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ROBUST OBSERVER DESIGN FOR SUGENO SYSTEMS WITH INCREMENTAL QUADRATIC NONLINEARITY IN THE CONSEQUENT

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This paper is concerned with observer design for nonlinear systems that are modeled by T–S fuzzy systems containing parametric and nonparametric uncertainties. Unlike most Sugeno models, the proposed method contains nonlinear functions in the consequent part of the fuzzy IF-THEN rules. This will allow modeling a wider class of systems with smaller modeling errors. The consequent part of each rule contains a linear part plus a nonlinear term, which has an incremental quadratic constraint. This constraint relaxes the conservativeness introduced by other regular constraints for nonlinearities such as the Lipschitz conditions. To further reduce the conservativeness, a nonlinear injection term is added to the observer dynamics. Simulation examples show the effectiveness of the proposed method compared with the existing techniques reported in well-established journals.

Keywords: nonlinear Sugeno model, incremental quadratic constraint, robust observer.

1. Introduction

Observer and observer-based controller design for uncertain Sugeno systems has been widely addressed by many researchers over the last decades (Tseng *et al.*, 2009; Yoneyama, 2009; Xu *et al.*, 2012; Ichalal *et al.*, 2012). Sugeno systems are popular for their local linear form, which allows one to use powerful existing tools (e.g., Linear Matrix Inequality (LMI)) for analysis and design of these systems. When uncertainties exist in the model, the observer-based controller design for Takagi–Sugeno (T–S) systems becomes harder as the problem results in Bilinear Matrix Inequalities (BMIs) instead of LMIs. Some researchers have tried to overcome this problem; examples of such works are reported by Asemani and Majd (2013), Chadli and Guerra (2012), as well as Dong *et al.* (2010; 2011).

As the complexity of the system increases, the number of rules in the fuzzy model and hence the number and dimensions of LMIs (used for stability analysis) increase and become harder to solve. Many works in the literature are devoted to decreasing the conservativeness of these LMIs in order to apply them to a wider class of systems (Guerra *et al.*, 2012; Abdelmalek *et al.*, 2007; Bernal and Hušek, 2005). Another possible solution is to use nonlinear local subsystems for the T–S model. Although it seems that this method increases the complexity of the fuzzy model, it decreases the number of rules and at the same time increases the model accuracy. The key idea of using nonlinear terms in the subsystems is to employ some kind of nonlinearity, which is less complicated than the nonlinearities of the main system.

A very simple form of these nonlinear T–S models is used by Rajesh and Kaimal (2007). The authors used linear form for the consequence part plus a sinusoidal term. A more advanced work is performed by Dong *et al.* (2010; 2011), who employed sector-bounded functions in the subsystems. Tanaka *et al.* (2009a; 2009b) proposed a T–S model with polynomial subsystems. For stability analysis, they used the Sum Of Squares (SOS) approach. This was the first use of the SOS instead of the LMI in fuzzy systems analysis. Sala and Arino (2009) as well as Sala (2009) represented a similar form of the Sugeno model and used the Taylor series expansion of the system for construction of the polynomial subsystems. The authors state that the nonlinear consequent part in the T–S model not only reduces the number of rules, but also reduces the conservativeness in the controller design.

In this paper, a similar form of Dong's model is employed. In other words, every subsystem in the Sugeno model contains a linear plus a nonlinear term in the consequent part of the fuzzy IF-THEN rules.

However, unlike in the previous works, this nonlinear term is not assumed to be Lipschitz, which is a mild condition but results in conservative designs. Instead, in this paper, the incremental Quadratic Constraint (δQC) is adopted (Açikmese and Corless, 2011). This constraint is less conservative compared with the Lipschitz condition. Hence, it can encompass a larger class of nonlinearities. In addition, for the first time, a nonlinear injection term is added to the fuzzy observer that provides more degrees of freedom to the design procedure. For further reduction in conservativeness, Fuzzy Lyapunov Functions (FLFs) are employed.

The FLF is one of the three classes of Lyapunov functions that are used to analyze T–S systems. The other two classes are the traditional quadratic Lyapunov functions and piecewise Lyapunov functions, which are usually more conservative than the FLF. A complete review of recent Lyapunov functions for discrete fuzzy systems is presented by Guerra *et al*. (2009). The FLF, as a non-quadratic Lyapunov function, has been of increasing interest in recent years. In this case, quadratic Lyapunov functions share the same membership functions with the T–S fuzzy model. For continuous-time systems, it is more difficult to obtain LMI conditions using the FLF, compared with discrete-time systems, because the stability conditions depend on the time derivative of the membership functions, which are usually handled with very conservative bounds.

Other approaches have been also investigated. Rhee and Won (2006) proposed a method which does not depend on the derivative of the membership functions. Mozelli *et al*. (2009) derived LMI conditions for state feedback controller design by adding some slack matrices. Results of Guerra and Bernal (2012), as well as Guerra *et al*. (2012), overcome the aforementioned deficiency by providing local asymptotic conditions at the price of computationally demanding LMIs. In this paper, the FLF is used based on the work by Faria *et al*. (2012).

The reminder of the paper is organized as follows. In Section 2, a nonlinear Sugeno model and an incremental quadratic constraint are introduced. Section 3 provides the problem of observer design for nonlinear T–S systems along with analytical results. Numerical examples are given in Section 4 to show effectiveness of the proposed method. Section 5 concludes the paper.

2. Problem statement

Consider the class of nonlinear systems described by

$$\begin{aligned}\dot{x}(t) &= f_a(x(t)) + f_b(x(t))\varphi\big(x(t), u(t), t\big)\\ &\quad + g(x(t))u(t),\\ y(t) &= f_{ya}(x(t)) + f_{yb}(x(t))\varphi\big(x(t), u(t), t\big),\end{aligned} \tag{1}$$

where $x(t) \in \mathbb{R}^{n_x}$ is the state, $u(t) \in \mathbb{R}^{n_u}$ is the control input, $y(t) \in \mathbb{R}^{n_y}$ is the measurable output, $f_n(x(t)) :$ $n \in \{a, b, ya, yb\}$ and $g(x(t)) \in \mathbb{R}^{(n_x \times n_u)}$ are nonlinear functions, and $\varphi\big(x(t), u(t), t\big) \in \mathbb{R}^{(n_x \times n_\varphi)}$ is a vector of nonlinear functions.

2.1. Incremental quadratic constraint. Suppose that the following relation exists:

$$\begin{aligned}\varphi\big(x(t), u(t), t\big) &= \phi\big(s(t), q(t)\big),\\ q(t) &= C_q x(t) + D_q \varphi\big(x(t), u(t), t\big),\end{aligned} \tag{2}$$

where $q \in \mathbb{R}^{n_q}$ and C_q and D_q are constant matrices with proper dimensions and $s(t) = \big(t, u(t), y(t)\big)$. For simplicity, in the rest of this paper, $s(t)$ and $q(t)$ are shown with s and q, respectively. Note that the term D_q is included to treat systems where the nonlinear term depends on the derivative of a state variable. Characterization of the nonlinear element $\phi(s, q)$ is based on a set of symmetric matrices $\mathcal{M}$, which is referred to as incremental multiplier matrices (Açikmese and Corless, 2011). Specifically, for all $M \in \mathcal{M}$ the following incremental quadratic constraint holds:

$$\begin{pmatrix} q_2 - q_1 \\ \phi(s, q_2) - \phi(s, q_1) \end{pmatrix}^T M \begin{pmatrix} q_2 - q_1 \\ \phi(s, q_2) - \phi(s, q_1) \end{pmatrix} \geq 0. \tag{3}$$

Defining $v := C_q x$, we have

$$\begin{aligned}\phi\Big(s, v + D_q \varphi\big(x(t), u(t), t\big)\Big) &= \psi(s, v),\\ \varphi\big(x(t), u(t), t\big) &= \psi\big(s, C_q x(t)\big).\end{aligned} \tag{4}$$

The implicit description of φ arises in many situations, for instance, when the nonlinear term depends on $\dot{x}$. As an example, consider the following plant (Açikmese and Corless, 2011):

$$\begin{aligned}\dot{x}_1 &= x_2,\\ \dot{x}_2 &= 0.5\sin(x_1 + \dot{x}_2).\end{aligned}$$

Letting $\varphi = \sin(x_1 + \dot{x}_2)$ yields $\dot{x}_1 = x_2, \dot{x}_2 = 0.5\varphi$. Now, let $q = x_1 + \dot{x}_2$ to obtain $\varphi = \phi(q) = \sin(q)$ with $q = C_q x + D_q \varphi$, where $C_q = [1, 0]$ and $D_q = 0.5$. For each $v = C_q x$, there exists a unique solution for $\varphi = \sin(v + 0.5\varphi)$, which can be denoted by $\varphi = \psi(v)$. In general, obtaining a δQC characterization for a nonlinearity is easier by using the function ϕ, rather than ψ when $D_q = 0$ (i.e., when φ is implicitly defined). In some cases, the only way to obtain ψ is via numerical methods, where ϕ may readily be shown to satisfy δQC.

Note that $\psi(s, v)$ satisfies the incremental quadratic constraint. That is,

$$\begin{pmatrix} \delta v \\ \delta \psi \end{pmatrix}^T N \begin{pmatrix} \delta v \\ \delta \psi \end{pmatrix} \geq 0, \tag{5}$$

where

$$N = \begin{pmatrix} I & D_q \\ 0 & I \end{pmatrix}^T M \begin{pmatrix} I & D_q \\ 0 & I \end{pmatrix}. \tag{6}$$

2.2. Nonlinear Sugeno model. The system (1) can be represented by a T–S fuzzy system with local nonlinear models and uncertainties as follows:

Plant Rule i:

if $z_1(t)$ is $\mu_{i1}(z), \dots,$ and $z_p(t)$ is $\mu_{ip}(z)$ then:

$$\begin{aligned}\dot{x}(t) &= (A_i + \Delta A_i)x(t) \\ &\quad + (G_{xi} + \Delta G_{xi})\varphi\big(x(t), u(t), t\big) \\ &\quad + (B_i + \Delta B_i)u(t) + D_{1i}\nu(t), \\ y(t) &= (C_i + \Delta C_i)x(t) \\ &\quad + (G_{yi} + \Delta G_{yi})\varphi\big(x(t), u(t), t\big) \\ &\quad + D_{2i}\nu(t),\end{aligned} \tag{7}$$

where $A_i \in \mathbb{R}^{(n_x \times n_x)}$, $B_i \in \mathbb{R}^{(n_x \times n_u)}$, $C_i \in \mathbb{R}^{(n_y \times n_x)}$, $G_{xi} \in \mathbb{R}^{(n_x \times n_\varphi)}$, $G_{yi} \in \mathbb{R}^{(n_y \times n_\varphi)}$, $D_{1i} \in \mathbb{R}^{(n_x \times n_\nu)}$, and $D_{2i} \in \mathbb{R}^{(n_y \times n_\nu)}$ $(i = 1, \dots, r)$ are constant matrices, in which r is the number of rules, n_x is the number of states, n_u is the number of inputs, n_y is the number of outputs, n_φ is the number of nonlinear functions in the vector φ, and n_ν is the dimension of ν. Moreover, $z_1(t), \dots, z_p(t)$ are the premise variables, the μ_{ij}'s denote the fuzzy sets, and $\nu(t)$ is a band-limited white noise.

The uncertainties are defined as

$$\begin{aligned}\Delta A_i &:= M_{1i}F_1N_1, \\ \Delta B_i &:= M_{1i}F_2N_2, \\ \Delta G_{xi} &:= M_{1i}F_3N_3, \\ \Delta C_i &:= M_{2i}F_4N_1, \\ \Delta G_{yi} &:= M_{2i}F_5N_3,\end{aligned} \tag{8}$$

where $F_i^T F_i < 1$ $(i = 1, \dots, 5)$, in which $F_i \in \mathbb{R}^{(n_F \times n_F)}$. In this case, the whole fuzzy system can be represented as

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r} \omega_i(z)\big[(A_i + \Delta A_i)x(t) + (B_i + \Delta B_i)u(t) \\ &\quad + (G_{xi} + \Delta G_{xi})\varphi(x(t), u(t), t) + D_{1i}\nu(t)\big], \\ y(t) &= \sum_{i=1}^{r} \omega_i(z)\big[(C_i + \Delta C_i)(x(t)) \\ &\quad + (G_{yi} + \Delta G_{yi})\varphi(x(t), u(t), t) + D_{2i}\nu(t)\big],\end{aligned} \tag{9}$$

where

$$\begin{aligned}\omega_i(z) &= \frac{h_i(z)}{\sum_{k=1}^{r} h_k(z)}, \\ h_i(z) &= \Pi_{j=1}^{p}\mu_{ij}(z).\end{aligned} \tag{10}$$

3. Observer design

The observer used in this paper is as follows:

Observer Rule i:

if $z_1(t)$ is $\mu_{i1}(z), \dots,$ and $z_p(t)$ is $\mu_{ip}(z)$ then:

$$\begin{aligned}\dot{\hat{x}}(t) &= A_i\hat{x}(t) + G_{xi}\hat{\varphi}\big(\hat{x}(t), u(t), t\big) + B_iu(t) \\ &\quad + L_i\big[\hat{y}(t) - y(t)\big], \\ \hat{y}(t) &= C_i\hat{x}(t) + G_{yi}\hat{\varphi}\big(\hat{x}(t), u(t), t\big), \\ \hat{\varphi}\big(\hat{x}(t), u(t), t\big) &= \psi\Big(s, C_q\hat{x}(t) + L_n[\hat{y}(t) - y(t)]\Big).\end{aligned} \tag{11}$$

Unlike other Sugeno observers, here the nonlinear injection term $L_n[\hat{y}(t) - y(t)]$ is used for better estimation of $\varphi(x(t), u(t), t)$, which in turn provides better estimation for all states of the system. For the system (7) and the observer (11), the following error dynamic can be stated:

$$\begin{aligned}&\dot{e}(t) \\ &= \sum_{i=1}^{r}\sum_{j=1}^{r} \omega_{ij}(z)\Big[(A_i + L_iC_j)e(t) \\ &\quad + (G_{xi} + L_iG_{yj})\delta\varphi(t) - \big[(\Delta A_i + L_i\Delta C_j)x(t) \\ &\quad + \Delta B_iu(t) + (\Delta G_{xi} + L_i\Delta G_{yj})\varphi\big(x(t), u(t), t\big) \\ &\quad + (D_{1i} + L_iD_{2j})\nu(t)\big]\Big],\end{aligned} \tag{12}$$

where

$$\begin{aligned}e(t) &= \hat{x}(t) - x(t), \\ \delta\varphi(t) &= \hat{\varphi}\big(\hat{x}(t), u(t), t\big) - \varphi\big(x(t), u(t), t\big), \\ \omega_{ij}(z) &= \omega_i(z)\omega_j(z).\end{aligned} \tag{13}$$

When there is no uncertainty in the model, from (11) it follows that $\hat{y}(t) - y(t) = \sum_{i=1}^{r} \omega_i(z)\big(C_ie(t) + G_{yi}\delta\varphi(t)\big)$. Hence, by defining two variables v_1 and v_2 as

$$\begin{aligned}v_1 &:= C_qx(t), \\ v_2 &:= C_q\hat{x}(t) + L_n\Big[\sum_{i=1}^{r} \omega_i(z)\big(C_ie(t) + G_{yi}\delta\varphi(t)\big)\Big],\end{aligned} \tag{14}$$

and based on (5), it follows that

$$\begin{pmatrix} e(t) \\ \delta\varphi(t) \end{pmatrix}^T \Phi^T M \Phi \begin{pmatrix} e(t) \\ \delta\varphi(t) \end{pmatrix} \geq 0, \tag{15}$$

where

$$\Phi := \sum_{i=1}^{r} \omega_i(z) \begin{pmatrix} C_q + L_nC_i & D_q + L_nG_{xi} \\ 0 & I \end{pmatrix}. \tag{16}$$

In order to analyze the system using LMIs, it is assumed that matrix M has the following form:

$$M = \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix}, \tag{17}$$

where $X \in \mathbb{R}^{(n_q \times n_q)}$, $Y \in \mathbb{R}^{(n_\varphi \times n_\varphi)}$, $X{=}X^T{>}0$, and $Y{=}Y^T{>}0$. Then, by defining $C_z := \sum_{i=1}^{r} \omega_i(z)C_i$ and using the same definition for G_{yz},

$$\Phi^T M \Phi = \begin{pmatrix} (C_q + L_n C_z)^T \\ (D_q + L_n G_{yz})^T \end{pmatrix} X \begin{pmatrix} C_q + L_n C_z & D_q + L_n G_{yz} \end{pmatrix} - \begin{pmatrix} 0 \\ I \end{pmatrix} Y \begin{pmatrix} 0 & I \end{pmatrix}. \tag{18}$$

3.1. Observer analysis. In this section, the conditions for the asymptotic convergence of the observer states in (11) to the system states in (7) will be given. The following lemmas are used in this paper.

Lemma 1. (Tuan *et al.*, 2001) *If*

$$\begin{aligned} M_{ii} &< 0, \quad 1 < i < r, \\ \frac{1}{r-1} M_{ii} + \frac{1}{2}(M_{ij} + M_{ji}) &< 0, \quad 1 < i \neq j < r, \end{aligned} \tag{19}$$

then

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \alpha_i \alpha_j M_{ij} < 0, \tag{20}$$

where $0 \leq \alpha_i \leq 1$ *and* $\sum_{i=1}^{r} \alpha_i = 1$.

Lemma 2. (Boyd *et al.*, 1994) *For any positive definite matrix* Π *with appropriate dimensions, the following property holds:*

$$X^T Y + Y^T X \leq X^T \Pi X + Y^T \Pi^{-1} Y. \tag{21}$$

In the following theorem, sufficient conditions for the stability of the error dynamic (12) will be given.

Theorem 1. *Assume* $|\dot{\omega}_i(z)| < \kappa_i$ *for known positive real numbers* κ_i*, where* $\dot{\omega}_i(z)$ *is the derivative of* $\omega_i(z)$ *with respect to time. The error dynamic (12) is asymptotically stable and with an* H_∞ *performance bound* $\gamma > 0$ *if there exist matrices* $P_{1\rho} = P_{1\rho}^T > 0$, $P_{2\rho} = P_{2\rho}^T > 0$ $(1 \leq \rho \leq r)$, X, Y, $X_1 = X_1^T$, $X_2 = X_2^T$, R_n, S_i, S_{Li} $(1 \leq i \leq 6)$*, and a scalar* $\eta > 0$ *such that* $P_{1\rho}, P_{2\rho}, X_1, X_2, S_1, S_2, S_3, S_4 \in \mathbb{R}^{n_x \times n_x}$, $X \in \mathbb{R}^{n_q \times n_q}$, $Y \in \mathbb{R}^{n_\varphi \times n_\varphi}$, $S_5 \in \mathbb{R}^{n_\varphi \times n_x}$, $S_6 \in \mathbb{R}^{n_u \times n_x}$, $S_{Li} \in \mathbb{R}^{n_x \times n_y}$, $R_n \in \mathbb{R}^{n_q \times n_y}$*, and*

$$\begin{aligned} P_{2\rho} + X_2 - P_{2\xi} &\geq 0, && \forall \rho \in 1, \dots, r - \xi, \\ P_{1\rho} + X_1 - P_{1\xi} &\geq 0, && \forall \rho \in 1, \dots, r - \xi, \\ \Xi_{ii} &< 0, && 1 < i < r, \\ \frac{1}{r-1} \Xi_{ii} + \frac{1}{2}(\Xi_{ij} + \Xi_{ji}) &< 0, && 1 < i \neq j < r, \end{aligned} \tag{22}$$

where ξ *is an arbitrary value in* $1, \dots, r$ *and*

$$\begin{aligned} \Xi_{ij} &= \begin{pmatrix} \mathcal{R}_{ij} & \Psi^T \\ \Psi & -X \end{pmatrix}, \\ \Psi &= \begin{pmatrix} XC_q + R_n C_i & 0_1 & XD_q + R_n G_{yi} & 0_2 \end{pmatrix}, \end{aligned} \tag{23}$$

in which 0_1 *and* 0_2 *are zero vectors with dimensions* $n_x \times (n_x + 2n_F)$ *and* $n_x \times (2n_x + n_\varphi + n_u + 4n_F)$*, respectively, and*

$$\mathcal{R}_{ij} = \begin{pmatrix} R_{11}^{ij} & * & * \\ \vdots & \ddots & * \\ R_{14,1}^{ij} & \cdots & R_{14,14}^{ij} \end{pmatrix}, \tag{24}$$

where

$$\begin{pmatrix} A & * \\ B & C \end{pmatrix} := \begin{pmatrix} A & B^T \\ B & C \end{pmatrix}$$

and

$$\begin{aligned}
R_{1,1}^{ij} &= E_1 - S_3 A_i - S_{Li} C_j - (S_3 A_i + S_{Li} C_j)^T + I, \\
R_{2,1}^{ij} &= (\sqrt{3} S_3 M_{1i} + \sqrt{2} S_{Li} M_{2j})^T, \\
R_{3,1}^{ij} &= P_{1i} - \eta S_3 A_i - \eta S_{Li} C_j + S_3^T, \\
R_{5,1}^{ij} &= -G_{xi}^T S_3^T - (S_{Li} G_{yj})^T, \\
R_{14,1}^{ij} &= -D_{1i}^T S_3^T - (S_{Li} D_{2j})^T, \quad R_{2,2}^{ij} = -I, \\
R_{3,3}^{ij} &= \eta(S_3 + S_3^T), \\
R_{4,3}^{ij} &= (\eta\sqrt{3} S_3 M_{1i} + \eta\sqrt{2} S_{Li} M_{2j})^T, \quad R_{4,4}^{ij} = -I, \\
R_{5,3}^{ij} &= -\eta G_{xi}^T S_3^T - \eta(S_{Li} G_{yj})^T, \\
R_{14,3}^{ij} &= -\eta D_{1i}^T S_3^T - \eta(S_{Li} D_{2j})^T, \quad R_{5,5}^{ij} = -Y, \\
R_{6,6}^{ij} &= E_2 - S_1 A_i - A_i^T S_1^T + 8 N_1^T N_1, \\
R_{7,6}^{ij} &= \sqrt{3} M_{1i}^T S_1^T, \quad R_{8,6}^{ij} = P_{2i} + S_1^T - S_2 A_i, \\
R_{10,6}^{ij} &= -S_5 A_i - G_{xi}^T S_1^T, \quad R_{7,7}^{ij} = -I, \\
R_{12,6}^{ij} &= -B_i^T S_1^T - S_6 A_i, \quad R_{14,6}^{ij} = -D_{1i}^T S_1^T, \\
R_{8,8}^{ij} &= S_2 + S_2^T, \quad R_{9,8}^{ij} = \sqrt{3} M_{1i}^T S_2^T, \quad R_{9,9}^{ij} = -I, \\
R_{10,8}^{ij} &= S_5 - G_{xi}^T S_2^T, \\
R_{12,8}^{ij} &= -B_i^T S_2^T + S_6, \quad R_{14,8}^{ij} = -D_{1i}^T S_2^T, \\
R_{10,10}^{ij} &= -S_5 G_{xi} - G_{xi}^T S_5^T + 8 N_3^T N_3, \\
R_{11,10}^{ij} &= \sqrt{3} M_{1i}^T S_5^T, \quad R_{11,11}^{ij} = -I, \\
R_{12,10}^{ij} &= -B_i^T S_5^T - S_6 G_{xi}, \quad R_{14,10}^{ij} = -D_{1i}^T S_5^T, \\
R_{12,12}^{ij} &= -S_6 B_i - B_i^T S_6^T + 6 N_2^T N_2, \\
R_{13,12}^{ij} &= \sqrt{3} M_{1i}^T S_6^T, \quad R_{13,13}^{ij} = -I, \\
R_{14,12}^{ij} &= -D_{1i}^T S_6^T, \quad R_{14,14}^{ij} = -\gamma I,
\end{aligned} \tag{25}$$

and

$$\begin{aligned} E_1 &= \pm \kappa_\xi X_1 + \sum_{\substack{\rho=1 \\ \rho \neq \xi}}^{r} \kappa_\rho (P_{1\rho} + X_1 - P_{1\xi}), \\ E_2 &= \pm \kappa_\xi X_2 + \sum_{\substack{\rho=1 \\ \rho \neq \xi}}^{r} \kappa_\rho (P_{2\rho} + X_2 - P_{2\xi}). \end{aligned} \tag{26}$$

Then, the observer gains are

$$L_i = S_3^{-1} S_{Li}, \quad L_n = X^{-1} R_n. \tag{27}$$

The $\pm$ sign means that the LMIs must be checked for both positive and negative signs. Note that parameters η and κ_i should be given in advance and LMIs can be solved to find the best value for γ.

Proof. Let us define the following fuzzy Lyapunov function:

$$V(t) := \sum_{i=1}^{r} \omega_i(z) \Big[e^T(t) P_{1i} e(t) + x^T(t) P_{2i} x(t) \Big]. \tag{28}$$

Its time derivative is

$$\begin{aligned} \dot{V}(t) &= x^T(t) \Big[\sum_{i=1}^{r} \dot{\omega}_i(z) P_{2i} \Big] x(t) \\ &+ 2x^T(t) \Big[\sum_{i=1}^{r} \omega_i(z) P_{2i} \Big] \dot{x}(t) \\ &+ e^T(t) \Big[\sum_{i=1}^{r} \dot{\omega}_i(z) P_{1i} \Big] e(t) \\ &+ 2e^T(t) \Big[\sum_{i=1}^{r} \omega_i(z) P_{1i} \Big] \dot{e}(t). \end{aligned} \tag{29}$$

In order to make the LMI representation possible, two zero terms are added to (29), which results in

$$\begin{aligned} \dot{V}(t) &= x^T(t) \Big[\sum_{i=1}^{r} \dot{\omega}_i(z) P_{2i} \Big] x(t) \\ &+ 2x^T(t) \Big[\sum_{i=1}^{r} \omega_i(z) P_{2i} \Big] \dot{x}(t) \\ &+ e^T(t) \Big[\sum_{i=1}^{r} \dot{\omega}_i(z) P_{1i} \Big] e(t) \\ &+ 2e^T(t) \Big[\sum_{i=1}^{r} \omega_i(z) P_{1i} \Big] \dot{e}(t) \\ &+ 2\big[x^T(t) S_1 + \dot{x}^T(t) S_2 \\ &+ \varphi(x(t), u(t), t)^T S_5 + u^T(t) S_6 \big] S_{x1} \\ &+ 2\big[e^T(t) S_3 + \dot{e}^T(t) S_4 \big] S_{e1}, \end{aligned} \tag{30}$$

where

$$\begin{aligned} S_{x1} = \dot{x}(t) - \Big[\sum_{i=1}^{r} \omega_i(z) \big\{ (A_i + \Delta A_i) x(t) \\ + (G_{xi} + \Delta G_{xi}) \varphi(x(t), u(t), t) \\ + (B_i + \Delta B_i) u(t) + D_{1i} \nu(t) \big\} \Big], \end{aligned}$$

$$\begin{aligned} S_{e1} = \dot{e}(t) - \Big[\sum_{i=1}^{r} \sum_{j=1}^{r} \omega_{ij}(z) \big\{ \big[(A_i + L_i C_j) e(t) \\ + (G_{xi} + L_i G_{yj}) \delta\varphi(t) \big] - \big[(\Delta A_i + L_i \Delta C_j) x(t) \\ + (\Delta G_{xi} + L_i \Delta G_{yj}) \varphi(x(t), u(t), t) \\ + \Delta B_i u(t) + (D_{1i} + L_i D_{2j}) \nu(t) \big] \big\} \Big]. \end{aligned} \tag{31}$$

Using Lemma 2, (8) and (30), $\dot{V}(t)$ becomes

$$\begin{aligned} \dot{V}(t) &\le x^T(t) \Big[\sum_{i=1}^{r} \dot{\omega}_i(z) P_{2i} \Big] x(t) \\ &+ 2x^T(t) \Big[\sum_{i=1}^{r} \omega_i(z) P_{2i} \Big] \dot{x}(t) \\ &+ e^T(t) \Big[\sum_{i=1}^{r} \dot{\omega}_i(z) P_{1i|} \Big] e(t) \\ &+ 2e^T(t) \Big[\sum_{i=1}^{r} \omega_i(z) P_{1i} \Big] \dot{e}(t) \\ &+ 2\big[x^T(t) S_1 + \dot{x}^T(t) S_2 \\ &+ \varphi^T(x(t), u(t), t) S_5 + u^T(t) S_6 \big] S_{x2} \\ &+ 2\big[e^T(t) S_3 + \dot{e}^T(t) S_4 \big] S_{e2} \\ &+ \sum_{i=1}^{r} \sum_{j=1}^{r} \omega_{ij}(z) \Big[3x^T(t) S_1 M_{1i} M_{1i}^T S_1^T x(t) \\ &+ 3\dot{x}^T(t) S_2 M_{1i} M_{1i}^T S_2^T \dot{x}(t) \\ &+ 3u^T(t) S_6 M_{1i} M_{1i}^T S_6^T u(t) \\ &+ 3\varphi(x(t), u(t), t)^T S_5 M_{1i} M_{1i}^T S_5^T \varphi(x(t), u(t), t) \\ &+ e^T(t) S_3 (3M_{1i} M_{1i}^T + 2L_i M_{2j} M_{2j}^T L_i^T) S_3^T e(t) \\ &+ \dot{e}^T(t) S_4 (3M_{1i} M_{1i}^T + 2L_i M_{2j} M_{2j}^T L_i^T) S_4^T \dot{e}(t) \\ &+ 8x^T(t) N_1^T N_1 x(t) + 6u^T(t) N_2^T N_2 u(t)) \\ &+ 8\varphi(x(t), u(t), t)^T N_3^T N_3 \varphi(x(t), u(t), t) \Big], \end{aligned} \tag{32}$$

where

$$\begin{aligned} S_{x2} = \dot{x}(t) - \Big[\sum_{i=1}^{r} \omega_i(z) \big[A_i x(t) + G_{xi} \varphi(x(t), u(t), t) \\ + B_i u(t) + D_{1i} \nu(t) \big] \Big], \end{aligned}$$

$$\begin{aligned} S_{e2} = \dot{e}(t) - \Big[\sum_{i=1}^{r} \sum_{j=1}^{r} \omega_{ij}(z) \big[\big((A_i + L_i C_j) e(t) \\ + (G_{xi} + L_i G_{yj}) \delta\varphi(t) \big) \\ - (D_{1i} + L_i D_{2j}) \nu(t) \big] \Big]. \end{aligned} \tag{33}$$

Then $\dot V(t) \le \sum_{i=1}^{r}\sum_{j=1}^{r}\omega_{ij}K^T\bar{\mathcal{R}}_{ij}K$, where

$$K = \left[e(t)\,\dot e(t)\,\delta\varphi(t)\,x(t)\,\dot x(t)\,\varphi(x(t),u(t),t)\,u(t)\,\nu(t)\right]^T$$

$$\bar{\mathcal{R}}_{ij} = \begin{pmatrix} \bar R_{11}^{ij} & * & * \\ \vdots & \ddots & * \\ \bar R_{81}^{ij} & \cdots & \bar R_{88}^{ij}, \end{pmatrix} \tag{34}$$

in which

$$\begin{aligned}
\bar R_{11}^{ij} &= \textstyle\sum_{\rho=1}^{r}\dot\omega_\rho(z)P_{1\rho} - S_3(A_i+L_iC_j)\\
&\quad - (A_i+L_iC_j)^TS_3^T\\
&\quad + S_3(3M_{1i}M_{1i}^T + 2L_iM_{2j}M_{2j}^TL_i^T)S_3^T,\\
\bar R_{21}^{ij} &= P_{1i} - S_4(A_i+L_iC_j) + S_3^T,\\
\bar R_{31}^{ij} &= -(G_{xi}+L_iG_{yj})^TS_3^T,\\
\bar R_{81}^{ij} &= -(D_1i+L_iD_2j)^TS_3^T,\\
\bar R_{22}^{ij} &= S_4+S_4^T+S_4(3M_{1i}M_{1i}^T+2L_iM_{2j}M_{2j}^TL_i^T)^TS_4,\\
\bar R_{32}^{ij} &= -(G_{xi}+L_iG_{yj})^TS_4^T,\\
\bar R_{82}^{ij} &= -(D_1i+L_iD_2j)^TS_4^T,\\
\bar R_{44}^{ij} &= \textstyle\sum_{\rho=1}^{r}\dot\omega_\rho(z)P_{2\rho} - S_1A_i - A_i^TS_1^T\\
&\quad + 8N_1^TN_1 + 3S_1M_{1i}M_{1i}^TS_1^T,\\
\bar R_{54}^{ij} &= P_{2i}+S_1^T - S_2A_i, \quad \bar R_{64}^{ij} = -S_5A_i - G_{xi}^TS_1^T,\\
\bar R_{74}^{ij} &= -B_i^TS_1^T - S_6A_i, \quad \bar R_{84}^{ij} = -D_{1i}^TS_1^T,\\
\bar R_{55}^{ij} &= S_2+S_2^T+3S_2M_{1i}M_{1i}^TS_2^T,\\
\bar R_{65}^{ij} &= S_5 - G_{xi}^TS_2^T,\\
\bar R_{75}^{ij} &= -B_i^TS_2^T + S_6, \quad \bar R_{85}^{ij} = -D_{1i}^TS_2^T,\\
\bar R_{66}^{ij} &= -S_5G_{xi} - G_{xi}^TS_5^T + 8N_3^TN_3 + 3S_5M_{1i}M_{1i}^TS_5^T,\\
\bar R_{76}^{ij} &= -B_i^TS_5^T - S_6G_{xi}, \quad \bar R_{86}^{ij} = -D_{1i}^TS_5^T,\\
\bar R_{77}^{ij} &= -S_6B_i - B_i^TS_6^T + 6N_2^TN_2 + 3S_6M_{1i}M_{1i}^TS_6,\\
\bar R_{87}^{ij} &= -D_{1i}^TS_6^T.
\end{aligned} \tag{35}$$

The other terms in $\bar{\mathcal{R}}_{ij}$ are equal to zero. Based on (10), we get

$$\dot\omega_\xi = -\sum_{\rho=1,\rho\ne\xi}^{r}\dot\omega_\rho(z), \tag{36}$$

and hence

$$\begin{aligned}\sum_{\rho=1}^{r}\dot\omega_\rho(z)P_{1\rho}\\ = \dot\omega_\xi X_1 + \sum_{\rho=1,\rho\ne\xi}^{r}\dot\omega_\rho(z)[P_{1\rho}+X_1-P_{1\xi}].\end{aligned} \tag{37}$$

Moreover, based on (22),

$$\dot\omega_\rho(z)[P_{1\rho}+X_1-P_{1\xi}] \le \kappa_\rho[P_{1\rho}+X_1-P_{1\xi}]. \tag{38}$$

By defining $S_4 := \eta S_3$ and $S_3L_i := S_{Li}$ and using the Schur complement lemma (Boyd *et al.*, 1994) on the diagonal elements of $\bar{\mathcal{R}}_{ij}$, and based on (26), (37) and (38), we have

$$\begin{aligned}&\dot V(t) + e^T(t)e(t) - \gamma\nu^T(t)\nu(t)\\ &\qquad\le \sum_{i=1}^{r}\sum_{j=1}^{r}\omega_{ij}\bar K^T\tilde{\mathcal{R}}_{ij}\bar K,\end{aligned} \tag{39}$$

where $\tilde{\mathcal{R}}_{ij}$ is equal to $\mathcal{R}_{ij}$ defined in (24) except that $\tilde{\mathcal{R}}_{5,5}^{ij} = 0$. Moreover, $\bar K$ is the same as K with augmented terms equal to one in order to have proper dimensions.

Adding the term $\binom{e(t)}{\delta\varphi(t)}^T\Phi^TM\Phi\binom{e(t)}{\delta\varphi(t)}$ to both the sides of (39), introducing $R_n = XL_n$ and using Schur complements on the right-hand side of the resulting inequality and based on the Schur complement of (18), we have

$$\begin{aligned}&\sum_{i=1}^{r}\sum_{j=1}^{r}\omega_{ij}(\bar K^T\tilde{\mathcal{R}}_{ij}\bar K) + \begin{pmatrix}e(t)\\ \delta\varphi(t)\end{pmatrix}^T\Phi^TM\Phi\begin{pmatrix}e(t)\\ \delta\varphi(t)\end{pmatrix}\\ &\qquad= \sum_{i=1}^{r}\sum_{j=1}^{r}\omega_{ij}\bar K^T\Xi_{ij}\bar K,\end{aligned} \tag{40}$$

in which Ξ_{ij} is defined in (23). Then, from Lemma 1 and (22), we get

$$\begin{aligned}&\dot V(t) + e^T(t)e(t) - \gamma\nu^T(t)\nu(t)\\ &\qquad+ \begin{pmatrix}e(t)\\ \delta\varphi(t)\end{pmatrix}^T\Phi^TM\Phi\begin{pmatrix}e(t)\\ \delta\varphi(t)\end{pmatrix} < 0,\end{aligned} \tag{41}$$

which, based on (15), guarantees that the error dynamic is asymptotically stable with an H_∞ performance bound γ. That is,

$$e^T(t)e(t) < \gamma\nu^T(t)\nu(t). \tag{42}$$

■

Remark 1. One of the main drawbacks of using the FLF for continuous-time systems is the existence of a derivative of the membership functions, which can be written as follows (Manai and Benrejeb, 2011):

$$\dot\omega_i(z) = \frac{\partial\omega_i}{\partial z(t)}\cdot\frac{\partial z(t)}{\partial x(t)}\cdot\frac{\mathrm{d}x(t)}{\mathrm{d}t}. \tag{43}$$

This derivative, if it exists at all, is usually hard to calculate. However, it can be assumed that it is limited by a bound (κ_i), which is used here. For the existence of a bound on $\partial\omega_i/\partial z(t)$, it is assumed that the membership functions are continuous. In addition, $\partial z(t)/\partial x(t)$ is known *a priori*. However, in general, it is hard to find a limit on $\dot x(t)$. Nevertheless, when $\dot x(t)$ increases from its bounds, it means that $x(t)$ has also been increased from its predefined bounds. Hence, the first derivative must approach to zero, which results in $\dot{\omega}_i(z) \to 0$. For more detail, the reader may refer to the work of Guerra and Bernal (2012).

Remark 2. It is also possible to assume the multiplier matrix M with the following form:

$$M = \begin{pmatrix} 0 & X \\ X^T & -Y \end{pmatrix}. \tag{44}$$

In this case, the equality in (18) changes to

$$\Phi^T M \Phi = \begin{pmatrix} \Upsilon_{11}^T X^{-1} \Upsilon_{11} & \Upsilon_{11}^T \\ \Upsilon_{11} & \Upsilon_{22} + \Upsilon_{22}^T - Y \end{pmatrix}, \tag{45}$$

where

$$\begin{aligned} \Upsilon_{11} &= XC_q + R_n C_z, \\ \Upsilon_{22} &= XD_q + R_n G_{yz}. \end{aligned} \tag{46}$$

Hence, it is possible to use Theorem 1 with the following changes:

$$\begin{aligned} R_{5,1}^{ij} &= R_{5,1}^{ij} + (XC_q + R_n C_i), \\ R_{1,5}^{ij} &= R_{1,5}^{ij} + (XC_q + R_n C_i)^T, \\ R_{5,5}^{ij} &= R_{5,5}^{ij} + (XD_q + R_n G_{yi})^T + (XD_q + R_n G_{yi}), \\ \Psi &= \begin{pmatrix} XC_q + R_n C_i & 0 & 0 & 0 \end{pmatrix}. \end{aligned} \tag{47}$$

In order to further enlarge the class of nonlinearities, the matrix M can have the following forms as well:

$$\begin{aligned} M &= T^T \begin{pmatrix} 0 & X \\ X^T & -Y \end{pmatrix} T, \\ M &= T^T \begin{pmatrix} X & 0 \\ 0 & -Y \end{pmatrix} T, \end{aligned} \tag{48}$$

where

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{pmatrix}. \tag{49}$$

In this case, $T_{22} + T_{21} D_q$ should be nonsingular. Then, it is possible to formulate the problem using LMIs. However, in order to use Theorem 1, L_n cannot be selected arbitrarily. This means that there are fewer degrees of freedom for the LMI solver. To solve this problem, an extra term should be added to the observer (Açikmese and Corless, 2011).

Remark 3. One essential point in Theorem 1 is the injection term $L_n\big(\hat{y}(t) - y(t)\big)$ in $\hat{\varphi}\big(\hat{x}(t), u(t), t\big)$ in (11). It should be mentioned that, if the nonlinearity $\varphi\big(x(t), u(t), t\big)$ is Lipschitz (instead of incremental quadratic), then L_n cannot be introduced. This case is shown in the following theorem.

Theorem 2. *Assume that $|\dot{\omega}_i(z)| < \kappa_i$ for known positive real numbers κ_i. The error dynamic (12) is asymptotically stable and with an H_∞ performance bound $\gamma > 0$ if $\varphi\big(x(t), u(t), t\big)$ satisfies the Lipschitz condition*

$$\begin{aligned} & e^T(t)\Gamma^T \theta \Lambda \Gamma e(t) \\ & - \delta\varphi^T\big(x(t), u(t), t\big) \Lambda \delta\varphi\big(x(t), u(t), t\big) \geq 0, \end{aligned} \tag{50}$$

where θ is the Lipschitz constant and Γ is a constant matrix with proper dimensions and there exist matrices $P_{1i} = P_{1i}^T > 0$, $P_{2i} = P_{2i}^T > 0$ $(1 \leq i \leq r)$, X_1, X_2, S_i, SL_i $(1 \leq i \leq 6)$, $\Lambda = \operatorname{diag}[\lambda_1, \dots, \lambda_s]$ and a scalar $\eta > 0$ such that

$$\begin{aligned} P_{2\rho} + X_2 - P_{2\xi} \geq 0, \quad & \forall \rho \in [1, \dots, r] - \xi, \\ P_{1\rho} + X_1 - P_{1\xi} \geq 0, \quad & \forall \rho \in [1, \dots, r] - \xi, \\ \mathcal{R}_{ii} < 0, \quad & 1 < i < r, \\ \frac{1}{r-1}\mathcal{R}_{ii} + \frac{1}{2}(\mathcal{R}_{ij} + \mathcal{R}_{ji}) < 0, \quad & 1 < i \neq j < r, \end{aligned} \tag{51}$$

where $\mathcal{R}_{ij}$ is defined in (24) with the following changes:

$$\begin{aligned} R_{1,1}^{ij} &= R_{1,1}^{ij} + \Gamma^T \theta \Lambda \Gamma, \\ R_{5,5}^{ij} &= -\Lambda. \end{aligned} \tag{52}$$

Then, the observer gains are

$$L_i = S_3^{-1} S_{Li}. \tag{53}$$

Proof. The proof is similar to that of Theorem 1. The only difference is adding (50), instead of (15), to both the sides of (39). Moreover, the nonlinear injection term is no longer defined here and hence (23) changes to $\Xi_{ij} = \mathcal{R}_{ij}$. ■

Remark 4. It should be noted that a common case for nonlinear T–S observers, which is considered in the literature, is a T–S model with Lipschitz nonlinearities along with a traditional Luenberger observer (Theorem 2). On the other hand, in this paper, the nonlinearity is of the δQC type, which encompasses a wider class of systems (Theorem 1). Moreover, by introducing a nonlinear injection term to the observer, better state estimates can be achieved. In the following section, it will be shown through simulations that Theorem 2 cannot be used for some systems, while Theorem 1 is a more general case and can be applied to a wider class of systems with almost no extra computation time.

4. Simulation examples

Example 1. In this example, the performance of two theorems in this paper is compared. Consider the system shown in Fig. 1, which represents a Translational Oscillator with an eccentric Rotational Actuator (TORA) system (Lee, 2004; Karagiannis *et al.*, 2005). The nonlinear coupling between the rotational motion of the actuator and the translational motion of the oscillator provides a mechanism for control. Let x_1 and x_2 denote the translational position and velocity of the cart, and x_3 and x_4 denote the angular position and velocity of the rotational mass, respectively. Then, the system dynamics can be described as (Tanaka and Wang, 2001)

$$\dot{x} = \begin{pmatrix} x_2 \\ \dfrac{-x_1 + \epsilon x_4^2 \sin x_3}{1 - \epsilon^2 \cos^2 x_3} \\ x_4 \\ \dfrac{\epsilon \cos x_3 (x_1 - \epsilon x_4^2 \sin x_3)}{1 - \epsilon^2 \cos^2 x_3} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{-\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} \\ 0 \\ \dfrac{1}{1 - \epsilon^2 \cos^2 x_3} \end{pmatrix} u + d_1, \quad y = \begin{pmatrix} x_1 & x_2 \end{pmatrix}^T + d_2, \tag{54}$$

where

$$d_1 = \begin{pmatrix} 0 \\ \dfrac{1}{1 - \epsilon^2 \cos^2 x_3} \\ 0 \\ \dfrac{-\epsilon \cos x_3}{1 - \epsilon^2 \cos^2 x_3} \end{pmatrix} \nu(t), \quad d_2 = \begin{pmatrix} 0.1 \\ 0.2 \end{pmatrix} \nu(t), \quad \epsilon = 0.05, \tag{55}$$

in which $\nu(t)$ represents band-limited white noise with power of 0.001. This system can be modeled as follows:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{1-\epsilon^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{\epsilon}{1-\epsilon^2} & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ \frac{-1}{1-\epsilon^2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{-\epsilon}{1-\epsilon^2} & 0 & 0 & 0 \end{pmatrix},$$

$$A_3 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & -a\epsilon \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & a\epsilon \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$B_1^T = \begin{pmatrix} 0 & \dfrac{-\epsilon}{1-\epsilon^2} & 0 & \dfrac{1}{1-\epsilon^2} \end{pmatrix}, \quad B_2^T = B_4^T = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}, \quad B_3^T = \begin{pmatrix} 0 & \dfrac{\epsilon}{1-\epsilon^2} & 0 & \dfrac{1}{1-\epsilon^2} \end{pmatrix},$$
$$G_{x1} = G_{x3} = \begin{pmatrix} 0 & 0 & 0 & 1 \end{pmatrix}^T, \quad G_{x2} = -G_{x4} = \begin{pmatrix} 0 & \epsilon & 0 & 0 \end{pmatrix}^T, \quad C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad \varphi(x) = x_4^2 + a x_4, \quad a = 4. \tag{56}$$

Also $x_4 \in [-a \;\; a]$. To introduce uncertainty in the model, ϵ is equal to 0.01. Note that the system is stabilized first and then an observer is designed for the stable system. The state feedback gains for the stabilizer are

$$\begin{aligned} K_1 &= \begin{pmatrix} -0.2000 & -5.9282 & -0.3682 & 0.6357 \end{pmatrix}, \\ K_2 &= \begin{pmatrix} -0.2443 & -6.3255 & -0.4499 & 0.7859 \end{pmatrix}, \\ K_3 &= \begin{pmatrix} -0.3245 & -7.3220 & -0.5987 & 1.0194 \end{pmatrix}, \\ K_4 &= \begin{pmatrix} -0.2278 & -5.7132 & -0.4192 & 0.7102 \end{pmatrix}. \end{aligned} \tag{57}$$

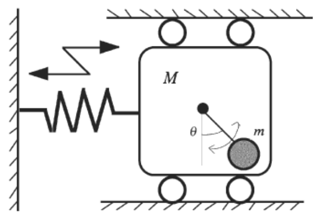

Fig. 1. TORA system.

The following gains are obtained for the observer based on Theorem 1:

$$L_1 = \begin{pmatrix} -5.42 & 2.71 \\ -11.96 & 0.94 \\ 3.09 & -1.53 \\ 2.02 & -0.95 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -5.57 & 2.81 \\ -13.53 & 2.06 \\ 3.44 & -1.79 \\ 2.45 & -1.31 \end{pmatrix}, \quad L_3 = \begin{pmatrix} -5.46 & 2.77 \\ -12.27 & 1.66 \\ 3.14 & -1.70 \\ 2.03 & -1.23 \end{pmatrix}, \quad L_4 = \begin{pmatrix} -5.47 & 2.76 \\ -12.40 & 1.56 \\ 3.20 & -1.69 \\ 2.14 & -1.18 \end{pmatrix}, \tag{58}$$
$$L_n = \begin{pmatrix} 0.5825 & -1.0597 \end{pmatrix},$$

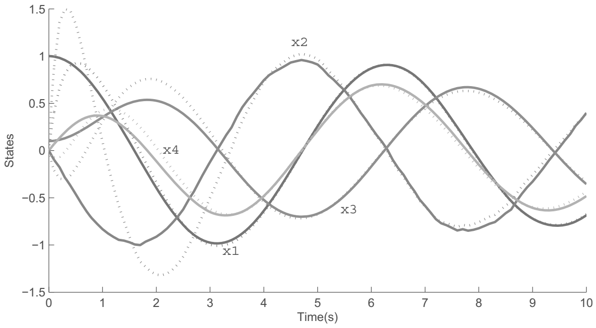

Fig. 2. States (solid lines) and their estimaties (dotted lines) of the TORA system based on Theorem 1.

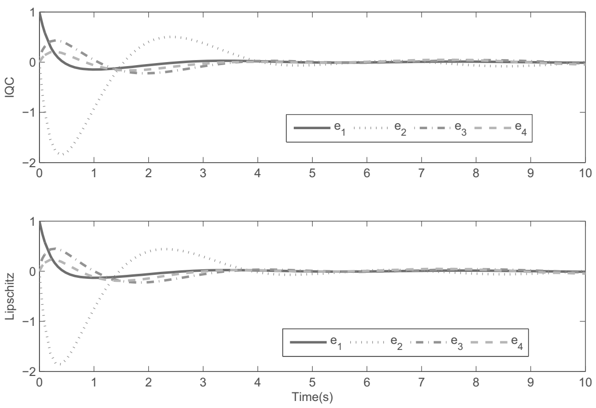

Fig. 3. Estimation errors of the TORA system based on Theorems 1 (top) and 2 (botom).

and the observer gains using Theorem 2 are

$$L_1 = \begin{pmatrix} -5.56 & 2.83 \\ -13.69 & 2.43 \\ 3.51 & -1.90 \\ 2.57 & -1.42 \end{pmatrix}, \quad L_2 = \begin{pmatrix} -5.72 & 2.90 \\ -15.35 & 3.12 \\ 3.88 & -2.05 \\ 3.01 & -1.64 \end{pmatrix}, \quad L_3 = \begin{pmatrix} -5.63 & 2.87 \\ -14.39 & 2.78 \\ 3.65 & -1.97 \\ 2.67 & -1.57 \end{pmatrix}, \quad L_4 = \begin{pmatrix} -5.68 & 2.88 \\ -15.01 & 2.97 \\ 3.84 & -2.03 \\ 2.95 & -1.62 \end{pmatrix}. \tag{59}$$

Comparing the gains, it is obvious that they are almost the same. Figure 2 shows the states and their estimates based on Theorem 1 and Fig. 3 shows the estimation errors of the two methods. It should be noted that using the δQC property instead of the Lipschitz condition neither alters the number of LMI variables, nor changes the size of LMIs. However, in order to add the nonlinear injection term to the observer, one extra variable is added and rows of LMIs are augmented by the dimension of X, which is equal to one in this example. Hence, it is clear that the computational burden will not increase much. ♦

Although in the above example the performance of two methods is similar, there are cases where Theorem 2 fails to find any answer. This is considered in the next example.

Example 2. In this example, the importance of the nonlinear injection term and the δQC condition is demonstrated. It will be shown that for some systems no answer can be obtained without the nonlinear injection term. In other words, Theorem 2 fails to provide a solution for some classes of systems. Consider the system (7) with the following parameters:

$$\begin{aligned} A_1 &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \\ G_{x1} &= G_{x2} = \begin{pmatrix} 0 & 1 & 0 \end{pmatrix}^T, \quad \Gamma = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} \\ B_1 &= B_2 = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}^T, \quad C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \\ D_{11} &= D_{12} = \begin{pmatrix} 0.1 & 0.1 & 0.1 \end{pmatrix}^T, \\ D_2 &= \begin{pmatrix} 0.1 & 0.2 \end{pmatrix}^T \; C_q = \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}, \quad D_q = 0, \\ M_1 &= \begin{pmatrix} 0.01 & 0.005 & 0.003 \end{pmatrix}, \\ M_2 &= \begin{pmatrix} 0.003 & 0.006 \end{pmatrix}, \\ N_1 &= \begin{pmatrix} 0.1 & 0.002 & 0 \end{pmatrix}, \\ N_2 &= 0.1, \quad N_3 = 0.002, \end{aligned} \tag{60}$$

where $\nu(t)$ is band-limited white noise with the power of 0.001 and $\varphi(x(t), u(t), t)$ is a nonlinear function of x_3. Clearly, $q = x_3$ in (2).

Suppose this nonlinearity satisfies the Lipschitz condition (50) as well as the incremental quadratic (δQC) condition (15) with a matrix multiplier of the form (17), where $Y = -(1/\theta)X$ and $\theta = 0.8$. Using Theorem 1 and employing the YALMIP toolbox (Löfberg, 2004), the following gains are obtained:

$$\begin{aligned} L_1 &= \begin{pmatrix} -5.51 & 0.57 \\ -2.98 & -0.67 \\ -0.02 & -0.74 \end{pmatrix}, \\ L_2 &= \begin{pmatrix} -6.58 & 1.33 \\ -4.05 & 0.08 \\ -1.35 & -0.39 \end{pmatrix}, \\ L_n &= \begin{pmatrix} 0.78 & -1.12 \end{pmatrix}, \end{aligned} \tag{61}$$

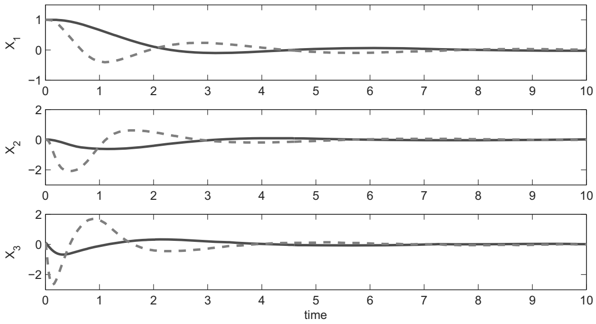

Fig. 4. State trajectory resulting from Theorems 1 (solid) and 2 (dashed).

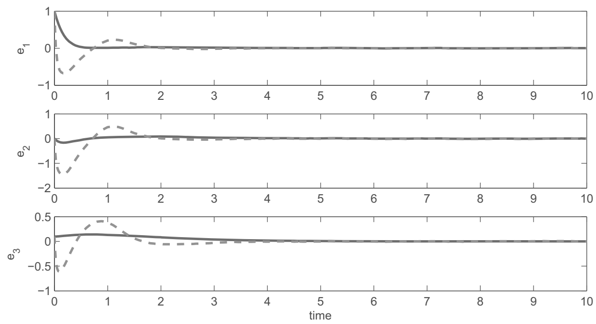

Fig. 5. Estimation errors resulting from Theorems 1 (solid) and 2 (dashed).

and the observer gains resulting from Theorem 2 are

$$
\begin{aligned}
L_1 &= \begin{pmatrix} -50.61 & 22.70 \\ -46.05 & 19.17 \\ -26.63 & 11.13 \end{pmatrix}, \\
L_2 &= \begin{pmatrix} -51.45 & 23.15 \\ -47.63 & 19.97 \\ -28.48 & 11.56 \end{pmatrix}.
\end{aligned} \tag{62}
$$

In order to show the performance of these two methods, the observed states are used in the controller loop. In this case, in order to better observe the performance of different methods, the controller gains are selected such that the states are not damped quickly. The state feedback controller gains are selected as $K_i = \begin{pmatrix} 30 & 20 & 30 \end{pmatrix}$, $i = 1, 2$. Simulation results are presented in Figs. 4 and 5. As these figures show, Theorem 1 provides better state estimates by faster convergence to the desired values.

Next, the vital role of the nonlinear injection term and hence the role of the δQC condition (compared with the Lipschitz condition) are shown by changing θ in the system model. Figure 6 shows the H_∞ performance bound γ versus changes in θ. It can be observed that by increasing θ from 0.1 to 0.9 the value of γ increases

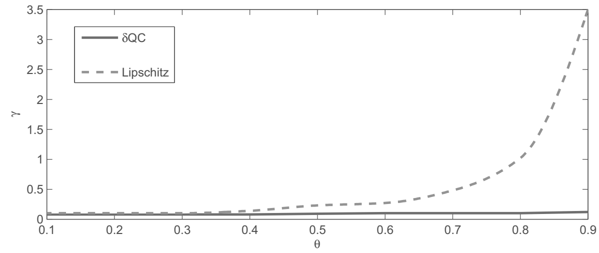

Fig. 6. H_∞ performance bound γ resulting from Theorems 1 (solid) and 2 (dashed) for different values of θ.

from 0.08 to 0.12 for Theorem 1 and from 0.1 to 3.5 for Theorem 2, respectively. Based on (42), higher values for γ can yield higher estimation errors and hence worse performance for the observer. For $\theta \geq 1$, Theorem 2 does not yield any feasible solution while Theorem 1 still provides good performances.

As a special case, consider $\varphi(x(t), u(t), t) = x_3|x_3|$, $x_3 \in [-3 \quad 3]$, which is Lipschitz as well as incremental quadratic (δQC) with a matrix multiplier of the form (17) and (44), respectively, with

$$
M_a = \begin{pmatrix} X & 0 \\ 0 & -\frac{1}{\theta}X \end{pmatrix}, \quad M_b = \begin{pmatrix} 0 & X \\ X & -\frac{2}{\theta}X \end{pmatrix}. \tag{63}
$$

In this case $\theta = 2$, so Theorem 2 has no solution. Using Theorem 1, the following gains are obtained for M_a:

$$
\begin{aligned}
L_1 &= \begin{pmatrix} -7.96 & 2.20 \\ 310.71 & -158.54 \\ 186.25 & -94.20 \end{pmatrix}, \\
L_2 &= \begin{pmatrix} -7.92 & 2.19 \\ 309.77 & -158.54 \\ 184.16 & -94.01 \end{pmatrix}, \\
L_n &= \begin{pmatrix} 0.33 & -0.63 \end{pmatrix}.
\end{aligned} \tag{64}
$$

Also, using Remark 2, the following gains are obtained for M_b:

$$
\begin{aligned}
L_1 &= \begin{pmatrix} -21.15 & 10.05 \\ -24.27 & 11.61 \\ -5.82 & 2.42 \end{pmatrix}, \\
L_2 &= \begin{pmatrix} -17.61 & 8.27 \\ -19.81 & 9.35 \\ -6.88 & 2.91 \end{pmatrix}, \\
L_n &= \begin{pmatrix} 2.58 & -2.13 \end{pmatrix}.
\end{aligned} \tag{65}
$$

To show the effectiveness of modeling a system using a nonlinear Sugeno model, the results are also compared with a traditional Sugeno model with linear local subsystems. Note that in this case the system model

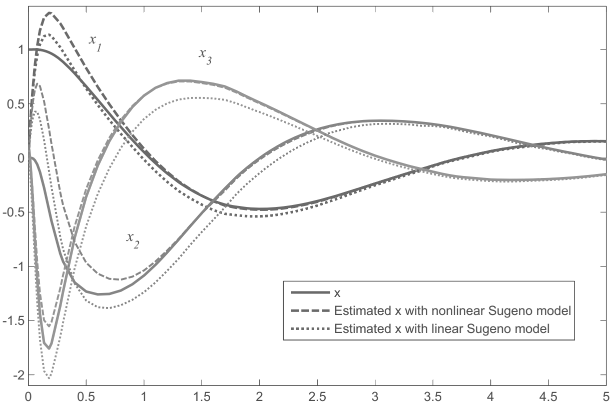

Fig. 7. States (solid line) and their estimations based on Remark 1 (dashed line) and linear Sugeno (dotted line) in Example 2.

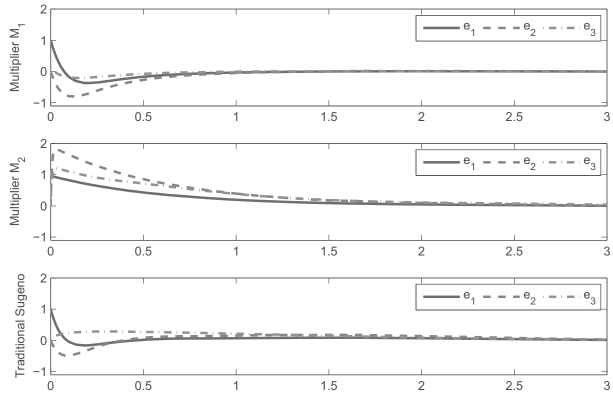

Fig. 8. Estimation error of Example 2, using multiplier M_a (up), M_b (middle) and linear Sugeno (down).

and hence the observer have the following four rules:

$$A_1 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{pmatrix},$$
$$A_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad A_4 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 4 \\ 1 & 0 & 0 \end{pmatrix}. \tag{66}$$

Here x_3 is one of the premise variables but it is not measured directly and must be estimated. The observer gains based on the method proposed by Faria *et al*. (2012) are

$$L_1 = \begin{pmatrix} -22.06 & 9.59 \\ -14.54 & 5.14 \\ 2.23 & -1.87 \end{pmatrix},$$
$$L_2 = \begin{pmatrix} -17.31 & 6.92 \\ -2.46 & -1.52 \\ 2.23 & -1.63 \end{pmatrix},$$
$$L_3 = \begin{pmatrix} -15.82 & 6.18 \\ -3.77 & -0.81 \\ 0.78 & -1.61 \end{pmatrix},$$
$$L_4 = \begin{pmatrix} -15.53 & 5.56 \\ -0.79 & -3.33 \\ 1.27 & -1.81 \end{pmatrix}. \tag{67}$$

The state variables of the system and their estimates with gains using the multiplier M_1 and a Sugeno model with linear rules are shown in Fig. 7. The observation errors of all the three methods are shown in Fig.8.

Simulation results show that the nonlinear Sugeno model simplifies the design. In other words, by reducing the number of rules in the model by two, the number of LMI variables is reduced by six. Although two variables are added for the nonlinear term and the nonlinear injection term, the number of rules considerably reduced. Moreover, by omitting unmeasured premise variables, the number of LMIs to be solved is drastically reduced. In addition, due to a better modeling of the system, a better performance of the observer can be achieved. Furthermore, the nonlinear Sugeno model with an incrementally quadratic constraint on the nonlinear term will result in less conservative design in contrast to the Lipschitz constraint. In other words, the proposed method encompasses a larger class of systems. Moreover, the performance of the observer can be improved by selecting a proper form for the multiplier matrix. ♦

5. Conclusion

In this paper, a Sugeno system with a nonlinear consequent part was considered to reduce the number of rules in the model. In addition, a novel observer was designed for these systems. In the Sugeno model, the consequent part was assumed to have an incremental quadratic nonlinearity. This property was compared with the well-known Lipschitz nonlinearity, which dominates in the literature. In simulation examples, it was shown that this property can reduce the conservativeness of the observer design. The observer-based controller design for such systems using a similar nonlinear injection term will be considered in future works.

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