

MINIMAL POSITIVE REALIZATIONS OF LINEAR CONTINUOUS-TIME FRACTIONAL DESCRIPTOR SYSTEMS: TWO CASES OF AN INPUT-OUTPUT DIGRAPH STRUCTURE

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In the last two decades, fractional calculus has become a subject of great interest in various areas of physics, biology, economics and other sciences. The idea of such a generalization was mentioned by Leibniz and L'Hospital. Fractional calculus has been found to be a very useful tool for modeling linear systems. In this paper, a method for computation of a set of a minimal positive realization of a given transfer function of linear fractional continuous-time descriptor systems has been presented. The proposed method is based on digraph theory. Also, two cases of a possible input-output digraph structure are investigated and discussed. It should be noted that a digraph mask is introduced and used for the first time to solve a minimal positive realization problem. For the presented method, an algorithm was also constructed. The proposed solution allows minimal digraph construction for any one-dimensional fractional positive system. The proposed method is discussed and illustrated in detail with some numerical examples.

Keywords: fractional system, positive systems, descriptor systems, realization, digraph structure, digraph mask, algorithm.

Nomenclature

A : matrices denoted in boldface,
 \mathbf{A}^{-1} : inverse matrix,
 $\det(\mathbf{A})$: determinant of the matrix **A**,
 $\mathbf{A} \geq 0$: non-negative matrix,
 \mathbb{A} : sets denoted by the double line,
 i : lower/upper indices and polynomial coefficients,
 ${}^C_a \mathcal{D}_t^\alpha$: Caputo fractional derivative,
 \mathcal{D} : one-dimensional digraph,
 v_i : vertex of a graph,
 (v_i, v_j) : arc of a digraph,
 $w(v_i, v_j)$: weight of an arc,
 $\mathfrak{M}(\mathcal{D})$: digraph mask,
 \cup : operation composition relative to vertices,
 \circ : operation superposition,
 $\mathbb{R}^{n \times m}$: set $n \times m$ of real matrices,
 \mathbb{R}_+ : set of non-negative numbers,
 \mathbb{Z}_+ : set of non-negative integers,
 \mathbf{I}_n : $n \times n$ identity matrix,
 $\mathcal{L}\{f(t)\}$: \mathcal{L} -transform of the function $f(t)$,
 $T(s)$: transfer function of one variable s ,
 $p(s)$: polynomial of the variable s .

1. Introduction

Descriptor non-positive and positive continuous-time systems have been considered in many papers and books (Luenberger, 1979; Lewis, 1984; 1986; Dai, 1989; Farina and Rinaldi, 2000; Kaczorek, 2001; Dodig and Stoi, 2009; Guang-Ren, 2010; Kaczorek and Sajewski, 2014). Matrix theory for the descriptor system has been presented by Luenberger (1979), Kublanovskaya (1983) as well as Horn and Johnson (1991). There are many problems associated with the analysis and synthesis of descriptor systems. One of the very important ones is the positive realization and minimal positive realization problem. In many research studies, we can find a constant matrix form, which satisfies the system described by the transfer function (Farina and Rinaldi, 2000; Benvenuti and Farina, 2004; Kaczorek, 2007; Guang-Ren, 2010; Sajewski, 2012; Kaczorek and Sajewski, 2014). In fact, there are many sets of matrices which fit into the system transfer function.

Integral and differential calculus of fractional order has become a subject of great interest in various areas of physics, biology, economics and other sciences. Fractional calculus is a generalization of traditional

integer order integration and differentiation actions onto non-integer order. The idea of such a generalization was mentioned in 1695 by Leibniz and L'Hospital. The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century (Nishimoto, 1984). However, it was only in the late 1960s, that this idea drew the attention of engineers. Fractional calculus was found to be a very useful tool for modelling the behaviour of many materials and systems. Mathematical fundamentals of the fractional calculus are given by Nishimoto (1984), Miller and Ross (1993), Podlubny (1999), Das (2011), Ortigueira (2011) as well as Kaczorek and Sajewski (2014). Some other applications of fractional-order systems can be found in the works of Ionescu *et al.* (2010), Magin *et al.* (2011), Kaczorek (2011), Petras *et al.* (2012), Vandoorn *et al.* (2013), Podlubny *et al.* (2014), Machado and Lopes (2015), Machado *et al.* (2015), Muresan *et al.* (2016a), Sikora (2016), or Muresan *et al.* (2016b).

The main purpose of this paper is to present a method based on one-dimensional digraph theory for computation of the set of a minimal positive realization. The problem is related to a one-dimensional continuous-time fractional positive descriptor system. A digraph structure corresponding to minimal realizations of the descriptor system is presented and discussed. In this paper, for the first time, a digraph mask is introduced and used. It is very important in the analysis of descriptor systems, with the use of digraph theory. By using the digraph mask, we can realize singularity in the system considered. It should be noted that there are many different possible structures of masks. Thus, we get a wider set of possible realizations. As a result of this discussion, sufficient conditions of digraph structures are established. Finally, we propose a procedure for determining the set of positive realizations in the class \mathcal{K}_1 of the digraph structure. The digraph classes of two-dimensional systems were presented by Hryniów and Markowski (2016a). Based on them, Markowski (2018) defined and presented in detail classes for one-dimensional systems. In the publications by Hryniów and Markowski (2016b), Markowski (2016; 2017a) or Markowski and Hryniów (2017a; 2017b), minimal realization of the continuous-time linear fractional system as a set of matrices $(\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D})$ was determined. This work is the next step in the research on the determination of the set of the realization in the form $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ of positive continuous-time non-fractional and fractional descriptor systems by using digraph theory, which was started by Markowski (2017b; 2017c).

This work is organised as follows. Section 2 presents models and representations of fractional continuous-time systems, basic properties of the descriptor continuous-time linear system defined as the state-space representation, as well as some notation

and basic definitions of digraph theory which will be used in further. In Section 3, we construct a method for determination of a minimal positive realization of the descriptor continuous-time system which consists of four parts: determination matrix \mathbf{A} (Section 3.1) and matrix \mathbf{E} (Section 3.2), choosing sets of the proper pairs (\mathbf{E}, \mathbf{A}) (Section 3.3) and determination matrices \mathbf{B} and \mathbf{C} (Section 3.4). Sufficient conditions of the digraph structure are also given. Finally, we demonstrate some numerical examples (Section 4), present concluding remarks, open problems and the related bibliography.

2. Background

2.1. Model and representation. The equation for a continuous-time dynamic system of a fractional order can be written as follows (Podlubny, 1999):

$$H(\mathfrak{D}^{\alpha_0 \alpha_1 \dots \alpha_r})(y_1, y_2, \dots, y_l) = G(\mathfrak{D}^{\beta_0 \beta_1 \dots \beta_q})(u_1, u_2, \dots, u_k),$$

where y_i , u_i represent functions of time and $H(\cdot)$, $G(\cdot)$ are combinations of the fractional-order derivative operators. For the linear time-invariant single-variable case, we obtain the following equation:

$$\underbrace{\left(\sum_{k=0}^r a_k \mathfrak{D}^{\alpha_k} \right)}_{H(\mathfrak{D}^{\alpha_0 \alpha_1 \dots \alpha_r})} (y(t)) = \underbrace{\left(\sum_{k=0}^q b_k \mathfrak{D}^{\beta_k} \right)}_{G(\mathfrak{D}^{\beta_0 \beta_1 \dots \beta_q})} (u(t)), \quad (1)$$

where $a_k \in \mathbb{R}$ and $b_k \in \mathbb{R}$.

Applying the Laplace transform to (1) with zero initial conditions, the input-output representation can be obtained. The fractional-order system as the transfer function has the following form:

$$G(s) = \frac{b_q s^{\beta_q} + b_{q-1} s^{\beta_{q-1}} + \dots + b_0 s^{\beta_0}}{a_r s^{\alpha_r} + a_{r-1} s^{\alpha_{r-1}} + \dots + a_0 s^{\alpha_0}}. \quad (2)$$

In the case of a commensurate-order system, the continuous-time transfer function (2) is given in the following form: $G(s) = \sum_{k=0}^q b_k (s^\alpha)^k / \sum_{k=0}^r a_k (s^\alpha)^k$, and can be considered a pseudo-rational function $H(\lambda)$ of the variable $\lambda = s^\alpha$ in the form

$$H(\lambda) = \frac{\sum_{k=0}^q b_k \lambda^k}{\sum_{k=0}^r a_k \lambda^k} = \frac{b_0 + b_1 \lambda + \dots + b_q \lambda^q}{a_0 + a_1 \lambda + \dots + a_r \lambda^r}. \quad (3)$$

2.2. Descriptor fractional systems. In this paper the following Caputo definition (Caputo, 1967) of the

fractional derivative will be used:

$${}^C \mathcal{D}_t^\alpha = \frac{d^\alpha}{dt^\alpha} = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau, \quad (4)$$

where $\alpha \in \mathbb{R}$ is the order of a fractional derivative, $f^{(n)}(\tau) = d^n f(\tau)/d\tau^n$ and $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$ is the gamma function.

Consider the fractional descriptor continuous-time linear system

$$\mathbf{E} {}_0 \mathcal{D}_t^\alpha x(t) = \mathbf{A}x(t) + \mathbf{B}u(t), \quad 0 < \alpha < 1, \quad (5)$$

$$y(t) = \mathbf{C}x(t),$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are the state, input and output vectors, respectively, and

$$\mathbf{E} \in \mathbb{R}^{n \times n}, \quad \mathbf{A} \in \mathbb{R}^{n \times n}, \quad \mathbf{B} \in \mathbb{R}^{n \times m}, \quad \mathbf{C} \in \mathbb{R}^{p \times n}.$$

It is assumed that $\det \mathbf{E} = 0$ and the pencil of the system (5) is regular, that is,

$$\det [\mathbf{E}s^\alpha - \mathbf{A}] \neq 0$$

for some $z \in \mathbb{C}$ (where \mathbb{C} is the field of complex numbers).

Definition 1. The system (5) is called a *descriptor system* if and only if $\det \mathbf{E} = 0$ ($\text{rank } \mathbf{E} = r < n$).

Definition 2. The fractional descriptor linear system (5) is called *internally positive* if and only if $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$ for $t \geq 0$ for any admissible initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$ for $t \geq 0$.

2.3. Digraphs. A directed graph (or just a digraph) \mathcal{D} consists of a non-empty finite set $\mathbb{V}(\mathcal{D})$ of elements called *vertices* and a finite set $\mathbb{A}(\mathcal{D})$ of ordered pairs of distinct vertices called *arcs* (Bang-Jensen and Gutin, 2009). We call $\mathbb{V}(\mathcal{D})$ the *vertex set* and $\mathbb{A}(\mathcal{D})$ the *arc set* of digraph \mathcal{D} . We will often write $\mathcal{D} = (\mathbb{V}, \mathbb{A})$, which means that \mathbb{V} and \mathbb{A} are the vertex set and the arc set of \mathcal{D} , respectively. The *order* of \mathcal{D} is the number of vertices in \mathcal{D} . The *size* of \mathcal{D} is the number of arcs in \mathcal{D} . For an arc (v_1, v_2) , the first vertex v_1 is its *tail* and the second vertex v_2 is its *head*.

There are two well-known methods of representation of a digraph: a list and an incidence matrix. In this paper we are using the incidence matrix to represent all digraphs. The manner of constructing digraphs with this method is presented, for example, by Godsil and Royle (2001) or Bang-Jensen and Gutin (2009).

Assume that the following matrices are given:

$$(\mathbf{A}, \mathbf{B}) = \left(\begin{array}{c|ccc|c|c} v_i \setminus v_j & v_1 & v_2 & v_3 & v_i \setminus s_m & s_1 \\ \hline v_1 & 0 & 0 & 1 & v_1 & 1 \\ v_2 & 1 & 0 & 1 & v_2 & 1 \\ v_3 & 0 & 1 & 1 & v_3 & 0 \end{array} \right). \quad (6)$$

There exists an arc from vertex v_j to vertex v_i (or from vertex s_m to vertex v_i) if and only if the (i, j) -th (or (i, m) -th) entry of the matrix is non-zero. A one-dimensional digraph corresponding to the matrices (6) is presented in Fig. 1.



Fig. 1. One-dimensional digraph of (6).

We present below some basic notions from graph theory which are used in our further discussion. A *walk* in a digraph \mathcal{D} is a finite sequence of arcs in which every two vertices v_i and v_j are adjacent or identical. A walk in which all of the arcs are distinct is called a *path*. The path that goes through all vertices is called a *finite path*. If the initial and terminal vertices of the path are the same, then the path is called a *cycle*. One of the very important operations on digraphs is composition relative to vertices.

Definition 3. (*Composition relative to vertices*) Let $\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n$ be digraphs with vertex sets $\mathbb{V}(\mathcal{G}_n) = \{v_{i(n)} : i \in \mathbb{N}\}$. The composition relative to vertices $\mathcal{D}[\mathcal{G}_1, \mathcal{G}_2, \dots, \mathcal{G}_n]$ is the digraph \mathcal{L} with the vertex set $\mathbb{V}(\mathcal{G}_1) \cup \mathbb{V}(\mathcal{G}_2) \cup \dots \cup \mathbb{V}(\mathcal{G}_n) = \{v_j : j = 1, \dots, \max\{i(n)\}, i \in \mathbb{N}\}$ and arc set $\mathcal{A}' = A(\mathcal{G}_1) \cup A(\mathcal{G}_2) \cup \dots \cup A(\mathcal{G}_n)$, where A' denotes the operation of deleting multiple arcs.

More information on the use of digraph theory in analysis of positive and non-positive system is given by Fornasini and Valcher (1997; 2005), Godsil and Royle (2001) or Bang-Jensen and Gutin (2009).

2.4. Problem formulation.

Theorem 1. (Podlubny, 1999) *The Laplace transform of the derivative-integral (4) has the form*

$$\mathcal{L} \left[\begin{array}{c} {}^C \mathcal{D}_t^\alpha \\ 0 \end{array} \right] = s^\alpha F(s) - \sum_{k=1}^n s^{\alpha-k} f^{(k-1)}(0^+).$$

After using the Laplace transform in (5), Theorem 1

and taking into account that

$$\begin{aligned} X(s) &= \mathcal{L}[x(t)] = \int_0^{\infty} x(t)e^{-st} dt, \\ \mathcal{L}[\mathfrak{D}^{\alpha}x(t)] &= s^{\alpha}X(s) - s^{\alpha-1}x_0, \end{aligned}$$

we obtain

$$\begin{aligned} X(s) &= [\mathbf{E}s^{\alpha} - \mathbf{A}]^{-1} [s^{\alpha-1}x_0 + \mathbf{B}U(s)], \quad (7) \\ Y(s) &= \mathbf{C}X(s), \quad U(s) = \mathcal{L}[u(t)]. \end{aligned}$$

From (7) we can determine the transfer matrix of the system in the following form:

$$\mathbf{T}(s) = \mathbf{C} [\mathbf{E}s^{\alpha} - \mathbf{A}]^{-1} \mathbf{B} \in \mathbb{R}^{p \times m}(s). \quad (8)$$

Definition 4. (Berman and Plemmons, 1979) A matrix $\mathbf{M} = [m_{i,j}] \in \mathbb{R}^{n \times n}$ is called a *Metzler matrix* if its off-diagonal entries are non-negative, $m_{ij} \geq 0$.

As has been shown by Mitkowski (2008), every non-negative matrix is a Metzler matrix. For every Metzler matrix there exists a real number $\eta \geq \min_i m_{ij}$ such that $\eta\mathbf{I} + \mathbf{M} = \mathbf{A}$ is a non-negative matrix.

The descriptor continuous-time linear system (5) is positive if and only if

$$\mathbf{E} \in \mathbb{R}_+^{n \times n}, \quad \mathbf{A} \in \mathbb{M}_n, \quad \mathbf{B} \in \mathbb{R}_+^{n \times m}, \quad \mathbf{C} \in \mathbb{R}_+^{p \times n}. \quad (9)$$

The matrices (9) are called a *realization* of a given transfer matrix $\mathbf{T}(s) \in \mathbb{R}^{p \times m}(s)$ if they satisfy the equality (8). The realization $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$ is called a *minimal realization* if the dimension of the state matrix \mathbf{A} is minimal among all possible realizations of $\mathbf{T}(s)$.

The task is as follows: For the given transfer matrix (8), determine a minimal positive realization of the fractional descriptor continuous-time system (5) using one-dimensional \mathcal{D} digraph theory. The dimension of the system must be minimal among all possible ones.

3. Problem solution

The solution to the minimal positive realization problem will be presented for a one-dimensional single-input single-output (SISO) system for $m = p = 1$. The proposed method will be based on one-dimensional digraph theory, and it will determine solutions in class \mathcal{K}_1 . The classes of the digraph structure are considered in detail by Markowski (2018).

Consider the irreducible transfer function

$$T(s) = \frac{n(s)}{d(s)} \quad (10)$$

with

$$\begin{aligned} n(s) &= b_q(s^{\alpha})^q + b_{q-1}(s^{\alpha})^{(q-1)} + b_{q-2}(s^{\alpha})^{(q-2)} \\ &\quad + \cdots + b_1s^{\alpha} + b_0, \\ d(s) &= a_r(s^{\alpha})^r + a_{r-1}(s^{\alpha})^{(r-1)} + a_{r-2}(s^{\alpha})^{(r-2)} \\ &\quad + \cdots + a_1s^{\alpha} + a_0, \end{aligned}$$

where $b_i, i = 0, 1, \dots, q$ and $a_j, j = 0, 1, \dots, r$ are given real coefficients, and $r < q$. We can rewrite the transfer function (10) in the form (3). In this case, the transferred function can be considered a pseudo-rational function of the variable $\lambda = s^{\alpha}$ in the form

$$T(\lambda) = \frac{n(\lambda)}{d(\lambda)} \quad (11)$$

with

$$\begin{aligned} n(\lambda) &= b_q\lambda^q + b_{q-1}\lambda^{q-1} + b_{q-2}\lambda^{q-2} \\ &\quad + \cdots + b_1\lambda + b_0, \\ d(\lambda) &= a_r\lambda^r + a_{r-1}\lambda^{r-1} + a_{r-2}\lambda^{r-2} \\ &\quad + \cdots + a_1\lambda + a_0. \end{aligned}$$

The proposed method consists of four parts:

- In Section 3.1, we determine state matrices \mathbf{A} with the use of digraph theory. The characteristic polynomial $d(\lambda)$ is considered in the temporary form $\tilde{d}(\lambda)$ which is needed to determine a digraph structure. It should be noted that this temporary form does not change coefficients and powers of the characteristic polynomial $d(\lambda)$.
- In Section 3.2, we determine matrix \mathbf{E} using digraph mask $\mathfrak{M}(\mathcal{D})$.
- Using the digraph structure determined in Section 3.1 and the digraph mask determined in Section 3.2, we can create (Section 3.3) all possible realizations as combinations of the characteristic polynomial realizations and a digraph mask. After this operation, we obtain the set of matrix pairs (\mathbf{E}, \mathbf{A}) . It should be noted that after using the digraph mask we obtain the characteristic polynomial in the desired form $d(\lambda)$.
- Finally, in Section 3.4, we should determine matrices \mathbf{B} and \mathbf{C} by extending the digraphs obtained in Section 3.3. Here we will consider two possible cases (Sections 3.4.1 and 3.4.2) of the digraph structure.

3.1. Determination of state matrix \mathbf{A} . In the first step, we must find matrix \mathbf{A} . After multiplying the denominator of a transfer matrix (11) by $\lambda^{-(q+1)}$, we

obtain the following characteristic polynomial:

$$d(\lambda) = a_r \lambda^{r-q-1} + a_{r-1} \lambda^{r-q-2} + a_{r-2} \lambda^{r-q-3} + \dots + a_1 \lambda^{-q} + a_0 \lambda^{-q-1}. \quad (12)$$

In order to determine the digraph structure corresponding to the characteristic polynomial (12), we must present them in the following temporary form, which is needed to draw the digraphs:

$$\begin{aligned} \tilde{d}(\lambda) &= 1 + d(\lambda) \\ &= 1 + a_r \lambda^{r-q-1} + a_{r-1} \lambda^{r-q-2} \\ &\quad + a_{r-2} \lambda^{r-q-3} + \dots + a_1 \lambda^{-q} + a_0 \lambda^{-q-1}. \end{aligned} \quad (13)$$

Remark 1. It should be noted that after this operation we can consider a characteristic polynomial of the descriptor system in the same way as a characteristic polynomial of the standard system thus $\mathbf{E} \Rightarrow \mathbf{I}$ (Layer 1 in Fig. 2). In the next steps (Section 3.2), we introduce a digraph mask $\mathfrak{M}(\mathcal{D})$ (Layer 2 in Fig. 2), which guarantees the singularity of the characteristic polynomial thus $\mathbf{I} \Rightarrow \mathbf{E}$ (Layer 3 in Fig. 2).

Then we turn the decomposition polynomial (13) into a set of binomials:

$$\begin{aligned} \tilde{d}(\lambda) &= \underbrace{(1 - a_r \lambda^{r-q-1})}_{B_r} \cup \underbrace{(1 - a_{r-1} \lambda^{r-q-2})}_{B_{r-1}} \\ &\quad \cup \underbrace{(1 - a_{r-2} \lambda^{r-q-3})}_{B_{r-2}} \cup \dots \cup \underbrace{(1 - a_1 \lambda^{-q})}_{B_1} \\ &\quad \cup \underbrace{(1 - a_0 \lambda^{-q-1})}_{B_0} \\ &= \bigcup_{i=0}^r (1 - a_r \lambda^{i-q-1}) = \bigcup_{i=0}^r B_i, \end{aligned} \quad (14)$$

where r is a number of binomials in the characteristic polynomial and \cup is the operation called a composition relative to vertices in digraphs presented in Definition 3.

For each binomial (14), using the following result, we create digraph representations.

Proposition 1. (Binomial digraph representation) *Digraph \mathcal{D} corresponding to binomial $1 - a_r \lambda^{r-q-1}$, where $r = 0, 1, \dots, q-1$, consists of one cycle, and it contains $|r - q - 1|$ -vertices.*

Proof. Suppose that the digraph consists of the vertices $v_1, v_2, \dots, v_q, v_{q+1}$. Assume that each of the arcs in the digraph we assigned weights in the form of $w(v_i, v_j) \lambda^{-1}$ for $i, j \in \mathbb{Z}_+ = \{1, 2, \dots, q+1\}$, where v_i is the tail and v_j is the head, and $i \neq j$. Consider the boundary points in the set:

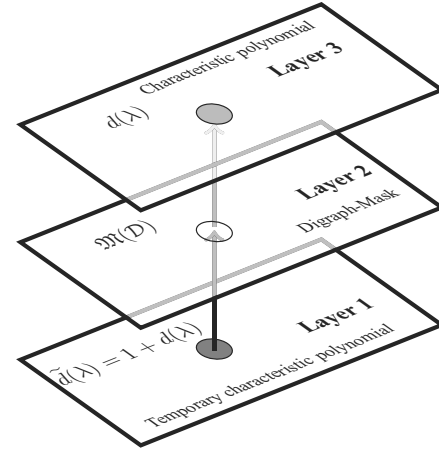


Fig. 2. Relation between $d(\lambda)$, $\tilde{d}(\lambda)$ and $\mathfrak{M}(\mathcal{D})$.

- $r = q - 1 \Rightarrow$ power of the binomial $1 - a_r \lambda^{r-q-1}$ is equal to $a_{q-1} \lambda^{-2}$. The cycle can be written as $w(v_i, v_j) \lambda^{-1} \cdot w(v_j, v_i) \lambda^{-1}$. Therefore, the maximum weight in a digraph is equal to $[w(v_i, v_j) \cdot w(v_j, v_i)] \lambda^{-2} = a_{q-1} \lambda^{-2}$.
- $r = 0 \Rightarrow$ power of the binomial $1 - a_r \lambda^{r-q-1}$ is equal to $a_0 \lambda^{-(q+1)}$. In this case the cycle can be written as $w(v_1, v_2) \lambda^{-1} \cdot w(v_2, :) \lambda^{-1} \cdot \dots \cdot w(:, v_i) \lambda^{-1} \cdot w(v_i, v_j) \lambda^{-1} \cdot w(v_j, v_1) \lambda^{-1}$, where ‘:’ is the next vertex in the cycle. Because $i, j = q + 1$ and $i \neq j$, the maximum weight in a digraph is equal to $[w(v_1, v_2) \cdot w(v_2, :) \cdot \dots \cdot w(:, v_i) \cdot w(v_i, v_j) \cdot w(v_j, v_1)] \lambda^{-(q+1)} = a_0 \lambda^{-(q+1)}$.

Using Proposition 1, Theorem 1 presented by Hryniów and Markowski (2014; 2015) and Definition 3, we can create all digraph realizations of the characteristic polynomials in a temporary form (13). Each digraph corresponding to a characteristic polynomial must satisfy two conditions. The first condition (C1) is related to the existence of the common part of the digraph (vertex in dark gray), the second condition (C2) is related to the non-existence of additional cycles in the digraph. To the vertices belonging to the common part of the digraph we assigned weight equal to 1 and to the other vertices—weight equal to 0. Then we must determine arc weights which appear in the obtained digraphs corresponding to the temporary polynomial (13).

One of the possible digraph structures, for the power of the nominator equal to q and the power of the denominator equal to $r = q - 1$, is presented in Fig. 3(a).

Remark 2. Other realizations can be obtained from the digraph presented in Fig. 3(a) by the following actions:

- *Renumbering vertices in the digraph.* This operation is similar to circle, rotation and synonymous with

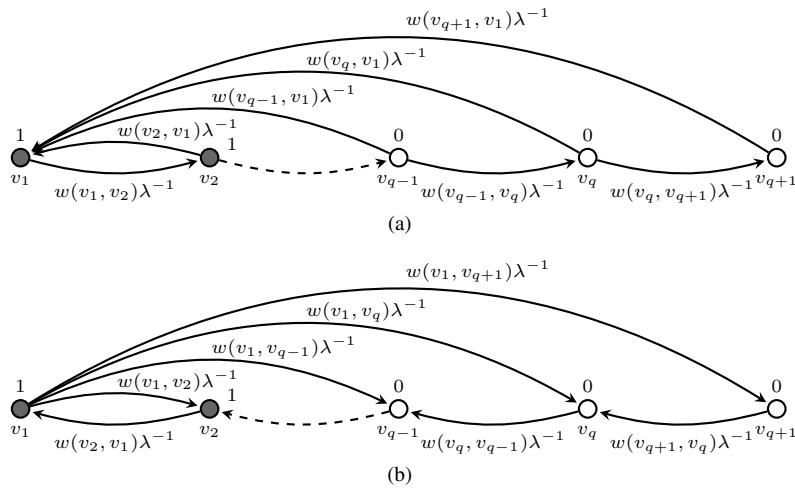


Fig. 3. Possible realization of the characteristic polynomial (13).

swapping columns / rows of a matrix used in matrix theory. The result of this operation is presented in Fig. 4.

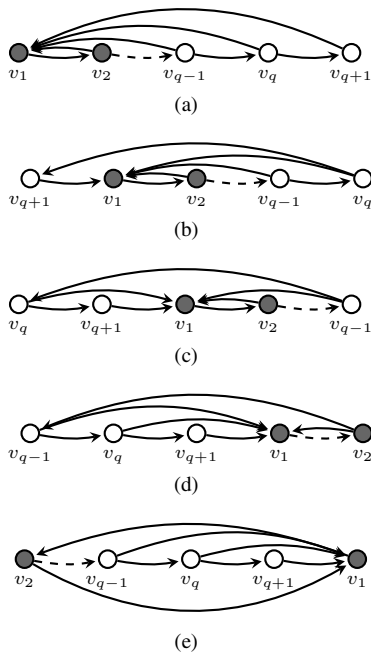


Fig. 4. Possible realization of the characteristic polynomial (13) obtained by renumbering vertices.

- *Change of the direction of the arcs in the graph.* This operation is similar to a transposition matrix used in matrix theory. The result of this operation is presented in Fig. 3(b).

Let us consider the following simple example.

Example 1. (Determination of a digraph structure corresponding to the characteristic polynomial) The

characteristic polynomial $d(\lambda) = \pm\lambda^{-2} \pm \lambda^{-3}$ is rewritten in the temporary form $\tilde{d}(\lambda) = 1 + d(\lambda) = 1 \pm \lambda^{-2} \pm \lambda^{-3}$, consisting of two binomials: $B_1 = 1 \pm \lambda^{-2}$ and $B_0 = 1 \pm \lambda^{-3}$. Possible realizations of the binomials are presented in Fig. 5.

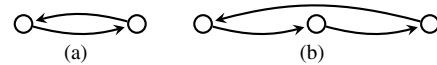


Fig. 5. Realization of the binomial: $B_1 = 1 \pm \lambda^{-2}$ (a), $B_0 = 1 \pm \lambda^{-3}$ (b).

Then, we can determine all possible realizations (Fig. 6) of the polynomial $\tilde{d}(\lambda)$ which satisfy conditions (C1) and (C2). Additionally, to the vertices belonging to the common part of the digraph we assigned weight equal to 1 (marked in dark gray) and 0 for other vertices.

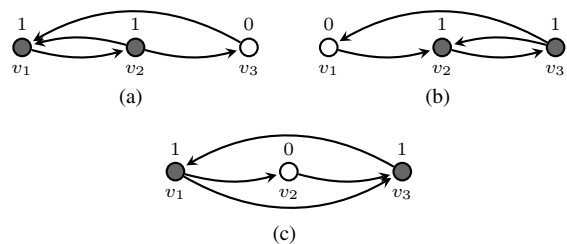


Fig. 6. All possible realizations of the characteristic polynomial $\tilde{d}(\lambda)$.

In the last step, we determine weights of the arcs by solving a set of equations. For example, from the digraph presented in Fig. 6(b), we obtain

$$\begin{cases} w(v_1, v_2) \cdot w(v_2, v_3) \cdot w(v_3, v_1) = \mp 1, \\ w(v_2, v_3) \cdot w(v_3, v_2) = \mp 1, \end{cases}$$

with the constraints $w(v_1, v_2) > 0$, $w(v_2, v_3) > 0$, $w(v_3, v_2) > 0$ and $w(v_3, v_1) > 0$ which must be satisfied if the realization is positive. ♦

3.2. Determination of matrix E. In the second step, we must find matrix **E**. The weight of the vertices in a digraph is associated with the digraph mask. If the characteristic polynomial is in the form of (13), then we can determine the digraph mask with the following proposition.

Proposition 2. (Digraph mask) *A digraph mask $\mathfrak{M}(\mathcal{D})$ corresponding to the characteristic polynomial (13) consists of $q+1$ vertices, and at least one vertex in the digraph must have zero weight.*

Proof. If a digraph mask $\mathfrak{M}(\mathcal{D})$ consists of all vertices with non-zero weights, this means that matrix **E** is diagonal with all non-zero entries, so that $\mathbf{E} = \mathbf{I}$. In this case, the condition $\det \mathbf{E} = 0$ is not satisfied and the system is not a descriptor. There appears at least one zero entry guaranteeing that the condition $\det \mathbf{E} = 0$ is satisfied and the system (5) is a descriptor. ■

All possible digraph masks of the characteristic polynomial (13) are presented in Fig. 7. It should be noted that vertices with weights equal to 1 are marked in dark gray and the other vertices have zero weights.

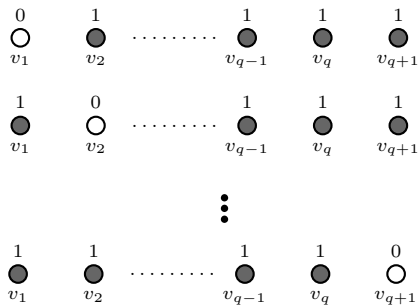


Fig. 7. All possible digraph masks of the characteristic polynomial (13).

Example 2. (Determination of a digraph mask—Example 1 continued) After using Proposition 2, we obtain the set of digraph masks (presented in Fig. 8) consisting of two vertices with weight equal to 1. ♦

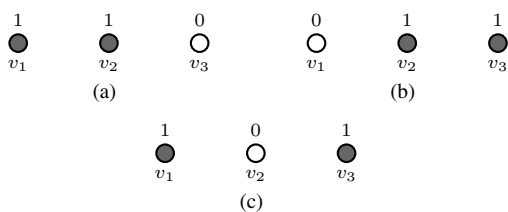


Fig. 8. All possible digraph masks of the characteristic polynomial $\tilde{d}(\lambda) = 1 \pm \lambda^{-2} \pm \lambda^{-3}$.

3.3. Choosing the proper pairs (E, A). With the use of Definition 3 and a digraph mask determined by Proposition 2, we can create all possible digraph realizations as combinations of the binomial representations and a digraph mask:

$$\begin{aligned} d_k^{(\lambda)} | \mathcal{D} &= \mathfrak{M}_j(G) \circ (\mathcal{G}_{B_r} \cup \mathcal{G}_{B_{r-1}} \dots \mathcal{G}_{B_1} \cup \mathcal{G}_{B_0}) \\ &= \mathfrak{M}_j(G) \circ \left(\bigcup_{i=0}^r \mathcal{G}_{B_i} \right), \end{aligned} \quad (15)$$

where $d_k^{(\lambda)} | \mathcal{D}$ means that we consider the k -th realization of the one-dimensional digraph \mathcal{D} corresponding to the characteristic polynomial $d(\lambda)$, $\mathfrak{M}_j(G)$ is the j -th possible digraph masks, \circ is the operation of the superposition of the digraph mask and one of the possible digraph realizations. It should be noted that the superposition \circ on digraph vertices corresponds to the logical operation OR. We choose a digraph structure in which all vertices have weights equal to 1. Other solutions are rejected. Let us consider the following simple example.

Example 3. (Determined pairs (E, A)—Example 2 continued) Using all possible realizations of the temporary polynomial $\tilde{d}(\lambda) = 1 \pm \lambda^{-2} \pm \lambda^{-3}$ presented in Fig. 6, a digraph mask presented in Fig. 8 and (15), we can create all possible digraph realizations which satisfy a characteristic polynomial $d(\lambda) = \pm \lambda^{-2} \pm \lambda^{-3}$. In Fig. 9 we present a realization for a digraph structure from Fig. 6(a). It should be noted that the digraphs presented in Figs. 9(f) and 9(i) have all vertices with weights equal to 1 and these digraph structures are correct. In the same way, we proceed with other structures from Figs. 6(b) and (c). Finally, we obtain six possible realizations.

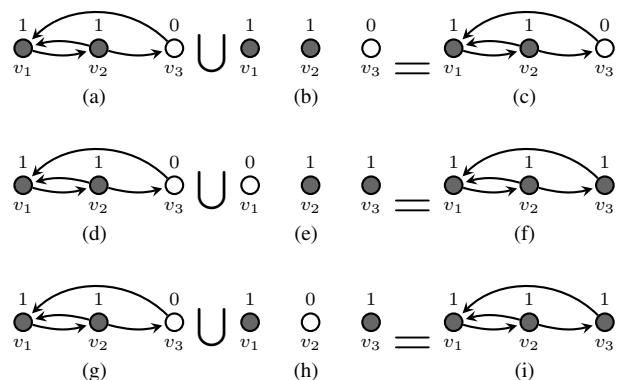


Fig. 9. All possible realizations of the characteristic polynomial $d(\lambda) = \pm \lambda^{-2} \pm \lambda^{-3}$.

Now we can write a matrix pair $(\mathbf{E}_1, \mathbf{A})$ and $(\mathbf{E}_2, \mathbf{A})$ corresponding to the digraph from Figs. 9(f) and (i),

$$\mathbf{A} = \begin{matrix} v_i \backslash v_j & v_1 & v_2 & \cdots & v_{q-1} & v_q & v_{q+1} \\ v_1 & 0 & w(v_2, v_1) & \cdots & w(v_{q-1}, v_1) & w(v_q, v_1) & w(v_{q+1}, v_1) \\ v_2 & w(v_1, v_2) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ v_{q-1} & 0 & 0 & \ddots & 0 & 0 & 0 \\ v_q & 0 & 0 & \cdots & w(v_{q-1}, v_q) & 0 & 0 \\ v_{q+1} & 0 & 0 & \cdots & 0 & w(v_q, v_{q+1}) & 0 \end{matrix} \in \mathbb{R}^{(q+1) \times (q+1)} \quad (16)$$

$$\tilde{\mathbf{A}} = \begin{matrix} v_i \backslash v_j & v_1 & v_2 & \cdots & v_{q-1} & v_q & v_{q+1} \\ v_1 & 0 & w(v_2, v_1) & \cdots & 0 & 0 & 0 \\ v_2 & w(v_1, v_2) & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ v_{q-1} & w(v_1, v_{q-1}) & 0 & \ddots & 0 & w(v_q, v_{q-1}) & 0 \\ v_q & w(v_1, v_q) & 0 & \cdots & 0 & 0 & w(v_{q+1}, v_q) \\ v_{q+1} & w(v_1, v_{q+1}) & 0 & \cdots & 0 & 0 & 0 \end{matrix} \in \mathbb{R}^{(q+1) \times (q+1)} \quad (17)$$

$$\mathbf{E}_1 = \begin{matrix} v_i \backslash v_j & v_1 & v_2 & \cdots & v_{q-1} & v_q & v_{q+1} \\ v_1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ v_2 & 0 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{q-1} & 0 & 0 & \cdots & 1 & 0 & 0 \\ v_q & 0 & 0 & \cdots & 0 & 1 & 0 \\ v_{q+1} & 0 & 0 & \cdots & 0 & 0 & 1 \end{matrix} \in \mathbb{R}^{(q+1) \times (q+1)} \text{ or } \mathbf{E}_2 = \begin{matrix} v_i \backslash v_j & v_1 & v_2 & \cdots & v_{q-1} & v_q & v_{q+1} \\ v_1 & 1 & 0 & \cdots & 0 & 0 & 0 \\ v_2 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ v_{q-1} & 0 & 0 & \cdots & 1 & 0 & 0 \\ v_q & 0 & 0 & \cdots & 0 & 1 & 0 \\ v_{q+1} & 0 & 0 & \cdots & 0 & 0 & 1 \end{matrix} \in \mathbb{R}^{(q+1) \times (q+1)} \quad (18)$$

respectively, in the following form:

$$(\mathbf{E}_1, \mathbf{A}) = \left(\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right),$$

$$(\mathbf{E}_2, \mathbf{A}) = \left(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \right).$$

In the general form, one of the possible digraph structures corresponds to the temporary polynomial (13) presented in Fig. 3(a). With the use of the digraph mask presented in Fig. 7 we can create all possible realizations which satisfy the characteristic polynomial (12). It should be noted that we always have minimum two possible realizations because two vertices belong to digraph common parts. Now we can write the set of matrices $(\mathbf{E}_1, \mathbf{A})$ and $(\mathbf{E}_2, \mathbf{A})$ in the form (16) and (18). \blacklozenge

Remark 3. If we change the direction of the arcs in the digraph presented in Fig. 3(a), we obtain the digraph presented in Fig. 3(b). This operation is similar to the

transposition of the matrix \mathbf{A} . Then, we can write the set of the matrices $(\mathbf{E}_1, \tilde{\mathbf{A}})$ and $(\mathbf{E}_2, \tilde{\mathbf{A}})$ in the form of (17) and (18).

From the above discussion we can write the following proposition.

Proposition 3. (State matrix entries) *The system (5) with matrices of the form (16) (or (17)) is positive if and only if the arc weight in a digraph satisfy the condition $w(v_i, v_j) \geq 0$ for $i, j \in \mathbb{Z}_+ = \{1, 2, \dots, q+1\}$ and $i \neq j$.*

3.4. Determination of matrices \mathbf{B} and \mathbf{C} . In the last step, we must find matrices \mathbf{B} and \mathbf{C} . After multiplying the numerator of the transfer function (11) by λ^{-q} , we obtain the following polynomial:

$$n(\lambda) = b_q + b_{q-1}\lambda^{-1} + b_{q-2}\lambda^{-2} + \cdots + b_1\lambda^{1-q} + b_0\lambda^{-q}. \quad (19)$$

We can determine two possible digraph structures corresponding to the polynomial (19). Below we discuss this in detail. It should be noted that these cases are discussed in terms of a digraph structure resulting from the input and output connection.

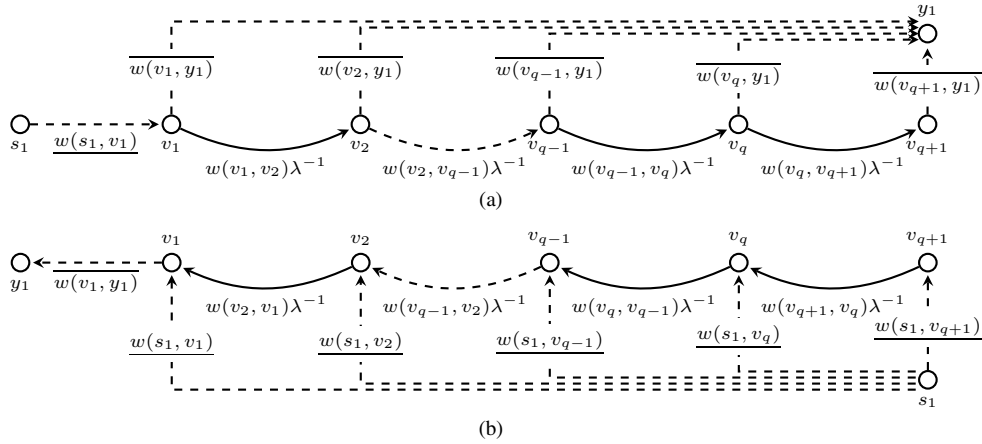


Fig. 10. Digraph corresponding to polynomial (19): Case 1 (a), Case 2 (b).

3.4.1. Case 1. For this purpose, we are expanding the digraph (see Fig. 3(a)) created in the first part of the algorithm. We add the source vertex s_1 corresponding to matrix \mathbf{B} and the output vertex y_1 corresponding to matrix \mathbf{C} . We connect the source vertex s_1 with vertices v_1 or v_2 which belong to the common part of the digraph, and vertices $v_1, v_2, \dots, v_q, v_{q+1}$ with the output vertex y_1 . Then we determine all paths p from the source vertex s_1 to the output vertex y_1 . It should be noted that paths have length $2 \leq p \leq q + 2$.

We obtain the digraph structure presented in Fig. 10(a) from which we can write the set of equations (20). After solving them we can write $\mathbf{B} \in \mathbb{R}^{(q+1) \times 1}$ and $\mathbf{C} \in \mathbb{R}^{1 \times (q+1)}$ in the form (22a).

The positive realization $(\mathbf{E}_1, \mathbf{A}, \mathbf{B}, \mathbf{C})$ of (11) is given by (16), (18) and (22a) if Proposition 3 applies and $\underline{w(s_1, v_1)} > 0$.

Remark 4. If we connect source s_1 with vertex v_2 ,

then we must determine a new digraph structure, write the corresponding set of equations and solve them. Afterwards we can write matrices $\tilde{\mathbf{B}}$ and $\tilde{\mathbf{C}}$. In this case, the positive realization is given by $(\mathbf{E}_2, \mathbf{A}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$.

3.4.2. Case 2. For this purpose, we are expanding the digraph presented in Fig. 3(b). We add the source vertex s_1 corresponding to matrix \mathbf{B} and the output vertex y_1 corresponding to matrix \mathbf{C} . We connect the source vertex s_1 with vertices $v_1, v_2, \dots, v_q, v_{q+1}$, and vertices v_1, v_2 which belong to the common part of the digraph with the output vertex y_1 . Then we determine all paths p from the source vertex s_1 to the output vertex y_1 . It should be noted that paths have length $2 \leq p \leq q + 2$.

We obtain the digraph structure presented in Fig. 10(b), from which we can write the set of the equations (21) and after solving them we can write $\mathbf{B} \in \mathbb{R}^{(q+1) \times 1}$, and $\mathbf{C} \in \mathbb{R}^{1 \times (q+1)}$ in the form of (22b).

$$\left\{ \begin{array}{l} \lambda^{-1} \\ \vdots \\ \lambda^{1-q} \\ \lambda^{-q} \end{array} \right\} \left\| \begin{array}{l} \underline{w(s_1, v_1)} \cdot \overline{w(v_1, y_1)} \\ \underline{w(s_1, v_1)} \cdot \overline{w(v_1, v_2)} \cdot \overline{w(v_2, y_1)} \\ \vdots \\ \underline{w(s_1, v_1)} \cdot \overline{w(v_1, v_2)} \cdot \dots \cdot \overline{w(v_{q-1}, v_q)} \cdot \overline{w(v_q, y_1)} \\ \underline{w(s_1, v_1)} \cdot \overline{w(v_1, v_2)} \cdot \dots \cdot \overline{w(v_{q-1}, v_q)} \cdot \overline{w(v_q, v_{q+1})} \cdot \overline{w(v_{q+1}, y_1)} \end{array} \right. \begin{array}{l} = b_q, \\ = b_{q-1}, \\ \vdots \\ = b_1, \\ = b_0. \end{array} \quad (20)$$

$$\left\{ \begin{array}{l} \lambda^{-1} \\ \vdots \\ \lambda^{1-q} \\ \lambda^{-q} \end{array} \right\} \left\| \begin{array}{l} \underline{w(s_1, v_1)} \cdot \overline{w(v_1, y_1)} \\ \underline{w(s_1, v_2)} \cdot \overline{w(v_2, v_1)} \cdot \overline{w(v_1, y_1)} \\ \vdots \\ \underline{w(s_1, v_q)} \cdot \overline{w(v_q, v_{q-1})} \cdot \dots \cdot \overline{w(v_2, v_1)} \cdot \overline{w(v_1, y_1)} \\ \underline{w(s_1, v_{q+1})} \cdot \overline{w(v_{q+1}, v_q)} \cdot \overline{w(v_q, v_{q-1})} \cdot \dots \cdot \overline{w(v_2, v_1)} \cdot \overline{w(v_1, y_1)} \end{array} \right. \begin{array}{l} = b_q, \\ = b_{q-1}, \\ \vdots \\ = b_1, \\ = b_0. \end{array} \quad (21)$$

$$\mathbf{B} = \begin{matrix} v_i \setminus s_m \\ v_1 \\ v_2 \\ \vdots \\ v_q \\ v_{q+1} \end{matrix} \begin{bmatrix} s_1 \\ w(s_1, v_1) \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{C} = \begin{matrix} y_p \setminus v_j \\ y_1 \end{matrix} \begin{bmatrix} v_1 & v_2 & \dots & v_q & v_{q+1} \\ \frac{b_q}{w(s_1, v_1)} & \frac{b_{q-1}}{w(s_1, v_1) \cdot w(v_1, v_2)} & \dots & \frac{b_1}{w(s_1, v_1) \cdot k_1} & \frac{b_0}{w(s_1, v_1) \cdot k_2} \end{bmatrix}, \quad (22a)$$

$$k_1 = w(v_1, v_2) \cdot \dots \cdot w(v_{q-1}, v_q), \quad k_2 = w(v_1, v_2) \cdot \dots \cdot w(v_{q-1}, v_q) \cdot w(v_q, v_{q+1})$$

$$\tilde{\mathbf{B}} = \begin{matrix} v_i \setminus s_m \\ v_1 \\ v_2 \\ \vdots \\ v_q \\ v_{q+1} \end{matrix} \begin{bmatrix} s_1 \\ \frac{b_q}{w(v_1, y_1)} \\ \frac{b_{q-1}}{w(v_1, y_1) \cdot w(v_2, v_1)} \\ \vdots \\ \frac{b_1}{w(v_1, y_1) \cdot w(v_q, v_{q-1}) \cdot \dots \cdot w(v_2, v_1)} \\ \frac{b_0}{w(v_1, y_1) \cdot w(v_{q+1}, v_q) \cdot w(v_q, v_{q-1}) \cdot \dots \cdot w(v_2, v_1)} \end{bmatrix}, \quad \tilde{\mathbf{C}} = \begin{matrix} y_p \setminus v_j \\ y_1 \end{matrix} \begin{bmatrix} v_1 & v_2 & \dots & v_q & v_{q+1} \\ w(v_1, y_1) & 0 & \dots & 0 & 0 \end{bmatrix}, \quad (22b)$$

The positive realization $(\mathbf{E}_1, \tilde{\mathbf{A}}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ of (11) is given by (17), (18) and (22b) if Proposition 3 applies and $w(v_1, y_1) > 0$. As in Case 1, Remark 4 is correct as well.

From the above discussion (Cases 1 and 2), we can formulate the following result.

Proposition 4. (Input and output matrix entries) *The system (5) with matrices of the form (22a) (or (22b)) is positive if and only if the arc weights in the digraph satisfy the condition $w(s_1, v_i) > 0$ and/or $w(v_i, y_1) > 0$ for $i \in \mathbb{Z}_+ = \{1, 2, \dots, q+1\}$.*

Remark 5. It should be noted here that the digraphs presented in Cases 1 and 2 have simple structures and we have almost immediately to write a set of matrices $(\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$. In other digraph structures this is not that immediate and additional conditions must be satisfied (see Section 4).

From the discussion led in Sections 3.1–3.4, we can write down the following sufficient conditions for digraph structures.

Proposition 5. *There exists a digraph structure in class \mathcal{K}_1 , which corresponds to a realization of a transfer function, if the following conditions are satisfied:*

- (C1) *in the digraph corresponding to the characteristic polynomial there exists a non-empty common part;*
- (C2) *the digraph corresponding to the characteristic polynomial does not have additional cycles;*
- (C3) *there exists a digraph mask corresponding to the characteristic polynomial;*

(C4) *there exists a digraph as a superposition operation on the digraph corresponding to the characteristic polynomial and a digraph mask in which all vertices have weights equal to 1;*

(C5) *in an extended digraph there are all paths from the source s to the output y , which have the length $p = 2, 3, \dots, q+2$.*

Proof. If conditions (C1) and (C2) are satisfied, then we can determine a digraph structure corresponding to a characteristic polynomial $\tilde{d}(\lambda)$ in the form of (13). Otherwise, the realization does not exist. Conditions (C1) and (C2) are presented in detail in Section 3.1. Then, if condition (C3) holds, then we can determine (by Proposition 2) $q+1$ possible digraph masks which guarantees the singularity of the solution. Condition (C3) is presented in detail in Section 3.2.

If conditions (C1)–(C3) are met, then by using a superposition operation on a digraph and the digraph mask we can determine all possible digraph structures. Then we choose digraphs in which all vertices have weights equal to 1. Other solutions are rejected. Condition (C4) is presented in detail in Section 3.3. Finally, to the digraph structures which met conditions (C1)–(C4) we add the source vertex s and the output vertex y . By connecting the source vertex s with vertices v_1, v_2, \dots, v_{q+1} and vertices v_1, v_2, \dots, v_{q+1} with the output vertex y , we can determine all paths from a source to an output. This type of connection guarantees that all polynomial coefficients b_q appear in (19). Condition (C5) is presented in detail in Section 3.4. If conditions (C1)–(C5) are satisfied, then

there exists a digraph structure which corresponds to a transfer function in the form of (11). ■

It should be noted that in Section 3.4 only two types of connection between the source vertex s and the output vertex y are shown. This type of connection guarantees that we will always be able to determine a digraph structure that corresponds to the transfer function (11). In general, there are many ways to connect the source s with the output y . Occasionally, there is a digraph structure corresponding to the transfer function (11), but we cannot calculate the weight values. In this case, we must reject this structure.

4. Numerical examples

Find a minimal realization of the transfer function

$$T(s) = \frac{b_3 s^{1.8} + b_2 s^{1.2} + b_1 s^{0.6} + b_0}{-a_2 s^{1.2} - a_1 s^{0.6} - a_0}. \quad (23)$$

Solution: Assuming that $\lambda = s^{0.6}$, we can rewrite the transfer function (23) in the following form:

$$T(\lambda) = \frac{b_3 \lambda^3 + b_2 \lambda^2 + b_1 \lambda + b_0}{-a_2 \lambda^2 - a_1 \lambda - a_0}. \quad (24)$$

After multiplying the denominator of (24) by $\lambda^{-q+1} = \lambda^{-4}$, we obtain the following characteristic polynomial:

$$d(\lambda) = -a_2 \lambda^{-2} - a_1 \lambda^{-3} - a_0 \lambda^{-4}, \quad (25)$$

and present it in a temporary form,

$$\begin{aligned} \tilde{d}(\lambda) &= 1 + d(\lambda) \\ &= 1 - a_2 \lambda^{-2} - a_1 \lambda^{-3} - a_0 \lambda^{-4} \\ &= \underbrace{(1 - a_2 \lambda^{-2})}_{B_2} \cup \underbrace{(1 - a_1 \lambda^{-3})}_{B_1} \cup \underbrace{(1 - a_0 \lambda^{-4})}_{B_0}, \end{aligned} \quad (26)$$

which is needed to draw the digraph.

For each simple binomial B_0 , B_1 and B_2 , using Proposition 1, we create digraph representations (Figs. 11(a)–11(c)).

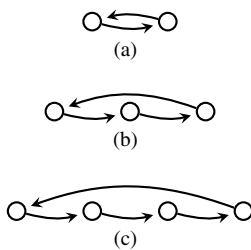


Fig. 11. Digraph corresponding to binomial: B_2 (a), B_1 (b), B_0 (c).

Now we can create all digraph realizations of the

characteristic polynomial (26). One of the possible digraph structures is presented in Fig. 12.

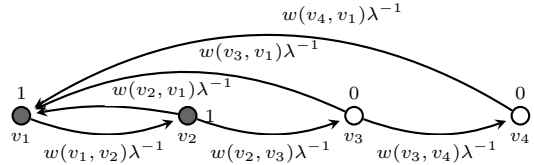


Fig. 12. One of the possible digraph structures corresponding to (26).

From the digraph we can write the following set of equations:

$$\begin{cases} \lambda^{-2} \left\| w(v_1, v_2) \cdot w(v_2, v_1) \right. &= a_2, \\ \lambda^{-3} \left\| w(v_1, v_2) \cdot w(v_2, v_3) \cdot w(v_3, v_1) \right. &= a_1, \\ \lambda^{-4} \left\| w(v_1, v_2) \cdot w(v_2, v_3) \cdot w(v_3, v_4) \cdot w(v_4, v_1) \right. &= a_0. \end{cases}$$

After solving them, we obtain the following arc weights:

$$\begin{aligned} w(v_1, v_2) &= 1, & w(v_2, v_3) &= 1, \\ w(v_3, v_4) &= 1, & w(v_2, v_1) &= a_2, \\ w(v_3, v_1) &= a_1, & w(v_4, v_1) &= a_0. \end{aligned} \quad (27)$$

We can write the state matrix in the following form:

$$\begin{aligned} \mathbf{A} &= \begin{bmatrix} 0 & w(v_2, v_1) & w(v_3, v_1) & w(v_4, v_1) \\ w(v_1, v_2) & 0 & 0 & 0 \\ 0 & w(v_2, v_3) & 0 & 0 \\ 0 & 0 & w(v_3, v_4) & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_2 & a_1 & a_0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}. \end{aligned} \quad (28)$$

It should be noted that the set of equations has many possible solutions. The proposed solution (27) is only one of them.

In the next step, using Proposition 2, we must determine (see Fig. 13) a set of digraph masks consisting of four vertices. Three of them have weights equal to 1 and one of them has zero weight.

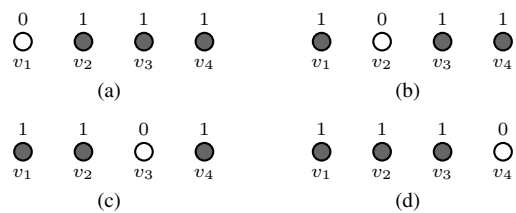


Fig. 13. All possible digraph masks.

Then, using (15), we can determine all possible realizations of the characteristic polynomial (25) as a combination of the digraph representation presented, for example, in Fig. 12 and the digraph mask presented in Fig. 13. After this, we obtain four possible realizations

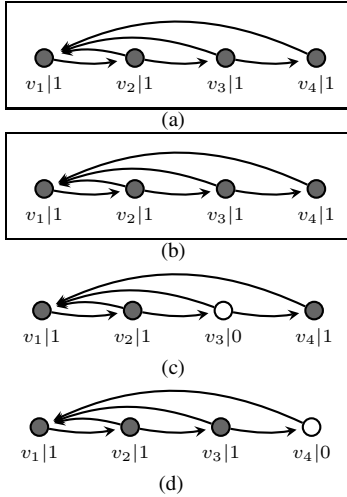


Fig. 14. All possible digraphs corresponding to (25).

(Fig. 14). It should be note that only two digraph structures presented in Figs. 14(a) and (b) satisfy the characteristic polynomial (25).

By using the digraph structure presented in Figs. 14(a) and (b), we can write the singular matrix in the following form:

$$\mathbf{E}_1 = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{E}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \quad (29)$$

In the second step, we should determine matrices \mathbf{B} and \mathbf{C} . For this purpose, we must multiply the numerator of the transfer function (24) by λ^{-3} . Then we obtain the following polynomial:

$$n(\lambda) = b_3 + b_2\lambda^{-1} + b_1\lambda^{-2} + b_0\lambda^{-3}, \quad (30)$$

and we expand the digraph created in the first step (Figs. 14(a) and 14(b)). We have two possible structures, which are considered in Sections 4.1 and 4.2.

4.1. Case 1. In this case, we connect the source vertex s_1 corresponding to matrix \mathbf{B} with the vertex v_1 or v_2 belonging to a set of common parts of the digraph presented in Fig. 12, and the vertex v_1, v_2, v_3, v_4 with the output vertex y_1 corresponding to matrix \mathbf{C} . Then we determine all paths from the source vertex s_1 to the output vertex y_1 . We obtain two possible digraph structures.

Digraph 1—connect vertex s_1 with vertex v_1 : After this

connection, we obtain a digraph presented in Fig. 15(a). Then, using the created digraph, we can write a set of equations in the form

$$\begin{cases} \frac{w(s_1, v_1) \cdot \overline{w(v_1, y_1)}}{w(s_1, v_1) \cdot \overline{w(v_2, y_1)}} = b_3, \\ \lambda^{-1} \frac{w(s_1, v_1) \cdot \overline{w(v_2, y_1)}}{w(s_1, v_1) \cdot \overline{w(v_3, y_1)}} = b_2, \\ \lambda^{-2} \frac{w(s_1, v_1) \cdot \overline{w(v_3, y_1)}}{w(s_1, v_1) \cdot \overline{w(v_4, y_1)}} = b_1, \\ \lambda^{-3} \frac{w(s_1, v_1) \cdot \overline{w(v_4, y_1)}}{w(s_1, v_1) \cdot \overline{w(v_4, y_1)}} = b_0. \end{cases}$$

Solving them, we can write an input and output matrix in the following form:

$$\mathbf{B} = \begin{bmatrix} \frac{w(s_1, v_1)}{0} \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad (31)$$

$$\mathbf{C} = \begin{bmatrix} \frac{b_3}{w(s_1, v_1)} & \frac{b_2}{w(s_1, v_1)} & \frac{b_1}{w(s_1, v_1)} & \frac{b_0}{w(s_1, v_1)} \end{bmatrix}.$$

The desired realization $(\mathbf{E}_1, \mathbf{A}, \mathbf{B}, \mathbf{C})$ of (23) is given by (29), (28) and (31). It should be noted that realization is positive if $\overline{w(s_1, v_1)} \in \mathbb{R}_+$.

Digraph 2—connect vertex s_1 with vertex v_2 : After this connection, we obtain the digraph presented in Fig. 15(b). Then, using the created digraph, we can write the following set of equations:

$$\begin{cases} \frac{w(s_1, v_2) \cdot \overline{w(v_2, y_1)}}{w(s_1, v_2) \cdot \left(\overline{w(v_3, y_1)} + a_2 \cdot \overline{w(v_1, y_1)} \right)} = b_3, \\ \lambda^{-1} \frac{w(s_1, v_2) \cdot \left(\overline{w(v_3, y_1)} + a_2 \cdot \overline{w(v_1, y_1)} \right)}{w(s_1, v_2) \cdot \left(\overline{w(v_4, y_1)} + a_1 \cdot \overline{w(v_1, y_1)} \right)} = b_2, \\ \lambda^{-2} \frac{w(s_1, v_2) \cdot \left(\overline{w(v_4, y_1)} + a_1 \cdot \overline{w(v_1, y_1)} \right)}{a_0 \cdot \overline{w(s_1, v_2)} \cdot \overline{w(v_1, y_1)}} = b_1, \\ \lambda^{-3} \frac{a_0 \cdot \overline{w(s_1, v_2)} \cdot \overline{w(v_1, y_1)}}{a_0 \cdot \overline{w(s_1, v_2)} \cdot \overline{w(v_1, y_1)}} = b_0. \end{cases}$$

Solving them, we can write an input and output matrix in the following form:

$$\tilde{\mathbf{B}} = \begin{bmatrix} 0 \\ \frac{w(s_1, v_2)}{0} \\ 0 \\ 0 \end{bmatrix}, \quad (32)$$

$$\tilde{\mathbf{C}} = \left[\frac{b_3}{\overline{w(v_1, y_1)}} \quad \frac{b_2}{\overline{w(v_2, y_1)}} \quad \frac{b_1}{\overline{w(v_3, y_1)}} \quad \frac{b_0}{\overline{w(v_4, y_1)}} \right],$$

where

$$\overline{w(v_1, y_1)} = \frac{b_0}{a_0 \cdot \overline{w(s_1, v_2)}},$$

$$\overline{w(v_2, y_1)} = \frac{b_3}{\overline{w(s_1, v_2)}},$$

$$\overline{w(v_3, y_1)} = \frac{a_0 \cdot b_2 - a_2 \cdot b_0}{a_0 \cdot \overline{w(s_1, v_2)}},$$

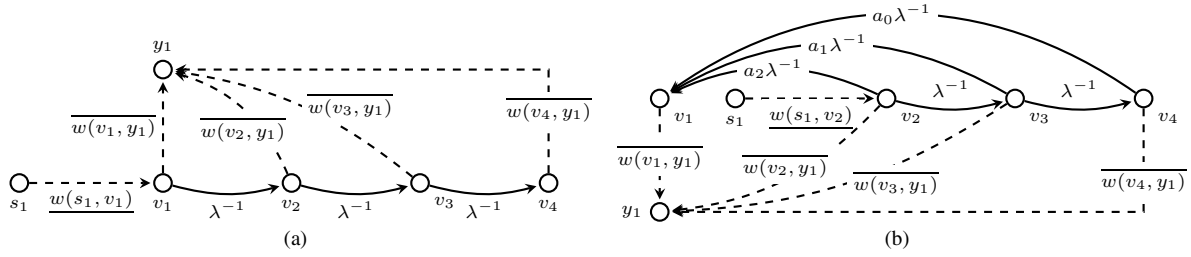


Fig. 15. Realization of (30) in Case 1: Digraph 1 (a), Digraph 2 (b).

$$\overline{w(v_4, y_1)} = \frac{a_0 \cdot b_1 - a_1 \cdot b_0}{a_0 \cdot \overline{w(s_1, v_2)}}.$$

$$\bar{\mathbf{C}} = [\overline{w(v_1, y_1)} \ 0 \ 0 \ 0], \quad (33)$$

The desired realization $(\mathbf{E}_2, \mathbf{A}, \tilde{\mathbf{B}}, \tilde{\mathbf{C}})$ of (23) is given by (29), (28) and (32). It should be noted that the realization is positive if $\overline{w(s_1, v_2)} \in \mathbb{R}_+$ and the coefficients of the polynomials (25) and (30) satisfy the following conditions:

$$\begin{aligned} a_0 \cdot b_2 - a_2 \cdot b_0 &> 0, \\ a_0 \cdot b_1 - a_1 \cdot b_0 &> 0. \end{aligned}$$

4.2. Case 2. In this case, we connect the source vertex s_1 corresponding to matrix \mathbf{B} with the vertex v_1, v_2, v_3 and v_4 , and the vertex v_1 or v_2 belonging to the set of common parts of the digraph presented in Fig. 12 with the output vertex y_1 corresponding to matrix \mathbf{C} . Then, we determine all paths from the source vertex s_1 to the output vertex y_1 . We obtain two possible digraph structures.

Digraph 1—connect vertex v_1 with vertex y_1 : After this connection, we obtain the digraph presented in Fig. 16(a). Then, using the created digraph, we can write the following set of equations:

$$\begin{cases} \left| \begin{array}{l} \overline{w(s_1, v_1)} \cdot \overline{w(v_1, y_1)} \\ \lambda^{-1} \left[\overline{w(v_1, y_1)} \cdot [a_0 \cdot \overline{w(s_1, v_4)} + a_1 \cdot \overline{w(s_1, v_3)} + a_2 \cdot \overline{w(s_1, v_2)}] \right] \end{array} \right. & = b_3, \\ \left| \begin{array}{l} \lambda^{-2} \left[\overline{w(v_1, y_1)} \cdot [a_1 \cdot \overline{w(s_1, v_2)} + a_0 \cdot \overline{w(s_1, v_3)}] \right] \\ \lambda^{-3} [a_0 \cdot \overline{w(s_1, v_2)} \cdot \overline{w(v_1, y_1)}] \end{array} \right. & = b_2, \\ & = b_1, \\ & = b_0. \end{cases}$$

Solving them, we can write an input and output matrix in the following form:

$$\bar{\mathbf{B}} = \begin{bmatrix} \overline{w(s_1, v_1)} \\ \overline{w(s_1, v_2)} \\ \overline{w(s_1, v_3)} \\ \overline{w(s_1, v_4)} \end{bmatrix},$$

where

$$\begin{aligned} \overline{w(s_1, v_1)} &= \frac{b_3}{\overline{w(v_1, y_1)}}, \\ \overline{w(s_1, v_2)} &= \frac{b_0}{a_0 \cdot \overline{w(v_1, y_1)}}, \\ \overline{w(s_1, v_3)} &= \frac{a_0 \cdot b_1 - a_1 \cdot b_0}{a_0^2 \cdot \overline{w(v_1, y_1)}}, \\ \overline{w(s_1, v_4)} &= \frac{a_0 \cdot (a_0 \cdot b_2 - a_2 \cdot b_0) - a_1 \cdot (a_0 \cdot b_1 - a_1 \cdot b_0)}{a_0^3 \cdot \overline{w(v_1, y_1)}}. \end{aligned}$$

The desired realization $(\mathbf{E}_1, \mathbf{A}, \bar{\mathbf{B}}, \bar{\mathbf{C}})$ of (23) is given by (29), (28) and (33). It should be noted that the realization is positive if $\overline{w(v_1, y_1)} \in \mathbb{R}_+$ and the coefficients of the polynomial (25) and (30) satisfy the following conditions:

$$\begin{aligned} a_0 \cdot b_1 - a_1 \cdot b_0 &> 0, \\ a_0 \cdot (a_0 \cdot b_2 - a_2 \cdot b_0) - a_1 \cdot (a_0 \cdot b_1 - a_1 \cdot b_0) &> 0. \end{aligned}$$

Digraph 2—connect vertex v_2 with vertex y_1 : After this connection, we obtain the digraph presented in Fig. 16(b). Then, using the created digraph, we can write a set of equations:

$$\begin{cases} \left| \begin{array}{l} \overline{w(s_1, v_2)} \cdot \overline{w(v_2, y_1)} \\ \lambda^{-1} \left[\overline{w(s_1, v_1)} \cdot \overline{w(v_2, y_1)} \right] \end{array} \right. & = b_3, \\ \left| \begin{array}{l} \lambda^{-2} \left[\overline{w(v_2, y_1)} \cdot [a_1 \cdot \overline{w(s_1, v_3)} + a_0 \cdot \overline{w(s_1, v_4)}] \right] \\ \lambda^{-3} [a_0 \cdot \overline{w(s_1, v_3)} \cdot \overline{w(v_2, y_1)}] \end{array} \right. & = b_2, \\ & = b_1, \\ & = b_0. \end{cases}$$

Solving them, we can write an input and output matrix in the following form:

$$\begin{aligned} \bar{\mathbf{B}} &= \begin{bmatrix} \overline{w(s_1, v_1)} \\ \overline{w(s_1, v_2)} \\ \overline{w(s_1, v_3)} \\ \overline{w(s_1, v_4)} \end{bmatrix}, \\ \bar{\mathbf{C}} &= [0 \ \overline{w(v_2, y_1)} \ 0 \ 0], \end{aligned} \quad (34)$$

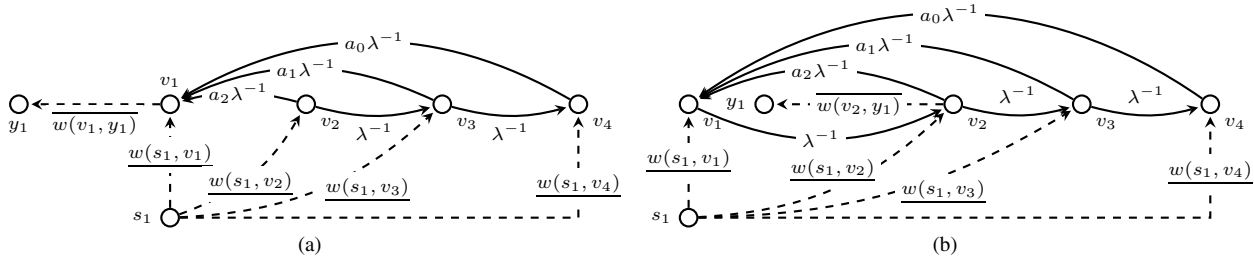


Fig. 16. Realization of (30) in Case 2: Digraph 1 (a), Digraph 2 (b).

where

$$\begin{aligned} \underline{w(s_1, v_1)} &= \frac{b_2}{\underline{w(v_2, y_1)}}, \\ \underline{w(s_1, v_2)} &= \frac{b_3}{\underline{w(v_2, y_1)}}, \\ \underline{w(s_1, v_3)} &= \frac{b_0}{a_0 \cdot \underline{w(v_2, y_1)}}, \\ \underline{w(s_1, v_4)} &= \frac{a_0 \cdot b_1 - a_1 \cdot b_0}{a_0^2 \cdot \underline{w(v_2, y_1)}}. \end{aligned}$$

The desired realization $(\mathbf{E}_2, \mathbf{A}, \check{\mathbf{B}}, \check{\mathbf{C}})$ of (23) is given by (29), (28) and (34). It should be noted that the realization is positive if $w(v_2, y_1) \in \mathbb{R}_+$ and the coefficients of the polynomials (25) and (30) satisfy the following conditions:

$$a_0 \cdot b_1 - a_1 \cdot b_0 > 0.$$

5. Concluding remarks

A method for computation of a minimal realization of a given proper transfer function of fractional positive descriptor one-dimensional continuous-time linear systems has been proposed. Sufficient conditions for the existence of a minimal set of realizations of a given proper transfer function have been established. A method based on one-dimensional digraph theory for the computation of minimal realizations has been proposed. It should be noted that in many research studies we can find a canonical form of the system, i.e., a constant matrix form, which satisfies the system described by the transfer function. With the use of this form, we are able to write only one realization of the system, while there exists a set of solutions. In this paper a solution to the problem of finding a set of possible minimal positive realizations has been proposed. Additionally, the possibility of an easy implementation in the form of a computer algorithm using CPU and/or GPU processors is the strength of this digraph method. The effectiveness of the algorithm has been illustrated with some numerical examples. An extension of this study to fractional descriptor discrete-time linear

systems and to a fractional descriptor hybrid model is possible.

Further work includes extension of the proposed method to find all possible solutions for other structures of matrix \mathbf{E} and extension of the algorithm to find the class of electrical circuits corresponding to the transmission matrix.

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