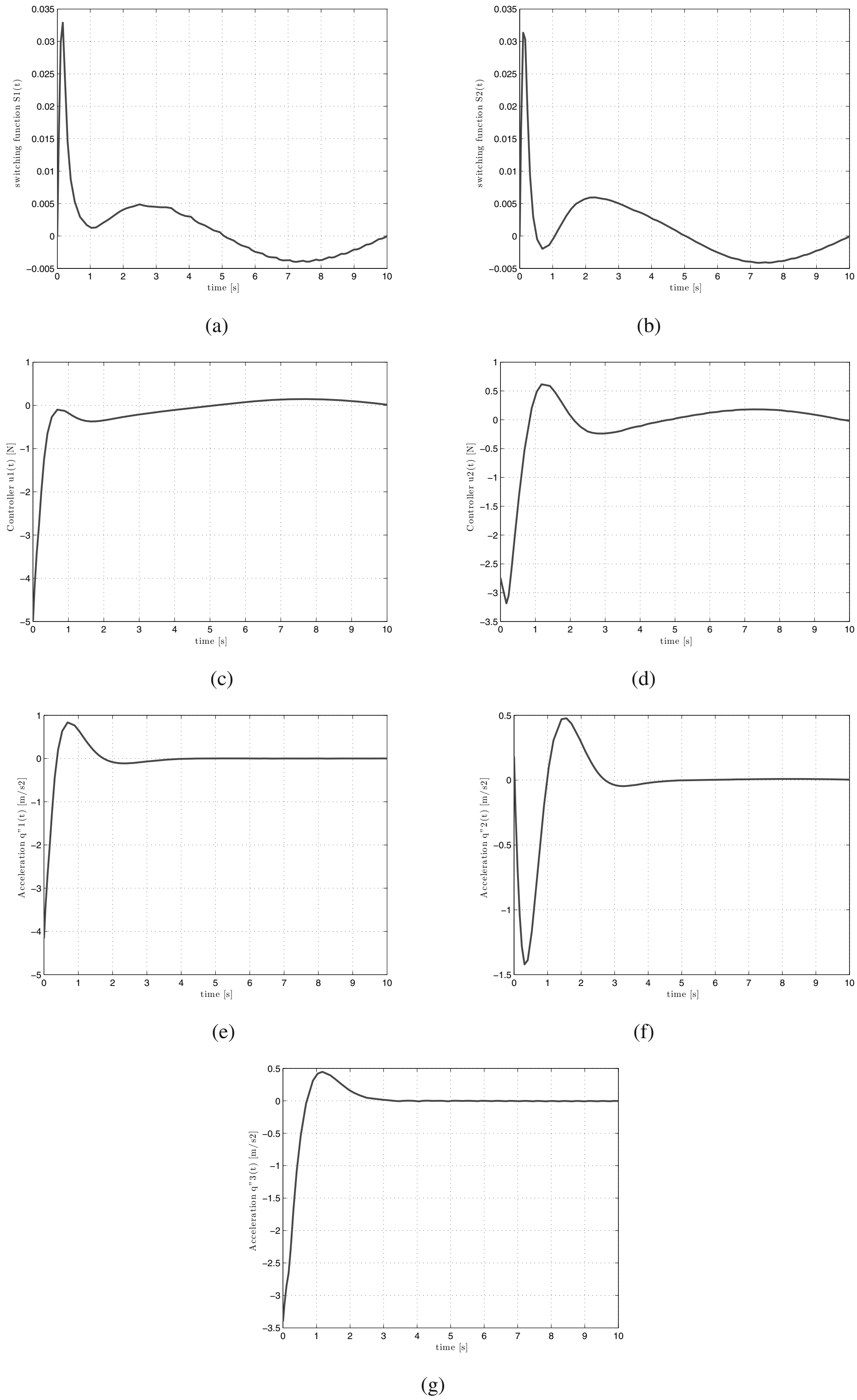

Fig. 4. Simulation results using AFISMC: switching function $S_1(t)$ (a), switching function $S_2(t)$ (b), controller $u_1(t)$ (c), controller $u_2(t)$ (d), acceleration $\ddot{q}_1(t)$ (e), acceleration $\ddot{q}_2(t)$ (f), acceleration $\ddot{q}_3(t)$ (g).

tem shown in Fig. 2. Suppose that

$$\begin{aligned} m_i &= m + \Delta m_i \\ &= m(1+\xi_i), \quad i = 1,\dots,3, \quad m = 1, \\ k_1 &= k_2 = 2, \quad c = 3, \\ x &= \begin{bmatrix} q_1 & \dot{q}_1 & q_2 & \dot{q}_2 & q_3 & \dot{q}_3 \end{bmatrix}^T, \\ u &= \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T, \quad \omega = \begin{bmatrix} f_1 & f_2 & f_3 \end{bmatrix}^T. \end{aligned}$$

The system can be described by the following state equation:

$$\dot{x} = (A + \Delta A)x + (B + \Delta B)u + H\omega,$$

with

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ -2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 2 & 0 & -4 & -3 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 2 & 3 & -2 & -3 \end{bmatrix},$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

$$\Delta A = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 \\ 2\zeta_1 & 0 & -2\zeta_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -2\zeta_2 & 0 & 4\zeta_2 & 3\zeta_2 & -2\zeta_2 & -3\zeta_2 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -2\zeta_3 & -3\zeta_3 & 2\zeta_3 & 3\zeta_3 \end{bmatrix},$$

$$\Delta B = \begin{bmatrix} 0 & 0 \\ -\zeta_1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & -\zeta_3 \end{bmatrix},$$

$$\zeta_i = \frac{\xi_i}{1+\xi_i}, \quad f_i = 0.1\sin(\pi t).$$

For $\xi_i = 0.1, i = 1,\dots,3$, applying the decomposition procedure given by (11)–(12), we get $a_m = 0.4711, a_u = 0.5892, b_m = 0.0909, b_u = 0$. Then a feasible solution of the LMI (18) is given by

$$X = \begin{bmatrix} 0.055 & -0.099 & -0.005 & 0.023 & -0.015 & 0.027 \\ -0.099 & 0.380 & 0.015 & -0.072 & -0.001 & -0.015 \\ -0.005 & 0.015 & 0.062 & -0.070 & -0.010 & 0.012 \\ 0.023 & -0.072 & -0.070 & 0.385 & 0.038 & -0.129 \\ -0.015 & -0.001 & -0.010 & 0.038 & 0.053 & -0.075 \\ 0.027 & -0.015 & 0.012 & -0.129 & -0.075 & 0.238 \end{bmatrix},$$

$$\epsilon_1 = 0.0385, \quad \gamma = 1.4534,$$

$$K = \begin{bmatrix} -2.661 & -3.862 & -2.453 & 0.286 & -4.266 & -1.972 \\ 2.285 & 0.922 & -3.025 & -3.742 & -0.740 & -0.637 \end{bmatrix}.$$

We remark that none of the traditional design methods of Kim *et al.* (2000), Choi (2001) or Xia and Jia (2003) nor the integral SMC methods given by Cao and Xu (2004), Shaocheng and Yongji (2006) or Choi (2007) is applicable to the example considered. The simulation results are obtained for the initial state vector

$$x(0) = \begin{bmatrix} 0.1 & 0.2 & 0.3 & 0.4 & 0.5 & 0.6 \end{bmatrix}^T.$$

Figures 3 and 4 indicate the evolution of switching function components, controller components, and accelerations using the proposed ISMC (28) and AFISMC (37)–(43), respectively. The displacement evolution for both ISMC and AFISMC methods is shown in Fig. 5. From these simulation results it is clear that the proposed schemes result in a stable sliding mode from the initial time. However, it is obvious from the controller evolution that the first approach is accompanied with the chattering phenomenon, which induces the appearance of an undesirable vibration as depicted by the evolution of accelerations. Fortunately, this disadvantage is overcome by AFISMC through the elimination of high frequency discontinuities in both the controller and acceleration. In addition, Fig. 5 shows that the last approach preserves the same dynamical performances of the closed loop system as the first one.

Figure 6 shows the displacement evolution for both the nominal and uncertain systems. The displacements $q_1(t)$ and $q_3(t)$, which are directly actuated by the control input, are superposed for both systems, while there exist a slight difference in the case of $q_2(t)$ representing a nonactuated variable. Therefore the robustness of the proposed approach is confirmed.

6. Conclusion

A robust ISMC design for mismatched uncertain systems has been studied. A sufficient condition for quadratic stability of sliding motion has been established in terms of LMIs. The immediate sliding mode existence has been guaranteed by the proposed ISMC law. The induced chattering phenomenon has been eliminated by the introduction of an AFISMC law. Finally, the effectiveness of the

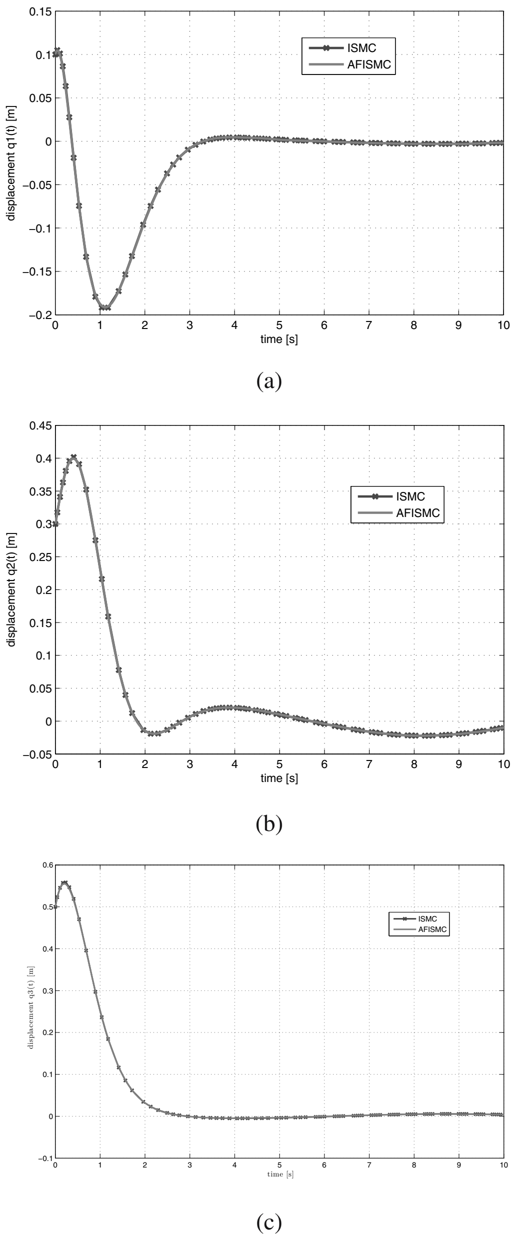

Fig. 5. Displacement evolution according to both ISMC and AFISMC: displacement $q_1(t)$ (a), displacement $q_2(t)$ (b), displacement $q_3(t)$ (c).

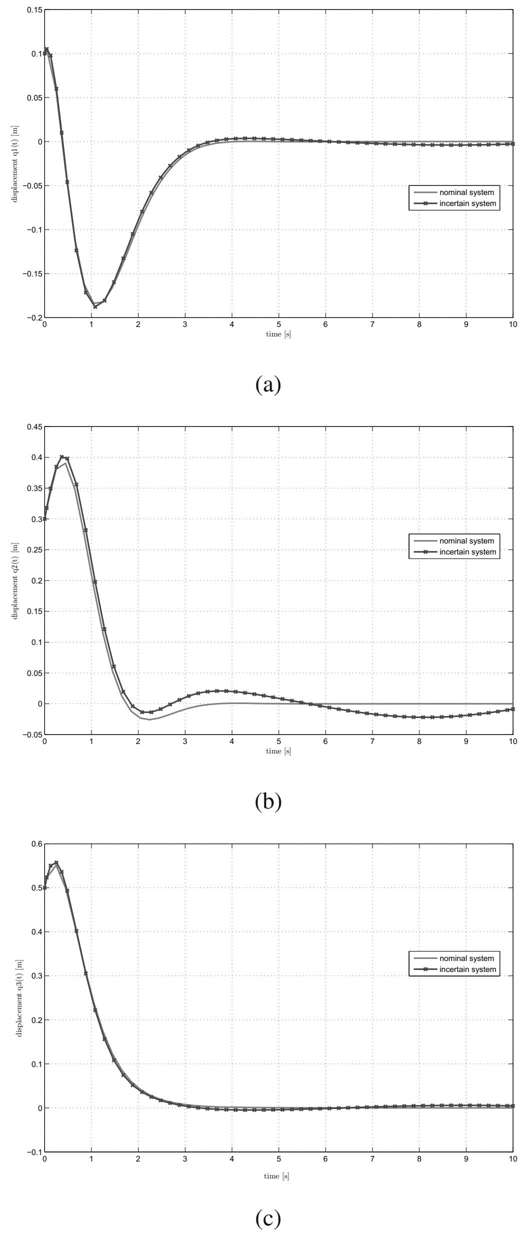

Fig. 6. Displacement evolution for the nominal and uncertain system according to AFISMC: displacement $q_1(t)$ (a), displacement $q_2(t)$ (b), displacement $q_3(t)$ (c).

proposed methods has been proved through a sixth order uncertain mechanical system example.

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