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STABILITY OF SOFTLY SWITCHED MULTIREGIONAL DYNAMIC OUTPUT CONTROLLERS WITH A STATIC ANTIWINDUP FILTER: A DISCRETE–TIME CASE

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This paper addresses the problem of model-based global stability analysis of discrete-time Takagi–Sugeno multiregional dynamic output controllers with static antiwindup filters. The presented analyses are reduced to the problem of a feasibility study of the Linear Matrix Inequalities (LMIs), derived based on Lyapunov stability theory. Two sets of LMIs are considered candidate derived from the classical common quadratic Lyapunov function, which may in some cases be too conservative, and a fuzzy Lyapunov function candidate, which has been proven to significantly reduce the conservatism level, although at the cost of increasing the number of LMIs. Two numerical examples illustrate the main result.

Keywords: Lyapunov, stability analysis, fuzzy control, antiwindup.

1. Introduction

In the literature, a majority of techniques utilized to analyse the stability of Takagi–Sugeno (TS) fuzzy control systems include Lyapunov theory, Linear Matrix Inequalities (LMIs) and bilinear matrix inequalities. In some of these approaches, invariant set theory has been also incorporated.

Stability analysis of the TS system was presented by Tanaka and Sugeno (1992) while the relaxed stability conditions for the continuous-time case by Tanaka *et al*. (1998). The design and stability of fuzzy logic multiregional output controllers (discrete-time case) were presented by Domański *et al*. (1999) and further developed by Tatjewski (2007). The latter introduced fuzzy Lyapunov functions into the analysis.

Stability analysis involving extended Lyapunov theory and systematic design of TS fuzzy control systems for the continuous-time case were considered by Xiu and Ren (2005). There were some trials to incorporate information about the shape of the membership functions of the TS system.

A fuzzy Lyapunov function approach to design and analysis of continuous-time domain TS fuzzy control systems was addressed by Rhee and Won (2006).

The state of the art in design and analysis of model-based fuzzy control systems was presented by Feng (2006). A stable indirect adaptive controller utilising TS system was developed by Qi and Brdyś (2008) for tracking control of uncertain nonlinear discrete-time systems. The controller parameter adaptation was extended by Qi and Brdys (2009) to cover also the adaptation of the TS plant model structure.

This work aims at the problem of model-based stability analysis of TS type fuzzy control systems which utilize the dynamic output controllers with static antiwindup filters through the feasibility study of LMIs. These conditions are derived based on Lyapunov stability theory. In the presented analyses it is assumed that the control process can be represented by a TS fuzzy model with an arbitrary small modelling error. This is based on the well known universal approximation theorem (see Yaochu, 2002). Two approaches are regarded: where the LMIs are derived based on a Common Quadratic

Lyapunov Function (CQLF) candidate and that based on a Fuzzy Lyapunov Function (FLF) due to certain conservatism reduction (see Feng, 2006; Tatjewski, 2007).

The main contribution of the paper is introduction of tools for global stability verification of the TS fuzzy output feedback controller with regional controllers that (regionally) may be regarded as PI controllers with static antiwindup filtration. The problem of the global stability verification is posed as a feasibility study of a set of LMIs. Two distinct sets of LMIs are introduced. The first is based on the CQLF and the second is derived from the FLF. Technically, a formal description of the PI with a static antiwindup filter was acquired from Gomes da Silva and Tarbouriech (2006). Throughout the technical analysis of the problem it occurred that the results obtained extended the above mentioned work to a much wider class of problem.

The multiregional approach undertaken in this paper allowed designing a globally stable nonlinear controller which utilised all the advantages of the PI controller. Although controller robustness with respect to uncertainty is not directly tackled it is achieved due to well-known robustness of regional PI controllers. Therefore, the paper offers not only a fresh approach to a nonlinear control design but also significantly extends the results obtained by Gomes da Silva and Tarbouriech (2006). Similarly as in their work, applying the sector condition by Khalil (1996) is a key technical step in deriving the LMI stability conditions. As nonlinear plant modelling is carried out within a fuzzy Takagi–Sugeno framework, the fuzzy Lyapunov function can be applied, which leads to less conservative LMI conditions.

The work by Gomes da Silva and Tarbouriech (2006) has been continued towards linear systems with time varying parameters which are measurement accessible (Castelan *et al.*, 2006; 2010; Gomes da Silva *et al.*, 2008; Klug *et al.*, 2011). Therefore, the issue of robustness has not been addressed. In the works of Castelan *et al.* (2010) and Klug *et al.* (2011), a linear compensator with time-varying parameters and an antiwindup filter have been obtained by applying Lyapunov contractive set theory and ellipsoidal approximation of the largest region of attraction of desired equilibrium in the LMI format.

This paper is organized as follows. Problem statement is described in Section 2. Section 3 addresses the main results by introducing derived theorems. Section 4 presents two numerical examples. Section 5 concludes the paper. To increase the legibility of the work, two appendices, containing the proofs of derived theorems are added.

2. Problem statement

Consider a nonlinear process given by

$$\dot{\mathbf{x}} = f(\mathbf{x}(t), \overline{\mathbf{u}}(t)), \tag{1}$$

where $\mathbf{x}(t)$ is a state vector, $\overline{\mathbf{u}}(t)$ is the controlled input and the function f is known.

Assume that, based on the universal approximation theorem plant dynamics (1) can be represented with an arbitrary small modelling error by a TS fuzzy model. In this case a discrete-time fuzzy approximation had been chosen. The corresponding inference rule set describing the system dynamics is as follows:

$$R^i : \text{If } z_1(k) \text{ is } MF_1^i \text{ and } \dots \text{ and } z_v \text{ is } MF_v^i \text{ then}$$
$$\mathbf{x}^i(k+1) = \mathbf{A}^i\mathbf{x}(k) + \mathbf{B}^i\overline{\mathbf{u}}(k) + \mathbf{a}^i; \tag{2}$$

where $\mathbf{x}(k) \in \mathbb{R}^n$, $\overline{\mathbf{u}}(k) \in \mathbb{R}^{nu}$ are state and input vectors respectively, $\mathbf{A}^i \in \mathbb{R}^{n\times n}$, $\mathbf{B}^i \in \mathbb{R}^{n\times nu}$, $\mathbf{a}^i \in \mathbb{R}^{n\times 1}$ are system matrices; $z_s(k)$ are the measurable variables of the system, i.e., state variables; MF_s^i are the fuzzy sets, $i = \overline{1,p}$ is the region indicator, p denotes the number of regions, $s = \overline{1,v}$ is the fuzzy set indicator, v denotes the number of fuzzy sets, $\mathbb{R}$ represents the set of real numbers.

The model (2) can equivalently be described by

$$\mathbf{x}(k+1) = \sum_{i=1}^{p} h^i(\mathbf{z}(k))[\mathbf{A}^i\mathbf{x}(k) + \mathbf{B}^i\overline{\mathbf{u}}(k) + \mathbf{a}^i], \tag{3}$$

where the firing strength of the i-th rule $h^i(\mathbf{z}(k))$ is defined as

$$h^i(\mathbf{z}(k)) = \prod_{s=1}^{v} w_s^i(z_s(k)) / \sum_{i=1}^{p}\prod_{s=1}^{v} w_s^i(z_s(k)), \tag{4}$$

where $w_s^i(z_s(k))$ is the weight resulting from MF_s^i, and the system output is given by

$$\mathbf{y}(k) = \sum_{i=1}^{p} h^i(\mathbf{z}(k))\mathbf{C}^i\mathbf{x}(k), \tag{5}$$

where the matrices and vectors are defined as follows: $\mathbf{y}(k) \in \mathbb{R}^{no}$ is the process output and $\mathbf{C^i}$ is the output matrix of appropriate dimensions. The control input might be subjected to the actuator saturation, thus

$$\overline{\mathbf{u}}(k) = g\left(\mathrm{sat}(\mathbf{u}(k)) + \mathbf{b}\right), \tag{6}$$

where the saturation function is defined as $\mathrm{sat}(\cdot) = \mathrm{sign}(\cdot)\min\{1, |\cdot|\}$, $\mathbf{u}(k)$ is the unconstrained controller output, g is the output scaling factor, $\mathbf{b}$ is the saturation function offset. Also, due to the definition of the saturation function and (6), it may be regarded as a decentralised function of the form $\mathrm{sat}(\cdot) = [\mathrm{sat}_1(\cdot)\ \dots\ \mathrm{sat}_{no}(\cdot)]^T$.

The controller chosen to stabilize the output of the plant (1), represented by (2) or equivalently (3), on a desired value is a multiregional PI type compensator based on TS fuzzy reasoning (Domański *et al.*, 1999; Tatjewski, 2007; Han *et al.*, 2008; Zubowicz *et al.*, 2010) and is given by the following inference rule set:

$$R^i : \text{If } z_1(k) \text{ is } MF_1^i \text{ and } \dots \text{ and } z_v \text{ is } MF_v^i \text{ then}$$
$$\begin{cases} \mathbf{u}^i(k) = \mathbf{x}_c^i(k) - \mathbf{D}_c^i\mathbf{y}(k) + \mathbf{D}_c^i\mathbf{r}(k), \\ \mathbf{x}_c^i(k+1) = \mathbf{x}_c^i(k) - \mathbf{B}_c^i\mathbf{y}(k) + \mathbf{B}_c^i\mathbf{r}(k) \\ \qquad + \mathbf{E}_c^i(\text{sat}(\mathbf{u}(k)) - \mathbf{u}(k)) + \mathbf{E}_c^i\mathbf{b}, \end{cases} \tag{7}$$

where $\frown_c^i(k) \in \mathbf{R}^{nc}$ and $\mathbf{u}^i(k)$ are the state and control signals generated by the i-th TS regional controller, respectively, $\mathbf{r}(k)$ is the reference trajectory input which is assumed to be piece-wise constant, $\mathbf{B}_c^i$, $\mathbf{E}_c^i$, $\mathbf{D}_c^i$ are controller matrices defined as

$$\begin{aligned} \mathbf{B}_c^i &= \left[K_I^i/c_{mx}\right], \\ \mathbf{D}_c^i &= \left[K_P^i/c_{mx}\right], \\ \mathbf{E}_c^i &= \left[K_I^i K_{AW}^i\right], \end{aligned} \tag{8}$$

where K_P^i, K_I^i, K_{AW}^i are the proportional, integral and antiwindup gains, respectively, c_{mx} is the controller input scaling factor.

The resulting controller representation is as follows:

$$\mathbf{u}(k) = \sum_{i=1}^{p} h^i \left[\mathbf{x}_c(k) - \mathbf{D}_c^i\mathbf{y}(k) + \mathbf{D}_c^i\mathbf{r}(k)\right], \tag{9}$$

$$\begin{aligned} \mathbf{x}_c(k+1) &= \sum_{i=1}^{p} h^i \left[\mathbf{x}_c(k) - \mathbf{B}_c^i\mathbf{y}(k) + \mathbf{B}_c^i\mathbf{r}(k)\right] \\ &\quad \times \sum_{i=1}^{p} h^i \left[\mathbf{E}_c^i(\text{sat}(\mathbf{u}(k)) - \mathbf{u}(k)) + \mathbf{E}_c^i\mathbf{b}\right], \end{aligned} \tag{10}$$

where $h^i \triangleq h^i(\mathbf{z}(k))$; the definitions of $\mathbf{x}_c(k)$ and $\mathbf{u}(k)$ result from a TS reasoning scheme, that is, $\mathbf{x}_c(k) \triangleq \sum_{i=1}^{p} h^i\mathbf{x}_c^i(k)$ and $\mathbf{u}(k) \triangleq \sum_{i=1}^{p} h^i\mathbf{u}^i(k)$.

3. Model-based stability analysis of the closed loop control system

In this section the main result of the paper is presented. Based on Lyapunov stability theory, sufficient conditions for stability of the Closed Loop (CL) system are presented in the form of a feasibility study of derived LMIs. The two presented approaches utilize the CQLF and the FLF as candidate functions respectively.

3.1. Closed loop system description. Extending the state vector as $\xi(k) = \begin{bmatrix} \mathbf{x}^T(k) & \mathbf{x}_c^T(k) \end{bmatrix}^T$, where $\xi(k) \in \mathbb{R}^{n+nu}$, and utilizing (3), (5), (6), (9) and (10) yields the CL system representation,

$$\begin{aligned} &\xi(k+1) \\ &= \sum_{i=1}^{p}\sum_{j=1}^{p} h^i h^j [\mathbf{A}_{CL}^{ij}\xi(k) + \mathbf{B}_{r1}^{ij}\mathbf{r}(k)] \\ &- \sum_{i=1}^{p}\sum_{j=1}^{p} h^i h^j \left[\mathbf{B}_\Psi^{ij}\mathbf{\Psi}\left(\sum_{i=1}^{p} h^j \left[\mathbf{K}^j\xi(k) + \mathbf{B}_{r2}^j\mathbf{r}(k)\right]\right)\right] \\ &- \sum_{i=1}^{p}\sum_{j=1}^{p} h^i h^j [\mathbf{B}_a^{ij}], \end{aligned} \tag{11}$$

where $\mathbf{\Psi}(\cdot)$ is a nonlinear vector function, resulting from antiwindup filtration, defined as $\mathbf{\Psi}(\cdot) = \text{sat}(\cdot) - (\cdot)$; $\mathbf{A}_{CL}^{ij}, \mathbf{B}_{r1}^{ij}, \mathbf{B}_\Psi^{ij}, \mathbf{K}^j, \mathbf{B}_{r2}^j, \mathbf{B}_a^{ij}$ are the CL system matrices defined as follows:

$$\begin{aligned} \mathbf{A}_{CL}^{ij} &= \begin{bmatrix} \mathbf{A}^i - g\mathbf{B}^i\mathbf{D}_c^j\mathbf{C}^i & g\mathbf{B}^i \\ -\mathbf{B}_c^j\mathbf{C}^i & \mathbf{I}^{no\times nu} \end{bmatrix}, \\ \mathbf{B}_\Psi^{ij} &= \begin{bmatrix} g\mathbf{B}^i \\ -\mathbf{E}_c^j \end{bmatrix}, \\ \mathbf{B}_{r1}^{ij} &= \begin{bmatrix} g\mathbf{B}^i\mathbf{D}_c^j\mathbf{C}^i \\ \mathbf{B}_c^j\mathbf{C}^i \end{bmatrix}, \\ \mathbf{K}^j &= \begin{bmatrix} -\mathbf{D}_c^j\mathbf{C}^i & \mathbf{I}^{1\times 1} \end{bmatrix}, \\ \mathbf{B}_{r2}^j &= \begin{bmatrix} \mathbf{D}_c^j\mathbf{C}^i \end{bmatrix}, \\ \mathbf{B}_a^{ij} &= \begin{bmatrix} \mathbf{b}^i \\ \mathbf{E}_c^j\mathbf{b}^i \end{bmatrix}, \end{aligned} \tag{12}$$

while the change in indexation results from

$$\sum_{i=1}^{p} h^i \sum_{j=1}^{p} h^j = \sum_{i=1}^{p}\sum_{j=1}^{p} h^i h^j.$$

A considerable simplification can be made by rewriting (11) as

$$\begin{aligned} &\xi(k+1) \\ &= \sum_{l=1}^{p} h^l [\mathbf{A}_{CL}^l\xi(k) + \mathbf{B}_{r1}^l\mathbf{r}(k)] \\ &- \sum_{l=1}^{K} h^l \left[\mathbf{B}_\Psi^l\mathbf{\Psi}\left(\sum_{l=1}^{K} h^l \left[\mathbf{K}^l\xi(k) + \mathbf{B}_{r2}^l\mathbf{r}(k)\right]\right)\right] \\ &- \sum_{l=1}^{K} h^l [\mathbf{B}_a^l], \end{aligned} \tag{13}$$

where $l = \overline{1, K}$ and $K = \sum_{k=1}^{p} k$.

For the purpose of stability validation only the internal dynamics of the extended state vector are important, so in further analysis an autonomous system given by

$$\xi(k+1) = \sum_{l=1}^{K} h^l \left[\mathbf{A}_{CL}^l\xi(k) - \mathbf{B}_\Psi^l\mathbf{\Psi}\left(\xi(k)\right)\right], \tag{14}$$

is considered where $\boldsymbol{\Psi}(\xi(k)) \equiv \boldsymbol{\Psi}\left(\sum_{l=1}^{K} h^l \left[\mathbf{K}^l \xi(k)\right]\right)$ is introduced to simplify the notation.

3.2. CQLF based stability analysis. Sufficient conditions for global stability of the system (14) are given in Theorem 1. The presented approach is an extension of that by Gomes da Silva and Tarbouriech (2006); see Section 1.

Theorem 1. *The system (14) is globally stable if the following set of LMIs (15) is satisfied:*

$$\begin{bmatrix} \mathbf{X} & -\mathbf{X}(\mathbf{K}^l)^T & -\mathbf{X}(\mathbf{A}_{CL}^l)^T \\ \mathbf{K}^l\mathbf{X} & 2\mathbf{Y} & \mathbf{Y}(\mathbf{B}_{\Psi}^l)^T \\ \mathbf{A}_{CL}^l\mathbf{X} & \mathbf{B}_{\Psi}^l\mathbf{Y} & \mathbf{X} \end{bmatrix} > 0, \quad (15)$$

with a diagonal matrix $\mathbf{Y} > 0$, $\mathbf{Y} \in \mathbb{R}^{nu\times nu}$, *and a matrix* $\mathbf{X} = \mathbf{X}^T > 0$, $\mathbf{X} \in \mathbb{R}^{(n+nu)\times(n+nu)}$. *Here* $l = \overline{1,K}$ *and* $K = \sum_{l=1}^{p} k$.

Finding a feasible solution to the LMI system (15) guarantees that the CQLF is strictly decreasing along the trajectory of the system, thus implying the asymptotic stability. A complete proof is given in Appendix A.

3.3. FLF based stability analysis. Sufficient and less conservative conditions for global stability of the system (14) are given in the following result.

Theorem 2. *Consider positive-definite matrices* $\mathbf{X}^\rho = (\mathbf{X}^\rho)^T$, $\mathbf{X}^l = (\mathbf{X}^l)^T$ *and* $\mathbf{X}^l, \mathbf{X}^\rho \in \mathbb{R}^{(n+nu)\times(n+nu)}$; *and diagonal positive-definite matrix* $\mathbf{Y}$, $\mathbf{Y} \in \mathbb{R}^{nu\times nu}$. *The CL system is globally stable if the LMI*

$$\begin{bmatrix} \mathbf{X}^l & -\mathbf{X}^l(\mathbf{K}^l)^T & -\mathbf{X}^l(\mathbf{A}_{CL}^l)^T \\ \mathbf{K}^l\mathbf{X}^l & 2\mathbf{Y} & \mathbf{Y}(\mathbf{B}_{\Psi}{}^l)^T \\ \mathbf{A}_{CL}^l\mathbf{X}^l & \mathbf{B}_{\Psi}{}^l\mathbf{Y} & \mathbf{X}^\rho \end{bmatrix} > 0, \quad (16)$$

is satisfied, where $\rho, l = \overline{1,K}$ *and* $K = \sum_{k=1}^{p} k$.

For the proof, see Appendix B.

Remark 1. Note that, by choosing fuzzy Lyapunov function matrices $\mathbf{P}^l$, where $l = \overline{1,K}$, according to

$$\mathbf{P}^1 = \mathbf{P}^2 = \cdots = \mathbf{P}^K = \mathbf{P}^*, \quad (17)$$

the following holds:

$$\mathbf{P}(\mathbf{z}(k)) = \sum_{l=1}^{K} h^l \mathbf{P}^l = \mathbf{P}^* \sum_{l=1}^{K} h^l = \mathbf{P}^*. \quad (18)$$

Hence the fuzzy Lyapunov function defined by (A4) becomes the Lyapunov function defined by (A1) with $\mathbf{P} = \mathbf{P}^*$. In other words, any solutions of the LMI (15) can be used to design the solution of the LMI (16). This simply means that by applying Theorem 2 certain conservatism reduction can be obtained. This reduction has been achieved by adding an extra degree of freedom in designing the Lyapunov function, namely, by applying Theorem 2. Instead of looking for a single matrix satisfying all LMI conditions (see Theorem 1), one can search for a different matrix in every separate region just ensuring that in the 'transition' (mixed) regions certain properties (given by the LMIs) hold. This actually results in greater freedom in choosing the controller gains in comparison to Theorem 1. This, however, is at the cost of an increase in the number of LMIs.

4. Numerical example

In this section two numerical examples are considered. One is an application of the derived theorems to verify the stability of the CL control system utilised to stabilise the pH level in a continuous stirred tank pH neutralisation reactor.

Example 1. Consider a nonlinear plant (modified example introduced originally by Tatjewski (2007)), whose dynamics can be represented by a TS fuzzy system,

$$\begin{aligned} &R_1: \text{If } z_1^1 \text{ is } MF_1^1 \text{ then} \\ &\qquad \mathbf{x}(k+1) = 0.7\mathbf{x}(k) + 0.8\mathbf{u}(k), \\ &R_2: \text{If } z_1^2 \text{ is } MF_1^2 \text{ then} \\ &\qquad \mathbf{x}(k+1) = 0.3\mathbf{x}(k) + 0.2\mathbf{u}(k), \end{aligned} \quad (19)$$

and a designed (according to the procedure presented by Han *et al.*, (2008) and Zubowicz *et al.* (2010), multiregional controller, whose parameters of which are as follows: $K_P^1 = 0.9$; $K_P^2 = 2$; $K_I^1 = 0.4$; $K_I^2 = 1$; $K_{AW}^1 = 0.3$; $K_{AW}^2 = 0.5$; $g = 2$; $c_{mx} = 1$.

Solving (15) with the data presented above resulted in finding the following matrices

$$\mathbf{P} = \begin{bmatrix} 0.1055 & 0.0286 \\ 0.0286 & 0.4541 \end{bmatrix},$$
$$\mathbf{T} = \begin{bmatrix} 1.108 \end{bmatrix}.$$

On the other hand, solving the LMI problem (16) with the same data set produced a solution with the following matrices

$$\mathbf{P}^1 = \begin{bmatrix} 0.1581 & 0.0490 \\ 0.0490 & 0.7663 \end{bmatrix},$$
$$\mathbf{P}^{12} = \begin{bmatrix} 0.1574 & 0.0402 \\ 0.0402 & 0.6440 \end{bmatrix},$$
$$\mathbf{P}^2 = \begin{bmatrix} 0.3646 & 0.1381 \\ 0.1381 & 0.7121 \end{bmatrix},$$
$$\mathbf{T} = 1.0\text{e+}7 \begin{bmatrix} 1.6737 \end{bmatrix}.$$

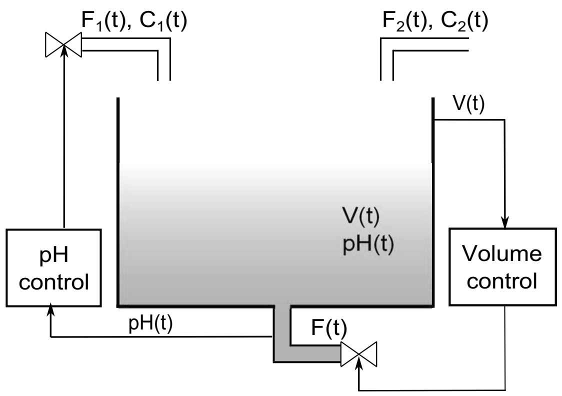

Fig. 1. Continuous stirred tank pH neutralisation reactor.

Clearly, the structure of the Lyapunov function determined by matrices $\mathbf{P}^1$, $\mathbf{P}^{12}$, $\mathbf{P}^2$ is less restrictive than the structure of such a function defined by matrix $\mathbf{P}$. ♦

Example 2. Consider a continuous stirred tank pH neutralisation reactor (see Fig. 1) given by the following set of nonlinear dynamic equations (following Domański *et al.*, 1999):

$$\frac{\mathrm{d}V\eta}{\mathrm{d}t} = F_1C_1 - (F_1+F_2)\eta, \tag{20}$$

$$\frac{\mathrm{d}V\zeta}{\mathrm{d}t} = F_2C_2 - (F_1+F_2)\zeta, \tag{21}$$

$$\frac{\mathrm{d}V}{\mathrm{d}t} = F_1 + F_2 - F, \tag{22}$$

$$\left[H^+\right]^3 + a\left[H^+\right]^2 + b\left[H^+\right] + c = 0, \tag{23}$$

where

$$\eta \cong [HAC] + \left[AC^-\right],$$
$$\zeta \cong \left[Na^+\right],$$
$$pH = -\log_{10}\left[H^+\right],$$
$$a = (K_a + \zeta),$$
$$b = (K_a(\zeta-\eta) - K_w),$$
$$c = -K_aK_w, \quad C_1 = 0.32\ \text{[mol/l]}$$

is the acid concentration in flow F_1, $C_2 = 0.05005$ [mol/l] is the acid concentration in flow F_2, $V^* = 1000$ [l], $K_a = 1.8\text{e}{-5}$ and $K_w = 1.0\text{e}{-14}$ are acid and water equilibrium constants respectively $F_1(0) = 81$ [l/min], $F_2(0) = 512$ [l/min].

In this example a discretisation time T_s equal to 0.1 s was utilised.

The CL matrices are as follows:

$$\mathbf{A}_{CL}^1 = \begin{bmatrix} 1.5447 & -0.7755 & 0.1542 & -0.2070 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.0142 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_{CL}^{12} = \begin{bmatrix} 0.3878 & -0.0098 & -0.2418 & -3.8835 \\ 1.0000 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.0074 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_{CL}^2 = \begin{bmatrix} 0.7832 & 0.7559 & -0.6378 & -7.5600 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.0007 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_{CL}^{23} = \begin{bmatrix} 0.0225 & 0.1832 & -0.3054 & -3.9195 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.0095 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{A}_{CL}^3 = \begin{bmatrix} 1.2439 & -0.3896 & 0.0270 & -0.2790 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0.0183 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{B}_\Psi^1 = \begin{bmatrix} -0.2070 & 0 & 0 & 0.0425 \end{bmatrix}^T,$$

$$\mathbf{B}_\Psi^{12} = \begin{bmatrix} -3.8835 & 0 & 0 & 0.0218 \end{bmatrix}^T,$$

$$\mathbf{B}_\Psi^2 = \begin{bmatrix} -7.5600 & 0 & 0 & 0.0011 \end{bmatrix}^T,$$

$$\mathbf{B}_\Psi^{23} = \begin{bmatrix} -3.9195 & 0 & 0 & 0.0417 \end{bmatrix}^T,$$

$$\mathbf{B}_\Psi^3 = \begin{bmatrix} -0.2790 & 0 & 0 & 0.0823 \end{bmatrix}^T,$$

$$\mathbf{K}^1 = \begin{bmatrix} 0.2222 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{K}^{12} = \begin{bmatrix} 0.1167 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{K}^2 = \begin{bmatrix} 0.0111 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{K}^{23} = \begin{bmatrix} 0.1472 & 0 & 0 & 1 \end{bmatrix},$$

$$\mathbf{K}^3 = \begin{bmatrix} 0.2833 & 0 & 0 & 1 \end{bmatrix},$$

Verifying system stability by solving the LMI conditions of Theorem 1 in the search of a CQLF for a CL system yields

$$\mathbf{P} = \begin{bmatrix} 0.1032 & 0 & 0 & 0.0001 \\ 0 & 0.7016 & 0 & 0 \\ 0 & 0 & 0.7016 & 0 \\ 0.0001 & 0 & 0 & 0.7016 \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} 22.8412 \end{bmatrix}.$$

Solving the LMI conditions of Theorem 2 gives

$$\mathbf{P}^1 = \begin{bmatrix} 0.6631 & 0 & 0 & 0.0054 \\ 0 & 1.4878 & 0 & 0 \\ 0 & 0 & 1.4878 & 0 \\ 0.0054 & 0 & 0 & 1.4877 \end{bmatrix},$$

$$\mathbf{P}^{12} = \begin{bmatrix} 0.1857 & 0 & 0 & 0.0002 \\ 0 & 1.2609 & 0 & 0 \\ 0 & 0 & 1.2609 & 0 \\ 0.0002 & 0 & 0 & 1.2608 \end{bmatrix},$$

$$\mathbf{P}^{2} = \begin{bmatrix} 0.1857 & 0 & 0 & 0.0002 \\ 0 & 1.2609 & 0 & 0 \\ 0 & 0 & 1.2609 & 0 \\ 0.0002 & 0 & 0 & 1.2608 \end{bmatrix},$$

$$\mathbf{P}^{23} = \begin{bmatrix} 0.1857 & 0 & 0 & 0.0002 \\ 0 & 1.2609 & 0 & 0 \\ 0 & 0 & 1.2609 & 0 \\ 0.0002 & 0 & 0 & 1.2608 \end{bmatrix},$$

$$\mathbf{P}^{3} = \begin{bmatrix} 0.6525 & 0 & 0 & 0.0104 \\ 0 & 1.4878 & 0 & 0 \\ 0 & 0 & 1.4878 & 0 \\ 0.0104 & 0 & 0 & 1.4876 \end{bmatrix},$$

$$\mathbf{T} = \begin{bmatrix} 41.4704 \end{bmatrix}.$$

Similarly as in Example 1, the approach utilising FLF (see Theorem 2) generates less restrictive solutions. This results in extra freedom during controller design (choosing controller gains). ♦

To sum up, notice that in both examples the two derived theorems were successfully applied to verify the stability of the CL systems. However, the approach resulting from Theorem 2 was identified as less conservative as it gives an extra degree of freedom (see Remark 1).

A standard MathWorks MATLAB LMI toolbox was applied to find solutions to LMI feasibility problems introduced in both presented examples.

5. Conclusions

The work presented in this paper addressed the stability verification problem of closed loop control systems comprising of Takagi–Sugeno multiregional dynamic output controllers with static antiwindup filters. Sufficient stability conditions were derived based on the common quadratic Lyapunov function and a fuzzy Lyapunov function, respectively, in the form of an LMI feasibility test. The main result of the paper was supported by two numerical examples, one of which was stability verification of the closed loop control system utilised for pH neutralisation in a continuous stirred tank reactor. The undergoing research is on deriving LMI conditions which will quantify controller robustness assured by robustness of regional PI controllers.

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Appendix A

Proof of Theorem 1

Consider the discrete-time CQLF candidate

$$V(\xi(k)) = \xi^T(k)\mathbf{P}\xi(k), \tag{A1}$$

where $\xi(k)$ is the trajectory of the CL system. The function $V(\xi(k))$ is considered to be positive-definite in $D \setminus 0$ and $\mathbf{P} = \mathbf{P}^T > 0$, $\mathbf{P} \in \mathbb{R}^{(n+nu)\times(n+nu)}$.

The difference along the trajectory of the system is given by

$$\begin{aligned}\Delta V(\xi(k)) &= V(\xi(k+1)) - V(\xi(k)) \\ &= \xi^T(k+1)\mathbf{P}\xi(k+1) - \xi^T(k)\mathbf{P}\xi(k).\end{aligned} \tag{A2}$$

By inserting (14) into (A2), we obtain

$$\begin{aligned}&\Delta V(\xi(k)) \\ &= \sum_{l=1}^{K} h^l \left[\xi^T(k)\left((\mathbf{A}^l_{CL})^T\mathbf{P}\mathbf{A}^l_{CL} - \mathbf{P}\right)\xi(k)\right] \\ &- \sum_{l=1}^{K} h^l \left[2\xi^T(k)\left((\mathbf{A}^l_{CL})^T\mathbf{P}\mathbf{B}^l_{\Psi}\right)\mathbf{\Psi}(\xi(k))\right] \\ &+ \sum_{l=1}^{K} h^l \left[\mathbf{\Psi}^T(\xi(k))\left((\mathbf{B}^l_{\Psi})^T\mathbf{P}\mathbf{B}^l_{\Psi}\right)\mathbf{\Psi}(\xi(k))\right].\end{aligned} \tag{A3}$$

Applying the sector condition (Khalil, 1996),

$$\mathbf{\Psi}^T(\xi(k))\mathbf{T}\left[\mathbf{\Psi}(\xi(k)) - \sum_{l=1}^{K} h^l \left[\mathbf{K}^l\xi(k)\right]\right], \tag{A4}$$

with a diagonal matrix $\mathbf{T} > 0$, $\mathbf{T} \in \mathbb{R}^{(nu)\times(nu)}$, into the right hand side of the inequality (A3), and knowing that

$0 \leq \sum_{l=1}^{K} h^l \leq 1$, yields the upper bound,

$$\begin{aligned}
&\Delta V(\xi(k)) \\
&\leq \xi^T(k)\left((\mathbf{A}_{CL}^l)^T \mathbf{P} \mathbf{A}_{CL}^l - \mathbf{P}\right)\xi(k) \\
&\quad - 2\xi^T(k)\left((\mathbf{A}_{CL}^l)^T \mathbf{P} \mathbf{B}_{\Psi}^l - (\mathbf{K}^l)^T \mathbf{T}\right)\mathbf{\Psi}(\xi(k)) \\
&\quad + \mathbf{\Psi}^T(\xi(k))\left((\mathbf{B}_{\Psi}^l)^T \mathbf{P} \mathbf{B}_{\Psi}^l - 2\mathbf{T}\right)\mathbf{\Psi}(\xi(k)),
\end{aligned} \tag{A5}$$

or

$$\begin{aligned}
&\Delta V(\xi(k)) \\
&\leq -\left[\xi^T(k)\ \mathbf{\Psi}(\xi(k))\right] \\
&\times \begin{bmatrix} \mathbf{P} - (\mathbf{A}_{\mathbf{CL}}^l)^T \mathbf{P} \mathbf{A}_{CL}^l & (\mathbf{K}^l)^T \mathbf{T} - (\mathbf{A}_{CL}^l)^T \mathbf{P} \mathbf{B}_{\Psi}^l \\ * & 2\mathbf{T} - (\mathbf{B}_{\Psi}^l)^T \mathbf{P} \mathbf{B}_{\Psi}^l \end{bmatrix} \\
&\times \begin{bmatrix} \xi^T(k) \\ \mathbf{\Psi}(\xi(k)) \end{bmatrix}.
\end{aligned} \tag{A6}$$

From (A6) it is straightforward that the candidate function (A1) is a global CQLF for the system (14) if the matrix in the quadratic form of (A6) is positive-definite with diagonal matrix $\mathbf{T} > 0$ and matrix $\mathbf{P} = \mathbf{P}^T > 0$. It is also straightforward that by applying Schur's compliment to (15) and pre-and post-multiplying it by $\left[\begin{smallmatrix} \mathbf{P} & \mathbf{0} \\ \mathbf{0} & \mathbf{T} \end{smallmatrix}\right]$ with $\mathbf{X} = \mathbf{P}^{-1}$ and $\mathbf{Y} = \mathbf{T}^{-1}$ one obtains an expression equivalent to the matrix present in the quadratic form (A6), which concludes the proof.

Appendix B
Proof of Theorem 2

Consider the discrete-time FLF candidate

$$V(\xi(k)) = \xi^T(k)\mathbf{P}(\mathbf{z}(k))\,\xi(k), \tag{B1}$$

where

$$\mathbf{P}(\mathbf{z}(k)) = \sum_{l=1}^{K} h^l(\mathbf{z}(k))\,\mathbf{P}^l,$$

$$\mathbf{P}^l = (\mathbf{P}^l)^T > 0,$$

$\mathbf{P}^l \in \mathbb{R}^{(n+nu)\times(n+nu)}$, for $\mathbf{z}(k) \in \mathbb{R}^v$.

The difference along the trajectory of the system is given by

$$\begin{aligned}
&\Delta V(\xi(k)) \\
&= \xi^T(k+1)\left(\sum_{\rho=1}^{K} h^\rho(\mathbf{z}(k+1))\,\mathbf{P}^\rho\right)\xi(k+1) \\
&\quad - \xi^T(k)\left(\sum_{l=1}^{K} h^l(\mathbf{z}(k))\,\mathbf{P}^l\right)\xi(k).
\end{aligned} \tag{B2}$$

By inserting (14) into (B2) one gets

$$\begin{aligned}
&\Delta V(\xi(k)) \\
&= \sum_{l=1}^{K}\sum_{\rho=1}^{K}\sum_{m=1}^{K} h^l h_+^\rho h^m \\
&\quad \times \left[\mathbf{A}_{CL}^l \xi(k) - \mathbf{B}_{\Psi}^l \Psi(\xi(k))\right]^T \mathbf{P}^\rho \qquad \text{(B3)} \\
&\quad \times \left[\mathbf{A}_{CL}^m \xi(k) - \mathbf{B}_{\Psi}^m \Psi\left(\sum_{m=1}^{K} h^m \mathbf{K}^m \xi(k)\right)\right] \\
&\quad - \sum_{l=1}^{K}\sum_{\rho=1}^{K}\sum_{m=1}^{K} h^l h_+^\rho h^m \left\{\xi^T(k)\mathbf{P}^l \xi(k)\right\}, \qquad \text{(B4)}
\end{aligned}$$

where $l, \rho, m = \overline{1,K}$, $h_+^\rho \equiv h^\rho(\mathbf{z}(k+1))$.

Knowing that

$$\mathbf{P}(\mathbf{z}(k)) = \sum_{l=1}^{K}\sum_{\rho=1}^{K}\sum_{m=1}^{K} h^l h_+^\rho h^m \mathbf{P}^l, \tag{B5}$$

the expression (B3) can be rewritten as

$$\begin{aligned}
&\Delta V(\xi(k)) \\
&= \xi^T(k)\sum_{l=1}^{K}\sum_{\rho=1}^{K}\sum_{m=1}^{K} h^l h_+^\rho h^m \\
&\quad \times \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^m - \mathbf{P}^l\right]\xi(k) \\
&\quad - \xi^T(k)\sum_{l=1}^{K}\sum_{\rho=1}^{K}\sum_{m=1}^{K} h^l h_+^\rho h^m \\
&\quad \times \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{B}_{\Psi}^m\right]\mathbf{\Psi}\left(\sum_{m=1}^{K} h^m \mathbf{K}^m \xi(k)\right) \qquad \text{(B6)} \\
&\quad - \sum_{l=1}^{K}\sum_{\rho=1}^{K}\sum_{m=1}^{K} h^l h_+^\rho h^m \\
&\quad \times \mathbf{\Psi}^T(\xi(k))\left[(\mathbf{B}_{\Psi}^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^m\right]\xi(k) \\
&\quad + \sum_{l=1}^{K}\sum_{\rho=1}^{K}\sum_{m=1}^{K} h^l h_+^\rho h^m \\
&\quad \times \mathbf{\Psi}^T(\xi(k))\left[(\mathbf{B}_{\Psi}^l)^T \mathbf{P}^\rho \mathbf{B}_{\Psi}^m\right]\mathbf{\Psi}\left(\sum_{m=1}^{K} h^m \mathbf{K}^m \xi(k)\right).
\end{aligned}$$

Since

$$\begin{aligned}
\mathbf{\Theta}_1 &= \sum_{m=1}^{K}\sum_{m=1}^{K} h^l h^m \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^m - \mathbf{P}^l\right] \\
&= \sum_{l=1}^{K} (h^l)^2 \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^l - \mathbf{P}^l\right]
\end{aligned} \tag{B7}$$

$$+\sum_{l=1}^{K}\sum_{m=1}^{K} h^l h^m \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^m - \mathbf{P}^l\right]$$
$$+\sum_{l=1}^{K}\sum_{m=1}^{K} h^l h^m \left[(\mathbf{A}_{CL}^m)^T \mathbf{P}^\rho \mathbf{A}_{CL}^l - \mathbf{P}^m\right], \quad \text{(B8)}$$

it follows that

$$\begin{aligned}\mathbf{\Theta}_1 \le &\sum_{l=1}^{K} \left(h^l\right)^2 \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^l - \mathbf{P}^l\right] \\ &+\sum_{l=1}^{K}\sum_{m=1}^{K} h^l h^l \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^l - \mathbf{P}^l\right] \\ &+\sum_{l=1}^{K}\sum_{m=1}^{K} h^l h^m \left[(\mathbf{A}_{CL}^m)^T \mathbf{P}^\rho \mathbf{A}_{CL}^m - \mathbf{P}^m\right] \\ =&\sum_{l=1}^{K} h^l \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^l - \mathbf{P}^l\right] = \widetilde{\mathbf{\Theta}}_1. \quad \text{(B9)}\end{aligned}$$

Other parts of the expression (B6) can be bounded similarly,

$$\mathbf{\Theta}_2 \le \sum_{l=1}^{K} h^l \left[(\mathbf{A}_{CL}^l)^T \mathbf{P}^\rho \mathbf{B}_\Psi^l\right] \mathbf{\Psi}(\xi(k)) = \widetilde{\mathbf{\Theta}}_2, \quad \text{(B10)}$$

$$\mathbf{\Theta}_3 \le \sum_{l=1}^{K} h^l \mathbf{\Psi}^T(\xi(k)) \left[(\mathbf{B}_\Psi^l)^T \mathbf{P}^\rho \mathbf{A}_{CL}^l\right] = \widetilde{\mathbf{\Theta}}_3, \quad \text{(B11)}$$

$$\begin{aligned}\mathbf{\Theta}_4 \le &\sum_{l=1}^{K} h^l \mathbf{\Psi}^T(\xi(k)) \left[(\mathbf{B}_\Psi^l)^T \mathbf{P}^\rho \mathbf{B}_\Psi^l\right] \mathbf{\Psi}(\xi(k)) \\ =& \widetilde{\mathbf{\Theta}}_4. \quad \text{(B12)}\end{aligned}$$

As in the work of Wang *et al*. (2004), by utilising (B9)–(B12), the following upper bound on the difference of the FLF candidate can be obtained:

$$\begin{aligned}\Delta V(\xi(k)) \le &\xi^T(k) \sum_{\rho=1}^{K} h_+^\rho \widetilde{\mathbf{\Theta}}_1 \xi(k) - \xi^T(k) \sum_{\rho=1}^{K} h_+^\rho \widetilde{\mathbf{\Theta}}_2 \\ &- \sum_{\rho=1}^{K} h_+^\rho \widetilde{\mathbf{\Theta}}_3 \xi(k) + \sum_{l=1}^{K} h^l \widetilde{\mathbf{\Theta}}_4. \quad \text{(B13)}\end{aligned}$$

The rest follows analogously to the proof of Theorem 1.

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