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SLIME MOULD GAMES BASED ON ROUGH SET THEORY

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We define games on the medium of plasmodia of slime mould, unicellular organisms that look like giant amoebae. The plasmodia try to occupy all the food pieces they can detect. Thus, two different plasmodia can compete with each other. In particular, we consider game-theoretically how plasmodia of *Physarum polycephalum* and *Badhamia utricularis* fight for food. Placing food pieces at different locations determines the behavior of plasmodia. In this way, we can program the plasmodia of *Physarum polycephalum* and *Badhamia utricularis* by placing food, and we can examine their motion as a *Physarum* machine—an abstract machine where states are represented as food pieces and transitions among states are represented as movements of plasmodia from one piece to another. Hence, this machine is treated as a natural transition system. The behavior of the *Physarum* machine in the form of a transition system can be interpreted in terms of rough set theory that enables modeling some ambiguities in motions of plasmodia. The problem is that there is always an ambiguity which direction of plasmodium propagation is currently chosen: one or several concurrent ones, i.e., whether we deal with a sequential, concurrent or massively parallel motion. We propose to manage this ambiguity using rough set theory. Firstly, we define the region of plasmodium interest as a rough set; secondly, we consider concurrent transitions determined by these regions as a context-based game; thirdly, we define strategies in this game as a rough set; fourthly, we show how these results can be interpreted as a Go game.

Keywords: slime mould games, *Physarum* machines, transition systems, rough set theory, simulation software.

1. Introduction

Slime mould is an informal name to denote different eukaryotic organisms. One of these organisms studied in details from the point of view of computer science is called *Physarum polycephalum*. It belongs to the species of order *Physarales*, subclass *Myxogastromycetidae*, class *Myxomycetes*, division *Myxostelida*. Plasmodium is a 'vegetative' phase of *Physarum polycephalum*. It is a single cell with a myriad of diploid nuclei. Therefore this organism can be really huge—up to 1 m^2. It behaves and moves as a giant amoeba connecting different pieces of food by protoplasmic veins (very long pseudopodia). Typically, the plasmodium forms a network of pseudopodia connecting the masses of protoplasm at the food sources which turns out to be efficient in terms of network length and resilience.

The notion of *Physarum* machines (machines based on the motion of plasmodia of *Physarum polycephalum*) was first introduced by Adamatzky (2007b). A formalization of these machines based on Kolomogorov–Uspensky automata was proposed by Adamatzky (2007a). *Physarum* machines are regarded as a biological sensing and computing device implemented in a vegetative stage of slime mould. It is programmed by spatio-temporal configurations of repelling and attracting gradients. Meanwhile, each dangerous place avoided by slime mould is treated as a repelling gradient and it is called a *repellent*, and each attracting place (first of all, a food piece) is considered as an attracting gradient and it is called an *attractant*. There are several classes of *Physarum* devices: morphological and sensing processors, bio-molecular and microfluidic

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logical circuits. In this paper, we focus on morphological processors—simple transitions of plasmodia from one place of food sources to another.

The plasmodium transitions can be treated as a game called a *slime mould game*, i.e., an experimental game, where, on the one hand, all basic definitions are verified in experiments with *Physarum polycephalum* and *Badhamia utricularis* and, on the other hand, they are regarded as nonsequential games: concurrent (with several attractants) or massive-parallel (with several hundred attractants). In this paper, we propose a technique for describing these games. Theoretically, they are defined as context-based games. In them, players can move concurrently as in concurrent games, but the set of actions is always infinite. In our experiments, we follow the following interpretations of basic entities: (i) attractants as payoffs; (ii) attractants occupied by the plasmodium as states of the game; (iii) active zones of plasmodium as players; (iv) logic gates for behaviors as moves (available actions) for the players; (v) propagation of the plasmodium as the transition table which associates, with a given set of states and a given move of the players, the set of states resulting from that move.

In these games, the following problems can be defined: (i) finding the shortest path in possible transitions (e.g., this task solved using Petri nets is considered by Clempner (2006)); (ii) defining leaders—concentrations of slime mould at some places (e.g., the task of computing the Stackelberg/Nash equilibria with respect to a three-player Stackelberg game consisting of a leader and two followers is examined by Trejo *et al.* (2015)). In this paper, we show that strategies in slime mould games are better defined by means of rough sets. The problem is that there is an ambiguity in finding the next direction of a plasmodium transition. The plasmodium can choose from one to several concurrent directions at each step and then it can change its own decision and move back. This ambiguity makes the games context-based and these contexts are defined as rough sets.

There are two different species of slime mould, *Physarum polycephalum* and *Badhamia utricularis*, to demonstrate context-based games experimentally. We assume that slime mould games are created over the formal underlying structures called *Physarum* machines described in Section 3. A *Physarum* machine is a programmable biological device implemented in the plasmodia of *Physarum polycephalum* and *Badhamia utricularis*. The plasmodial stage of such organisms can be treated as a natural transition system. Therefore, transition systems can be considered as biological models for strategic games (Schumann *et al.*, 2014). The descriptions of the behavior of *Physarum* machines in the form of transition systems can be considered in terms of rough set theory that enables us to model some ambiguities and uncertainties in motions of plasmodia. To model uncertainties in games, we can also use fuzzy sets (Saipara and Kumam, 2016). However, rough sets are an appropriate tool to model uncertainties in discrete situations. In our approach, rough set models of behavior of *Physarum* machines have been created on the basis of transition systems (Pancerz and Schumann, 2015). Moreover, we have used rough sets to describe strategy games based on *Physarum* machines (Pancerz and Schumann, 2017).

In order to illustrate slime mould games defined theoretically, we introduce a special rough-set version of an ancient Chinese Go game (Schumann and Pancerz, 2015) with a standard definition of rough sets (Pawlak, 1991) and a definition using VPRSM (variable precision rough set model) (Ziarko, 1993). In this paper, the presented approach to rough set modeling of slime mould games is based on the generalized rough set theory (Yao and Lin, 1996) in which arbitrary binary relations defining neighborhood systems are used (cf. Section 2.1). The neighborhood is determined with respect to direct propagation of plasmodia from one active group of points to another. In transition system models, direct propagations are represented by transition relations.

In this paper, we focus on theoretical aspects of slime mould games. In Section 2, we recall basic notions concerning both rough sets and transition systems. In Section 3, we define *Physarum* machines. In Section 4, we consider rough set aspects of transition systems (natural implementations of *Physarum* machines). In Section 5, we examine fundamentals of slime mould games. In Section 6, we analyze one of the possible examples of slime mould games presented as a Go game.

2. Basic notions

2.1. Rudiments of rough sets. Rough set theory proposed by Pawlak (1991) is a mathematical approach to imperfect knowledge. In his original definition of rough sets, an equivalence binary relation was considered. However, the original rough set model can be generalized in several directions (Yao *et al.*, 1997). One of the generalizations can be obtained by using arbitrary binary relations. Rough set theory built from binary relations may be related to neighborhood systems (Yao and Lin, 1996).

Let U be a nonempty set of objects (a universe of discourse). Any subset $R \subseteq U \times U$ is called a binary relation on U. Let $u, v \in U$. If $(u, v) \in R$, we say that v is R-related to u, u is a predecessor of v, whereas v is a successor of u. A successor neighborhood of $u \in U$ is given by $R_s(u) = \{v \in U : (u, v) \in R\}$.

The idea of rough sets consists of the approximation of a given set $X \subseteq U$ of objects by a pair of approximations (lower and upper) of X. In the case of an arbitrary binary relation R, we define $\underline{R}(X) = \{u \in U :$

$R_s(u) \subseteq X\}$ and $\overline{R}(X) = \{u \in U : R_s(u) \cap X \neq \emptyset\}$, called the R-lower and R-upper approximation of X, respectively. The R-lower approximation of X consists of each element $u \in U$ whose successor neighborhood is wholly included in X. The R-upper approximation of X consists of each element $u \in U$ whose successor neighborhood has a nonempty intersection with X. If $\underline{R}(X) \subset \overline{R}(X)$, then the set X is rough otherwise (i.e., $\underline{R}(X) = \overline{R}(X)$), the set X is sharp.

The roughness of a set can be characterized numerically. To this end, the accuracy of approximation of X with respect to R is defined as

$$\alpha_R(X) = \frac{\mathrm{card}(\underline{R}(X))}{\mathrm{card}(\overline{R}(X))},$$

where card denotes the cardinality of the set and $X \neq \emptyset$.

Some relaxed definition was proposed by Ziarko (1993) in the variable precision rough set model (VPRSM). The standard set inclusion is replaced with the majority set inclusion in definitions of approximations. Let $0 \leq \beta < 0.5$. In the case of an arbitrary binary relation R, we define

$$\underline{R}^{\beta}(X) = \{u \in U : R_s(u) \overset{\beta}{\subseteq} X\},$$

$$\overline{R}^{\beta}(X) = \Big\{u \in U : 1 - \frac{R_s(u) \cap X}{R_s(u)} < 1 - \beta\Big\},$$

called the R_β-lower and R_β-upper approximation of X, respectively, where

$$R_s(u) \overset{\beta}{\subseteq} X \Longleftrightarrow 1 - \frac{\mathrm{card}(R_s(u) \cap X)}{\mathrm{card}(R_s(u))} \leq \beta.$$

In the case of the VPRSM, the accuracy of the approximation of X with respect to R is defined as

$$\alpha_R(X) = \frac{\mathrm{card}(\underline{R}^{\beta}(X))}{\mathrm{card}(\overline{R}^{\beta}(X))},$$

where and $X \neq \emptyset$.

2.2. Rudiments of transition systems. Transition systems are a commonly used and understood model of computation. A given transition system consists of a set of states, with an initial state (or initial states), together with transitions between states. Transitions are labeled to specify the kind of events they represent (Winskel and Nielsen, 1995). In this paper, we use the following definition of the transition system: A transition system is the quadruple TS $= (S, E, T, S_{\mathrm{init}})$, where S is the non-empty set of states, E is the set of events, $T \subseteq S \times E \times S$ is the transition relation, and $S_{\mathrm{init}} \subseteq S$ is the set of initial states.

Usually transition systems are based on actions which may be viewed as labeled events. If $(s, e, s') \in T$, then the idea is that TS can go from s to s' as a result of the event e occurring at s. A single element $(s, e, s') \in T$ is briefly called a transition. We can write a transition as $s \xrightarrow{e} s'$. It is sometimes convenient to consider transitions between states as strings of events. We write $s_1 \xrightarrow{v} s_k$, where $v = e_1 e_2 \ldots e_{k-1}$ is a (possibly empty), string of some events from E, to mean $s_1 \xrightarrow{e_1} s_2 \xrightarrow{e_2} \ldots \xrightarrow{e_{k-1}} s_k$ for some states $s_1, s_2, \ldots, s_k$ from S.

Any transition system $TS = (S, E, T, S_{\mathrm{init}})$ can be presented in the form of a labeled directed graph with nodes corresponding to states from S, edges representing the transition relation T, and labels of edges corresponding to events from E. Initial states are encircled to distinguish them.

In the transition systems mentioned earlier, it is assumed, that all events happen instantaneously. In timed transition systems, timing constraints restrict the times at which events may occur (Henzinger *et al*., 1992). The timing constraints are classified into two categories: lower-bound and upper-bound requirements. Let N be a set of nonnegative integers. A timed transition system TTS $= (S, E, T, S_{\mathrm{init}}, l, u)$ consists of an underlying transition system TS $= (S, E, T, S_{\mathrm{init}})$, a minimal delay function (a lower bound) $l : E \to N$ assigning each event a nonnegative integer, and a maximal delay function (an upper bound) $u : E \to N \cup \{\infty\}$ assigning each event a nonnegative integer or infinity.

3. *Physarum* machines

In this section, we give a formal description of *Physarum* machines that are biological computing devices experimentally implemented in the plasmodium of *Physarum polycephalum*. An analogous description can be made for the plasmodium of *Badhamia utricularis*.

A *Physarum* machine is a formal underlying structure to create a slime mould game. It comprises an amorphous yellowish mass with networks of protoplasmic veins, programmed by spatial configurations of attracting and/or repelling stimuli.

Let $t = 0, 1, 2, \ldots$ be a discrete time. Formally, a structure of the *Physarum* machine can be described as a triple $\mathcal{PM} = (Ph, Attr, Rep)$ (Pancerz and Schumann, 2015), where $Ph = \{ph_1, ph_2, \ldots, ph_k\}$, is the set of original points of plasmodium,

$$Attr = \bigcup_{t=1}^{\infty} Attr_t,$$

with $Attr_t = \{attr_1, attr_2, \ldots, attr_m\}$, is the set of attractants at time step t, and $Rep = \bigcup_{t=1}^{\infty} Rep_t$, where

$$Rep_t = \{rep_1, rep_2, \ldots, rep_n\},$$

is the set of repellents at time step t.

Let us define the set $Attr_{t+1}$ as a union of subsets,

$$\bigcup_{attr_x \in Attr_t} Attr_{t+1}^{attr_x},$$

such that each subset $Attr_{t+1}^{attr_x}$ is the set of attractants at time step $t+1$ seen from the point $attr_x \in Attr_t$ reached at time step t. If the cardinal number $\mathrm{card}(Attr_{t+1}^{attr_x}) \leq p-1$ for each time step $t > 0$ at each point $attr_x \in Attr_t$, then the machine $\mathcal{PM}$ is called *p-adic valued*. In the same way, we can divide Rep_{t+1} into subsets

$$\bigcup_{attr_x \in Attr_t} Rep_{t+1}^{attr_x}$$

such that each $Rep_{t+1}^{attr_x}$ is the set of repellents at time step $t+1$ seen from the point $attr_x \in Attr_t$. Both divisions are needed to describe the machine $\mathcal{PM}$ as p-adic valued arithmetic functions.

In a standard case, positions of original points of plasmodium, attractants, and repellents are considered in the two-dimensional space (e.g., at a Petri dish (Petri, 1887)).

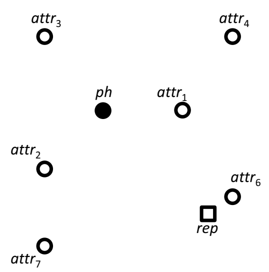

Fig. 1. Structure $\mathcal{PM} = (Ph, Attr, Rep)$ of the *Physarum* machine.

Consider a structure $\mathcal{PM} = (Ph, Attr, Rep)$ of the *Physarum* machine given in Fig. 1.

It is worth noting that, in the graphical presentation of structures of *Physarum* machines, we will use the following symbols: closed circles corresponding to original points of plasmodium, open circles corresponding to attractants, and open rectangles corresponding to repellents.

One can see that the components of the structure $\mathcal{PM} = (Ph, Attr, Rep)$ are as follows:

$$Ph = \{ph\},$$

$$Attr = Attr_{t=0}^{ph} \cup Attr_{t=1}^{attr_2} \cup Attr_{t=1}^{attr_1},$$

where

$$Attr_{t=0}^{ph} = \{attr_1, attr_2, attr_3\},$$

$$Attr_{t=1}^{attr_2} = \{attr_3, attr_7\},$$

$$Attr_{t=1}^{attr_1} = \{attr_4, attr_5, attr_6\},$$

and

$$Rep = Rep_{t=2}^{attr_6} = \{rep\}.$$

Hence, we see that for $t = 0, 1, 2$ this machine is 4-adic valued.

In general, dynamics (behavior) of the *Physarum* machine $\mathcal{PM}$ can be described by the family $V = \{V^t\}_{t \in \{t_0, t_1, t_2, \dots\}}$ of the sets of protoplasmic veins formed by plasmodium during its action, where $V^t = \{v_1^t, v_2^t, \dots, v_{\mathrm{card}(V^t)}^t\}$ is the set of all protoplasmic veins of plasmodium present at the time step t in $\mathcal{PM}$. Each vein $v_i^t \in V^t$, where $i = 1, 2, \dots, \mathrm{card}(V^t)$, is the pair $\langle \pi_{i^s}^t, \pi_{i^e}^t \rangle$ of active points in $\mathcal{PM}$, i.e., $\pi_{i^s}^t \in Ph \cup Attr_t$ and $\pi_{i^e}^t \in Ph \cup Attr_t$. Here $\pi_{i^s}^t$ is the start point of the vein v_i^t whereas $\pi_{i^e}^t$ is the end point of the vein v_i^t.

The starting point in modeling the behavior of a given *Physarum* machine TS($\mathcal{PM}$) is a transition system describing plasmodium propagation (Pancerz and Schumann, 2015). To build a model in the form of the transition system TS($\mathcal{PM}$) $= (S, E, T, S_{\text{init}})$, of the behavior of the *Physarum* machine $\mathcal{PM} = \{Ph, Attr, Rep\}$, we take into consideration a stable state, i.e., the state at a given time step t (e.g., the last one), when the set of all protoplasmic veins formed by plasmodium is fixed, i.e., $V^t = \{v_1^t, v_2^t, \dots, v_{\mathrm{card}(V^t)}^t\}$. The following bijective functions are used:

$$\sigma_t : \bigcup_{attr_x \in Attr_{t-1}} Attr_t^{attr_x} \to S_t$$

assigning a state to each attractant at t, $\epsilon_t : V^t \to E_t$ assigning an event to each protoplasmic vein at t, $\tau : V^t \to T$ assigning a transition to each protoplasmic vein, and $\iota : Ph \to S_{init}$ assigning an initial state to each original point of plasmodium. One can see that

$$S = Ph \cup \bigcup_{t=0}^{\infty} S_t, \quad E = \bigcup_{t=0}^{\infty} E_t.$$

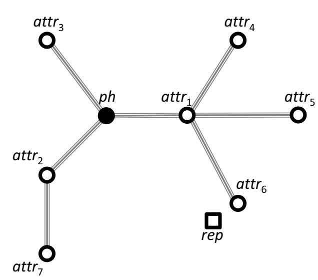

Fig. 2. Stable state of the *Physarum* machine $\mathcal{PM} = (Ph, Attr, Rep)$.

Consider a stable state of the *Physarum* machine $\mathcal{PM} = (Ph, Attr, Rep)$ from Fig. 2. One can see that protoplasmic veins were formed by plasmodium.

A model, in the form of a transition system $\text{TS}(\mathcal{PM}) = (S, E, T, S_{init})$, of the behavior of the *Physarum* machine $\mathcal{PM} = (Ph, Attr, Rep)$, where $S = S_{t=0} \cup S_{t=1} \cup S_{t=2}$, i.e., $S = \{s_1, s_2, s_3, s_4, s_5, s_6, s_7, s_8\}$, $E = E_{t=0} \cup E_{t=1} \cup E_{t=2}$, i.e., $E = \{e_1, e_2, e_3, e_4, e_5, e_6, e_7, s_8\}$, $T = \{(s_1, e_1, s_2), (s_1, e_2, s_3), (s_1, e_3, s_4), (s_2, e_4, s_5), (s_2, e_5, s_6), (s_2, e_6, s_7), (s_3, e_7, s_8)\}$, and $S_{\text{init}} = \{s_1\}$, is shown in Fig. 3.

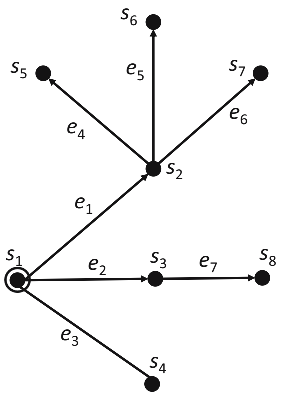

Fig. 3. Model, in the form of a transition system $TS(\mathcal{PM}) = (S, E, T, S_{\text{init}})$, of the behavior of the *Physarum* machine $\mathcal{PM} = \{Ph, Attr, Rep\}$.

In the *Physarum* machine $\mathcal{PM}$ we can identify five full paths of plasmodium propagation. These paths are determined by strings of events in the transition system model $\text{TS}(\mathcal{PM})$ of $\mathcal{PM}$, i.e.: $s_1 \xrightarrow{e_1} s_2 \xrightarrow{e_4} s_5$, $s_1 \xrightarrow{e_1} s_2 \xrightarrow{e_5} s_6$, $s_1 \xrightarrow{e_1} s_2 \xrightarrow{e_6} s_7$, $s_1 \xrightarrow{e_2} s_3 \xrightarrow{e_7} s_8$, and $s_1 \xrightarrow{e_3} s_4$.

Timed transition systems can be used to model the behavior of *Physarum* machines. In timed transition systems, the quantitative lower-bound and upper-bound timing constraints are imposed on events. This ability of modeling the behavior of *Physarum* machines is important because attracting and repelling stimuli can be activated and/or deactivated for proper time periods to perform given computational tasks. In the case of a model in the form of a timed transition system $\text{TTS}(\mathcal{PM}) = (S, E, T, S_{\text{init}}, l, u)$, the bijective functions are slightly modified, i.e.

- $\sigma : Ph \cup Attr \to S$ assigning a state to each original point of plasmodium as well as to each attractant,
- $\epsilon : \bigcup_{t \in \{t_0, t_1, t_2, \dots\}} V_t \to E$ assigning an event to each protoplasmic vein,
- $\tau : \bigcup_{t \in \{t_0, t_1, t_2, \dots\}} V_t \to T$ assigning a transition to each protoplasmic vein,
- $\iota : Ph \to S_{\text{init}}$ assigning an initial state to each original point of plasmodium.

4. Rough sets and transition systems describing plasmodium propagation

To describe the behavior of *Physarum* machines in terms of rough sets, we can use a model related to neighborhood systems.

Let $\text{TS}(\mathcal{PM}) = (S, E, T, S_{\text{init}})$ be a transition system modeling the behavior of a given *Physarum* machine $\mathcal{PM}$. In the set of states of $\text{TS}(\mathcal{PM})$, we can distinguish some regions of interest (ROIs), i.e., selected subsets of states identified for a particular purpose. Let $\Omega = \{\omega_1, \omega_2, \dots, \omega_v\}$ be the set of all regions of interest. $\mathcal{N} = \{N_{\omega_1}, N_{\omega_2}, \dots, N_{\omega_v}\}$ denotes a family of sets of states corresponding to regions of interest.

For each node $n \in N_{\omega_1} \cup N_{\omega_2} \cup \dots \cup N_{\omega_v}$, we define its inter-region neighborhood:

$$\text{IRN}(n) = \Big\{n' : (n, n') \in E \wedge \underset{\omega \in \Omega}{\exists} (n' \in N_\omega \wedge n \notin N_\omega)\Big\}.$$

Let $\omega_i, \omega_j \subseteq N$ be two distinguished regions of interest. The lower approximation $\underline{\text{IRN}}(\omega_i \to \omega_j)$ of the inter-region neighborhood, from ω_i to ω_j, is defined as

$$\underline{\text{IRN}}(\omega_i \to \omega_j) = \Big\{n \in N_{\omega_i} : \text{IRN}(n) \neq \emptyset \wedge \text{IRN}(n) \subseteq N_{\omega_j}\Big\}.$$

The upper approximation $\overline{\text{IRN}}(\omega_i \to \omega_j)$ of the inter-region neighborhood, from ω_i to ω_j, is defined as

$$\overline{\text{IRN}}(\omega_i \to \omega_j) = \{n \in N_{\omega_i} : \text{IRN}(n) \cap N_{\omega_j} \neq \emptyset\}.$$

The accuracy of approximation of the inter-region neighborhood can be defined analogously to the accuracy of approximation in rough set theory, i.e.,

$$\alpha_{\text{IRN}}(\omega_i \to \omega_j) = \frac{\text{card}(\underline{\text{IRN}}(\omega_i \to \omega_j))}{\text{card}(\overline{\text{IRN}}(\omega_i \to \omega_j))}.$$

We slightly modify definitions of the lower and upper approximations of the inter-region neighborhood in the case of the variable precision rough set model (VPRSM). Let $\omega_i, \omega_j \subseteq N$ be two distinguished regions of interest, and $0 \leq \beta < 0.5$. The β-lower approximation $\underline{\text{IRN}}(\omega_i \to \omega_j)$ of the inter-region neighborhood, from ω_i to ω_j, is defined by

$$\underline{\text{IRN}}^\beta(\omega_i \to \omega_j) = \Big\{n \in N_{\omega_i} : \text{IRN}(n) \neq \emptyset \wedge \text{IRN}(n) \overset{\beta}{\subseteq} N_{\omega_j}\Big\},$$

where the majority set inclusion $\text{IRN}(n) \overset{\beta}{\subseteq} N_{\omega_j}$ is understood analogously to that shown in Section 2.1. The

β-upper approximation $\overline{\text{IRN}}(\omega_i \to \omega_j)$ of the inter-region neighborhood, from ω_i to ω_j, is defined as

$$\overline{\text{IRN}}^{\beta}(\omega_i \to \omega_j) = \Big\{ n \in N_{\omega_i} : 1 - \frac{\text{card}(\text{IRN}(n) \cap N_{\omega_j})}{\text{card}(\text{IRN}(n))} < 1 - \beta \Big\}.$$

The accuracy of approximation of the inter-region neighborhood is defined as

$$\alpha^{\beta}_{\text{IRN}}(\omega_i \to \omega_j) = \frac{\text{card}(\underline{\text{IRN}}^{\beta}(\omega_i \to \omega_j))}{\text{card}(\overline{\text{IRN}}^{\beta}(\omega_i \to \omega_j))}$$

for $0 \le \beta < 0.5$.

Consider now the transition system $\text{TS}(\mathcal{PM}) = (S, E, T, S_{\text{init}})$ displayed in Fig. 3. For instance, three regions of interest are marked out with the following sets of states: $N_{\omega_1} = \{s_1\}$—a set containing the initial state representing the original point ph of plasmodium, $N_{\omega_2} = \{s_2, s_3\}$—a set of states corresponding to attractants $attr_1$ and $attr_2$ when they are occupied by plasmodium, and $N_{\omega_3} = \{s_6, s_7, s_8\}$—a set of states corresponding to attractants $attr_5$, $attr_6$, and $attr_7$ when they are occupied by plasmodium.

The lower approximations are as follows: $\underline{\text{IRN}}(\omega_1 \to \omega_2) = \emptyset$ and $\underline{\text{IRN}}(\omega_2 \to \omega_3) = \{s_3\}$, whereas the upper approximations are as follows: $\underline{\text{IRN}}(\omega_1 \to \omega_2) = \{s_1\}$ and $\underline{\text{IRN}}(\omega_2 \to \omega_3) = \{s_2, s_3\}$. Hence $\alpha_{\text{IRN}}(\omega_1 \to \omega_2) = 0$ and $\alpha_{\text{IRN}}(\omega_2 \to \omega_3) = 0.5$. In the case of VPRSM, for $\beta = 0.6$, we obtain: $\underline{\text{IRN}}^{0.6}(\omega_1 \to \omega_2) = \{s_1\}$ and $\underline{\text{IRN}}^{0.6}(\omega_1 \to \omega_2) = \{s_1\}$, $\underline{\text{IRN}}^{0.6}(\omega_2 \to \omega_3) = \{s_2, s_3\}$, and $\underline{\text{IRN}}^{0.6}(\omega_2 \to \omega_3) = \{s_2, s_3\}$. Hence $\alpha^{0.6}_{\text{IRN}}(\omega_1 \to \omega_2) = 1$ and $\alpha^{0.6}_{\text{IRN}}(\omega_2 \to \omega_3) = 1$.

5. Fundamentals of slime mould games

In slime mould games, we can have two or more players. In antagonistic games, we deal with two: the first, *Physarum polycephalum* plasmodium and the second, the *Badhamia utricularis* plasmodium. Locations of original points of both plasmodia are randomly generated. We can control motions of plasmodia via attracting or repelling stimuli. Locations of attractants and repellents are contexts during the game.

Traditionally, a play of the game is formalized as a sequence of moves in sequential games and as a concurrency of moves in concurrent games. This method assumes the polarization of two-person games, when in each position there is only one player's turn to move in sequential games and both players can move concurrently in concurrent games. The sequential games can hold just on models of fragments of linear logic such as multiplicative (Abramsky and Jagadeesan, 1994) or multiplicative-exponential fragments (Baillot *et al.*, 1997). In concurrent games introduced by Abramsky and Mellies (1999), process calculi can be used.

The propagation of *Physarum polycephalum* plasmodium is understood here as a transition system $\text{TS}(\mathcal{PM}) = (S, E, T, S_{\text{init}})$. States S will be regarded as possible payoffs for *Physarum polycephalum*. Events E will be examined as allowed moves in slime mould games. Transitions T are represented as a set of states resulting from the moves and initial states S_{init} as different players.

Among all the possible actions in plasmodium propagations, we can perceive attracting (Fig. 4(a)), repelling (Fig. 4(b)), splitting (Fig. 4(c)), and fusing (Fig. 4(d)) actions. Now we can define logical operations on S as follows. *Negation* $\neg s_x$ is true if and only if there is no s_y such that $s_y \xrightarrow{e_{yx}} s_x$ and it is false otherwise. *Conjunction* $s_x \wedge s_y = \min(s_x, s_y)$ is true if and only if for both s_x and s_y there is s_z such that $s_z \xrightarrow{e_{zx}} s_x$ **and** $s_z \xrightarrow{e_{zy}} s_y$ and it is false otherwise. *Disjunction* $s_x \vee s_y = \max(s_x, s_y)$ is true if and only if for both s_x and s_y there is s_z such that $s_z \xrightarrow{e_{zx}} s_x$ **or** $s_z \xrightarrow{e_{zy}} s_y$ and it is false otherwise.

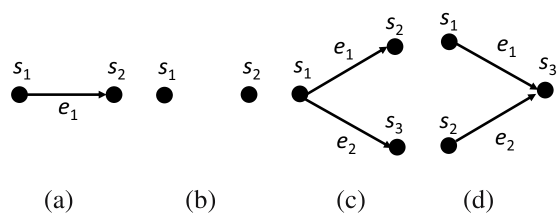

Fig. 4. Four primitive actions in the plasmodium propagation for $S = \{s_1, s_2, s_3\}$ and $E = \{e_1, e_2\}$: attracting (a), repelling (b), splitting (c), fusing (d).

A (finite) *concurrent game on* Physarum *polycephalum* is the sextuple

$$\mathcal{G} = (States, Agt, Act_n, Mov^n, Tab^n, (\preceq_A)_{A \in Agt}),$$

where

- $States = \{s_1, \dots, s_m\} = S$ is a (finite) set of *states* presented by attractants occupied by the plasmodium;
- $Agt = \{1, \dots, k\} = S_{\text{init}}$ is a finite set of *players* presented by different active zones of plasmodium (original points);
- Act_n is a nonempty set of *actions* presented by logical operations $\neg$, $\wedge$, and $\vee$ over $State$ or their inductive combinations with n inputs and one output, an element of Act_n^{Agt} is called a *move*;
- $Mov^n\colon States^n \times Act_n^{Agt} \to 2^{Act} \setminus \{\emptyset\}$ is the mapping indicating the *available* sets of actions to

a given player in a given set of states, $n > 0$ is said to be a radius of plasmodium actions, a move $m^n_{Agt} = (m^n_A)_{A \in Agt}$ is legal at $(s_1, \dots, s_n)$ if $m^n_A \in Mov^n(\mathbf{s}, A)$ for all $A \in Agt$, where $\mathbf{s} = (s_1, \dots, s_n)$ (note that if there exists a minimal p such that $n < p$ for all moves, this game is called *p-adic valued*);

- $Tab^n\colon States^n \times Act^{Agt}_n \to States$ is the transition table which associates, with a given set of states and a given move of the players, the set of states resulting from that move;

- for each $A \in Agt$, $\preceq_A$ is a preorder (reflexive and transitive relation) over $States^\omega$, called the *preference relation* of player A, indicating the intensity of attractants; for each $\pi, \pi' \in States^\omega$, by $\pi \preceq_A \pi'$ we mean that π' is *at least as good as* π for A and when it is not $\pi \preceq_A \pi'$, we say that A *prefers* π over π'.

Assume there is a game G and its states are formulated as some propositions, whose set is denoted by $Prop$.

A concurrent game on *Physarum polycephalum* $\mathcal{G}$ is a *model* $\mathcal{M}$ for the game G if $\mathcal{M} = \langle \mathcal{G}, \varphi \rangle$, where $\varphi\colon States \to \mathcal{P}(Prop)$ is a labeling function such that it labels the states in $\mathcal{G}$ by proposition symbols from the set $Prop$ of the game G.

Usually, for each player (plasmodium) named 1, …, k, where $k = |Agt|$, we have a separate space of attractants. However, the space can be joint, too. The point is that if two players (plasmodia) move to the same state, then their transitions from this state are the same. In order to avoid this property, we should design different moves of different players in different spaces.

The set of *outcomes* $Out_{\mathcal{G}}(s)$ of the concurrent game $\mathcal{G}$ from the state s is a set of all infinite paths $s_0 s_1 \ldots \in States^\omega$ such that $s_0 = s$ and for all $j > 0$, there exists a move $m \in \prod_{k=1}^{|Agt|} Mov(s_j, k)$ and $Tab(s_j, m) = s_{j+1}$. The *set of all outcomes* is as follows: $Out_{\mathcal{G}} = \bigcup_{s' \in States} Out_{\mathcal{G}}(s)$. The set of *histories* $Hist_{\mathcal{G}}(s)$ starting in s is a set of all finite paths $s_0 s_1 \ldots s_l$ such that $s_0 = s$ and there exists $\sigma \in Out_{\mathcal{G}}(s)$ which starts with $s_0 s_1 \ldots s_l$. The set $Hist_{\mathcal{G}} = \bigcup_{s' \in States} Hist_{\mathcal{G}}(s)$ is said to be the *set of all histories*.

A *strategy* of a player j in $\mathcal{G}$ is a mapping $strat_j\colon Hist_{\mathcal{G}} \to Act$ such that for any history $\sigma \in Hist_{\mathcal{G}}$ it is true that $strat_j(\sigma) \in Mov(last(\sigma), j)$, where $last(\sigma)$ denotes the last state in the finite path σ. In other words, a strategy $strat_j(\sigma)$ is the choice of a legal action in the last state of the history σ which was observed by player j. If we have one space for all players and they have the same last state $last(\sigma)$, then for all of them the next legal action will be the same, too. Therefore, in this case, histories of players are unimportant for choosing the next action, i.e., starting from $last(\sigma)$ their histories will be the same.

Adamatzky and Grube have performed some experiments showing that there are cases when sets of strategies for players in the same space are always disjoint (Schumann *et al.*, 2014). Let us suppose that we have only two agents. The first is presented by a usual *Physarum polycephalum* plasmodium, the second by its modification called a *Badhamia utricularis* plasmodium (references on this new culture are included in the work of Neubert *et al*. (1995)). *Physarum polycephalum* grows definitely faster than *Badhamia utricularis* and overtakes more flakes at the same time than the latter; see the photos in the work of Schumann *et al*. (2014). Only if the inoculum were "fatter" for *Badhamia utricularis*, it might grow faster. Moreover, if the invasive growth front of *Badhamia utricularis* is well nourished by oat, it easily overgrows the opposing tube system of *Physarum polycephalum*. Thus, at the microscopic level we can find out that in most observations *Physarum polycephalum* could grow into branches of *Badhamia utricularis*, while *Badhamia utricularis* could grow over *Physarum polycephalum* strands (Schumann *et al*., 2014). We can see that somehow *Physarum polycephalum* feeds on small branches of *Badhamia utricularis*. Thus, in the case of *Physarum polycephalum* and *Badhamia utricularis* we observe a competition in the small branches. For them the sets of strategies are disjoint. They never meet the same states.

Based on competitions between *Physarum polycephalum* and *Badhamia utricularis*, we can study the simplest biological forms of *zero-sum games*.

A *strategy for several players* A is defined as the following tuple $(strat_j)_{j \in A}$ of strategies for all players of A. The set of all outcomes if the players in A follow the strategy $strat_A = (strat_j)_{j \in A}$ is denoted by $Out_{\mathcal{G}}(s, strat_A)$. All possible outcomes if the players in A obey $strat_A$ is denoted by $Out_{\mathcal{G}}(strat_A) = \bigcup_{s \in States} Out_{\mathcal{G}}(s, strat_A)$. A strategy for Agt: $(strat_j)_{j \in Agt}$ is called a *strategy profile*.

We will say that a strategy $strat_A$ is *memoryless* for players of A at a state s if they choose their joint action based only on s as the last state of the play. This holds if they simultaneously meet s in a joint space. Notice that in a game with two players presented by *Physarum polycephalum* plasmodium 1 and *Badhamia utricularis* plasmodium 2 in a joint space, their strategies $strat_{Agt=\{1,2\}}$ cannot be memoryless at any time.

Let us take a move m_{Agt} and an action m' for some player B. The move n_{Agt} with $n_A = m_A$ when $A \neq B$ and $n_B = m'$ is denoted by $m_{Agt}[B \to m']$. Then a Nash equilibrium for concurrent games is defined as follows (for more details on equilibria in concurrent games, see the works of Brenguier (2013) and Bouyer *et al.* (2012; 2011)).

Let $\mathcal{G}$ be a concurrent game with a preference relation $(\preceq_A)_{A\in Agt}$ and let s be a state of $\mathcal{G}$. A *Nash equilibrium* of $\mathcal{G}$ from s is a strategy profile $strat_{Agt}$ such that $Out(s, strat_{Agt}[B \to strat']) \preceq_B Out(s, strat_{Agt})$ for all players $B \in Agt$ and all strategies $strat'$ of B.

Example 1. Let

$$\mathcal{G} = (States, Agt, Act_n, Mov^n, Tab^n, (\preceq_A)_{A\in Agt})$$

and $n = 9$ (i.e., our game is 10-adic valued), $Agt = \{A_1, A_2\}$, $States = \{s_1, s_2, \dots, s_m\}$ ($m \geq p = 10$), $Act_n = \{\max, \min\}$. Assume that A_1 follows only max at all legal moves and A_2 follows only min at its legal moves. This means that in a transition system TS($\mathcal{PM}$), $E_{A_i} = \{(\mathbf{s}, \mathbf{s}') \in S^9 \times S^9 \colon Tab^9(\mathbf{s}, m^9_{A_i}) = \mathbf{s}'\}$, where $i = 1, 2$, $m^9_{A_1} = \max$, and $m^9_{A_2} = \min$. This game can be illustrated in the cellular-automaton form with the neighborhood $|N| = 8$ (Schumann, 2014). Let us take cells belonging to the set $\mathbb{Z}^2$, therewith each cell takes its value in $States$. Let transitions depend on a local transition rule $\delta\colon States^9 \to States$ that transforms states of cells taking into account the states of 8 neighboring cells. Each step of dynamics is fixed by discrete time t = 0, 1, 2, ... At the moment t, the configuration of the whole system (or the global state) is given by the mapping x^t from $\mathbb{Z}^2$ into $States$, and the evolution is the sequence $x^0x^1x^2x^3\dots$ defined as $x^{t+1}(z) = \delta(x^t(z), x^t(z+\alpha_1), x^t(z+\alpha_2), \dots, x^t(z+\alpha_8))$, where $\langle\alpha_1, \alpha_2, \dots, \alpha_8\rangle$ are neighbors of z.

At each move we will write an occupied attractant from $States$ as 1 and an unoccupied attractant as 0. Suppose that at $t = 0$ we have the following states, where, given $\langle s_i, s_j\rangle$, s_i means a state for A_1 and s_j means a state for A_2:

$\langle 0,1\rangle$	$\langle 0,0\rangle$	$\langle 1,1\rangle$	$\langle 0,1\rangle$	$\langle 1,0\rangle$
$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,0\rangle$
$\langle 0,0\rangle$	$\langle 1,0\rangle$	$\langle 0,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 0,0\rangle$

Then at $t = 1$ we obtain the following states:

$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$
$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$

Hence, we deal with the set of actions represented by logical operations over $States$ or their inductive combinations which give n outputs for n inputs and we take into account time t. ♦

Example 2. Let

$$\mathcal{G} = (States_t, Agt, Act_{t,n}, Mov^n_t, Tab^n_t, (\preceq_A)_{A\in Agt}),$$

where

- $States_t$ means states for consecutive times $t = 0, 1, \dots$;
- Agt is a finite set of players;
- $Act_{t,n}$ is an infinite set of actions represented by logical operations or their inductive combinations with n inputs and n outputs; these actions can be applicable only to states at time t and give states at time $t+1$;
- Mov^n_t is a set of legal moves at time $t = 0, 1, \dots$;
- $Tab^n_t\colon States^n_t \times Act^{Agt}_{t,n} \to States^n_{t+1}$ is the transition table which associates, with a given set of states and a given move of the players, the set of states resulting from that move;
- for each $A \in Agt$, $\preceq_A$ is a preorder over $States^\omega$, called the *preference relation* of player A.

Suppose that $n = 9$ (i.e., the game is 10-adic valued still), $Agt = \{A_1, A_2\}$, $States = \{s_1, s_2, \dots, s_m\}$ ($m \geq p = 10$) such that each state has its value in $\{0, 1\}$ (i.e., it is occupied or not), $Act_{t,9} = \{\max \Rightarrow \min, \min \Rightarrow \max\}$. Assume that A_1 follows the rule: $\max\{$the states of A_1 at $t\} \Rightarrow \min\{$the states of A_2 at $t\}$, and A_2 follows the rule: $\min\{$the states of A_2 at $t\} \Rightarrow \max\{$the states of A_1 at $t\}$. This means that in an appropriate transition system TS($\mathcal{PM}$), $E_{A_i} = \{(\mathbf{s}, \mathbf{s}') \in S^9_t \times S^9_{t+1} \colon Tab^9(\mathbf{s}, m^9_{A_i}) = \mathbf{s}'\}$, where $i = 1, 2$, $m^9_{A_1} = \max\{$the states of A_1 at $t\} \Rightarrow \min\{$the states of A_2 at $t\}$, and $m^9_{A_2} = \min\{$the states of A_2 at $t\} \Rightarrow \max\{$the states of A_1 at $t\}$.

In the cellular-automaton form, at $t = 0$, we get

$\langle 0,1\rangle$	$\langle 0,0\rangle$	$\langle 1,1\rangle$	$\langle 0,1\rangle$	$\langle 1,0\rangle$
$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$	$\langle 0,0\rangle$
$\langle 0,0\rangle$	$\langle 1,0\rangle$	$\langle 0,0\rangle$	$\langle 1,1\rangle$	$\langle 1,0\rangle$
$\langle 0,1\rangle$	$\langle 1,0\rangle$	$\langle 1,1\rangle$	$\langle 1,1\rangle$	$\langle 0,0\rangle$

At $t = 1$, we have

$\langle 1,0\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$
$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$	$\langle 0,1\rangle$

We have just shown that we can involve actions which are different for all $t = 0, 1, \ldots$ and are mutually dependent. ♦

Example 3. Let us consider an example, when $n = 1$ (i.e., the game is 2-adic valued) and $Agt = \{A_1, A_2\}$ in $\mathcal{G}$, to demonstrate that the set of actions for slime mould games can be uncountably infinite.

Assume that agent A_1 moves from state s_1^t to state s_1^{t+1}, $t = 0, 1, \ldots$ and agent A_2 moves from state s_2^t to state s_2^{t+1}, $t = 0, 1, \ldots$ by the following transition rules: $s_1^{t+1} = (s_1^t \Rightarrow s_2^t)$ and $s_2^{t+1} = (s_2^t \Rightarrow s_1^t)$. Then we obtain the following infinite streams: $\langle s_1^0 \Rightarrow s_2^0, s_1^1 \Rightarrow s_2^1, s_1^2 \Rightarrow s_2^2, \ldots\rangle$ and $\langle s_2^0 \Rightarrow s_1^0, s_2^1 \Rightarrow s_1^1, s_2^2 \Rightarrow s_1^2, \ldots\rangle$. Notice that the stream $\langle s_1^0 \Rightarrow s_2^0, s_1^1 \Rightarrow s_2^1, s_1^2 \Rightarrow s_2^2, \ldots\rangle$ (resp., $\langle s_2^0 \Rightarrow s_1^0, s_2^1 \Rightarrow s_1^1, s_2^2 \Rightarrow s_1^2, \ldots\rangle$) may be understood as an infinite propositional formula $(((s_1^0 \Rightarrow s_2^0) \Rightarrow s_2^1) \Rightarrow s_2^2) \Rightarrow \ldots$ (resp. $(((s_2^0 \Rightarrow s_1^0) \Rightarrow s_1^1) \Rightarrow s_1^2) \Rightarrow \ldots$). Both formulas are mutually dependent and they cannot be represented as a linear sequence (inductive composition). Hence, we have a 2-adic valued formula presented by two infinite mutually dependent propositional formulas. This formula is called *non-well-founded* (Aczel, 1988). Also, we can show that any non-well-founded formula of radius $n = |Agt|$ can be formulated as n infinite mutually dependent propositional formulas. In concurrent games, there is no reflection of players. They do not pay attention to which actions are involved in transitions by others. A non-well-founded action of radius $n = |Agt|$ means that each player of Agt coordinates their action with others by using their reasoning in the form they can foresee, maybe wrongly. Reflexive games are considered by Schumann (2014). They are strong extensions of concurrent games. ♦

In the case of slime mould, non-well-founded actions are not results of predictions of the others as in the case of human reflexive games. The matter is that one plasmodium can follow some non-well-founded actions (Khrennikov and Schumann, 2014), since for $n \geq 2$ inputs, there is in fact an uncertainty, which logic gates with n inputs are involved.

A (finite) *context-based game on Physarum polycephalum* is the sextuple

$$\mathcal{G} = (States_t, Agt, Act_{t,n}, Mov_t^n, Tab_t^n, (\preceq_A)_{A\in Agt}),$$

where

- $States_t = \{s_1, \ldots, s_m\}$ is a (finite) set of *states* represented by attractants occupied by the plasmodium at time $t = 0, 1, 2, \ldots$;
- $Agt = \{1, \ldots, k\}$ is a finite set of *players* presented by different active zones of plasmodium;
- $Act_{t,n}$ is a nonempty set of *non-well-founded actions* with radius n at $t = 0, 1, 2, \ldots$; an element of $Act_{t,n}^{Agt}$ is called a *move* at time $t = 0, 1, 2, \ldots$; for this n the game is called $n+1$-adic valued;
- $Mov_t^n\colon States_t^n \times Act_{t,n}^{Agt} \to 2^{Act} \setminus \{\emptyset\}$ is the mapping indicating the *available* sets of actions to a given player in a given set of states, $n > 0$ is said to be a radius of plasmodium actions, a move $m_{Agt}^n = (m_A^n)_{A\in Agt}$ is legal at $\langle s_1, \ldots, s_n\rangle$ if $m_A^n \in Mov^n(\mathbf{s}, A)$ for all $A \in Agt$, where $\mathbf{s} = \langle s_1, \ldots, s_n\rangle$;
- $Tab_t^n\colon States_t^n \times Act_{t,n}^{Agt} \to States_{t+1}^n$ is the transition table which associates, with a given set of states at t and a given move of the players at t, the set of states at $t+1$ resulting from that move;
- for each $A \in Agt$, $\preceq_A$ is a preorder (reflexive and transitive relation) over $States^\omega$, called the *preference relation* of player A, indicating the intensity of attractants.

It is worth noting that context-based games are *massively parallel*. In one experiment (game) we can use even several hundred attractants considered as ‘processors’ and then the plasmodium builds several thousand connections among attractants and these connections change permanently.

All other notions such as outcomes, histories, and strategies are defined just as in the previous section. We can prove a statement that for any concurrent game $\mathcal{G}$ on the medium of slime mould, there is an appropriate context-based game as the greatest fixed point for all uncertain modifications of $\mathcal{G}$ in experiments with plasmodia. This statement allows us to build logic circuits on the slime mould using context-based game notions.

In the concurrent games, as well as in the context-based games, we can use rough sets for defining strategies. Let $\Omega^t = \{\omega_1^t, \omega_2^t, \ldots, \omega_k^t\}$ be a set of all *nearest strategies* at t for all agents $Agt = \{1, 2, \ldots, k\}$, i.e., strategies performed only one time at the actual time step t. Let, $\mathcal{N} = \{N_{\omega_1^t}, N_{\omega_2^t}, \ldots, N_{\omega_k^t}\}$ denote a family of payoffs corresponding to the nearest strategies such that $N_{\omega_i^t} \subset States$ represents the states obtained by player $i = \overline{1,k}$ by applying strategy ω_i^t at the actual time step t.

If for each i, ω_i^t yields only a singleton $N_{\omega_i^t}$, i.e., $\mathrm{card}(N_{\omega_i^t}) = 1$, the game is called *concurrent*. If for some i, ω_i^t yields $N_{\omega_i^t}$ such that $\mathrm{card}(N_{\omega_i^t}) > 1$, the game is called *context-based*.

For each state $s \in N_{\omega_1^t} \cup N_{\omega_2^t} \cup \cdots \cup N_{\omega_k^t}$, we define its p-adic valued inter-region neighborhood at t:

$$\mathrm{IRN}_t^p(s) = \Big\{ s' : (s, s') \in E \wedge \underset{\omega\in\Omega^t}{\exists} (s' \in N_\omega \wedge s \notin N_\omega) \Big\}.$$

The cardinal number $\mathrm{card}(\mathrm{IRN}_t^p(s)) \leq p$.

We assume that each player can change its strategies at each new step t. This means that we deal with a transition $\omega_i^t \to \omega_i^{t+1}$.

The p-adic valued lower approximation $\underline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1})$ of the strategy change at $t+1$ is defined as follows:

$$\underline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1}) = \Big\{ s \in N_{\omega_i^t} : \mathrm{IRN}^p_t(s) \neq \emptyset \wedge \mathrm{IRN}^p_t(s) \subseteq N_{\omega_i^{t+1}} \Big\}.$$

The p-adic valued upper approximation $\overline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1})$ of the strategy change at $t+1$ is defined as follows:

$$\overline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1}) = \Big\{ s \in N_{\omega_i^t} : \mathrm{IRN}^p(s) \wedge N_{\omega_i^{t+1}} \neq \emptyset \Big\}.$$

The *intentionality of player* i by a strategy change at $t+1$ is defined in the following manner:

$$\alpha_{\mathrm{IRN}}(\omega_i^t \to \omega_i^{t+1}) = \frac{\mathrm{card}(\underline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1}))}{\mathrm{card}(\overline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1}))}.$$

We see that $0 \leq \alpha_{\mathrm{IRN}}(\omega_i^t \to \omega_i^{t+1}) \leq 1$.

The situation $\alpha_{\mathrm{IRN}}(\omega_i^t \to \omega_i^{t+1}) = 0$ means that $\mathrm{card}(\underline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1})) = 0$, i.e., it means that new payoffs at $t+1$, if they take place, are obtained not intentionally. The case $\alpha_{\mathrm{IRN}}(\omega_i^t \to \omega_i^{t+1}) = 1$ means that $\mathrm{card}(\underline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1})) = \mathrm{card}(\overline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1}))$, i.e., it means that all the new payoffs at $t+1$ are obtained intentionally. Hence, the measure $\alpha_{\mathrm{IRN}}(\omega_i^t \to \omega_i^{t+1})$ tells us about intentionality of player i in moving from t to $t+1$.

Now, let us define the *intentionality of player* i through the whole game, assuming that it is infinite. Let

$$\alpha_{\mathrm{IRN}}(i) = \frac{\sum_{t=0}^{\infty} \mathrm{card}(\underline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1})) \cdot p^t}{\sum_{t=0}^{\infty} \mathrm{card}(\overline{\mathrm{IRN}}^p_{t+1}(\omega_i^t \to \omega_i^{t+1})) \cdot p^t}.$$

Then also $0 \leq \alpha_{\mathrm{IRN}}(i) \leq 1$. But now this measure runs over the set of p-adic integers $\mathbb{Z}_p$.

We suppose that *somebody wins* if (s)he has occupied more payoffs (attractants) at the majority steps $t \to \infty$ than each other player separately. Somebody *loses* if (s)he has occupied less payoffs (attractants) at the majority steps $t \to \infty$ than each other player separately. Let us define the same formally.

Let $\mathcal{N} = \{N_{\omega_1^t}, N_{\omega_2^t}, \ldots, N_{\omega_k^t}\}$ be a family of payoffs corresponding to the nearest strategies ω_i^t at the actual time step t for each player $i = \overline{1,k}$. Let $\mathcal{N}^\emptyset = \{N^\emptyset_{\omega_1^t}, N^\emptyset_{\omega_2^t}, \ldots, N^\emptyset_{\omega_k^t}\}$ be a family of vacant attractants at t such that

$$N^\emptyset_{\omega_i^t} = \Big\{ s_{t+1} : \underset{s_t \in N_{\omega_i^t}}{\forall} (s_t, s_{t+1}) \in E \wedge \underset{\omega \in \Omega^t}{\forall} (s_{t+1} \notin N_\omega) \Big\},$$

i.e., it is a set of all accessible attractants for player i at time t which can be occupied at $t+1$ from the set $N_{\omega_i^t}$.

The *probability of winning* for i is defined as follows:

$$Win(i) = \frac{\sum_{t=0}^{\infty} \mathrm{card}(N_{\omega_i^{t+1}}) \cdot p^t}{\sum_{t=0}^{\infty} \mathrm{card}(N^\emptyset_{\omega_i^t}) \cdot p^t}.$$

Player i wins if $Win(i) \geq Win(j)$ for any player $j \in (Agt - \{i\})$. Player i loses if $Win(i) \leq Win(j)$ for any player $j \in (Agt - \{i\})$.

In that case, $N_{\omega_1^t}, N_{\omega_2^t}, \ldots, N_{\omega_k^t}$ are pairwise disjoint, the game is being carried out independently of the competitors' strategies—each player plays in a parallel manner without intercommunication. Suppose now that $N_{\omega_1^t} \cap N_{\omega_2^t} \cap \cdots \cap N_{\omega_k^t} = N_t^c \neq \emptyset$.

The lower approximation of payoffs for player i at $t+1$ is

$$\underline{N_{\omega_i^{t+1}}} = \{ s \in N_{\omega_i^t} : N_{\omega_i^{t+1}} \neq \emptyset \wedge N_{\omega_i^{t+1}} \subseteq N^c_{t+1} \}.$$

The upper approximation of payoffs for player i at $t+1$ is

$$\overline{N_{\omega_i^{t+1}}} = \Big\{ s \in N_{\omega_i^t} : N_{\omega_i^{t+1}} \cap N^c_{t+1} \neq \emptyset \Big\}.$$

The *probability of winning in competitions for attractants* for i is thus defined as

$$Win_c(i) = \frac{\sum_{t=0}^{\infty} \mathrm{card}(\underline{N_{\omega_i^{t+1}}}) \cdot p^t}{\sum_{t=0}^{\infty} \mathrm{card}(\overline{N_{\omega_i^{t+1}}}) \cdot p^t}.$$

Player i wins in competitions for food if $Win_c(i) \geq Win_c(j)$ for any player $j \in (Agt - \{i\})$. Player i loses if $Win_c(i) \leq Win_c(j)$ for any player $j \in (Agt - \{i\})$.

Hence, in context-based games, we have the following main features: (i) the game can be infinite and its measures are set up by rough sets with values running over p-adic integers $\mathbb{Z}_p$; (ii) the game is concurrent if each move for each player gives only one payoff, otherwise the game is context-based; (iii) each player can change their strategy at each time step t; (iv) if the set of payoffs for agents i and j are intersected at t, this means that the strategies of i and j are intersected also at t.

6. Go games

Go is a game, originated in ancient China, in which two persons play with a Go board and Go stones (Kim and Jeong, 1994). In general, two players alternately place black and white stones, on the vacant intersections of a board with a 19×19 grid of lines, to surround the territory. Whoever has more territory at the end of the game is the winner. Vertically and horizontally adjacent stones of the same color form a group. One of the basic principles of Go is that stones must have at least one *liberty* to remain on the board. A liberty of a given stone is a vacant intersection adjacent to it. If a stone has at least one liberty, then the next stone of a given player can be placed on it to extend their group. The rough set based Go game was implemented in the module simulating slime mould (*Physarum*) games.

As was shown by Schumann (2016), the Go game can be simulated by means of a 5-adic valued *Physarum* machine. Let us consider an antagonistic game implemented in plasmodia of *Physarum polycephalum* and *Badhamia utricularis* in the 5-adic valued universe (at each step *Physarum polycephalum* and *Badhamia utricularis* can see not more than 4 attractants). In the Go game, payoffs can be assessed by means of the measure defined on the basis of rough set theory (Schumann and Pancerz, 2015).

Let a board for the Go game, with a 19×19 grid of lines, be in use. The set of all intersections of the grid is denoted by I. At the beginning, the fixed numbers of original points of both the plasmodia of *Physarum polycephalum* and the plasmodia of *Badhamia utricularis* are randomly deployed on intersections. An example of the initial configuration of the Go game is shown in Fig. 5. In this case, two original points of the plasmodia of *Physarum polycephalum* (Ph_1 and Ph_2), treated as black stones, as well as two original points of the plasmodia of *Badhamia utricularis* (Ba_1 and Ba_2), treated as white stones, are deployed on intersections.

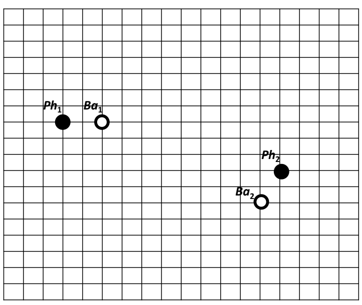

Fig. 5. Example of the initial configuration of the Go game implemented on the *Physarum* machine.

During the game, two players alternately place attractants on the vacant intersections of the board. The first player plays for the *Physarum polycephalum* plasmodia, the second one for the *Badhamia utricularis* plasmodia. The attractants occupied by plasmodia of *Physarum polycephalum* are treated as black stones whereas the attractants occupied by plasmodia of *Badhamia utricularis* as white stones.

We assume the following formal structure of the 5-adic valued *Physarum* machine $\mathcal{PM} = (P, B, A)$: $P = \{Ph_1, Ph_2, \dots, Ph_k\}$ is the set of original points of the plasmodia of *Physarum polycephalum*, $B = \{Ba_1, Ba_2, \dots, Ba_l\}$ is the set of original points of the plasmodia of *Badhamia utricularis*, and $A = \{A^t\}_{t=0,1,2,\dots}$ is the family of the sets of attractants, where $A^t = \{A_1^t, A_2^t, \dots, A_{r_t}^t\}$ is the set of all attractants present at time step t in $\mathcal{PM}$. The structure of the *Physarum* machine $\mathcal{PM}$ is changing in time (new attractants are added by the players). Each intersection $i \in I$ is identified by two coordinates x and y written as $i(x, y)$. For each intersection $i(x, y)$, we can distinguish its neighborhood, called the intersection neighborhood, i.e., $\mathrm{IN}(i) = \{i'(x', y') \in I : (x' = x - 1 \vee x' = x + 1) \wedge (x' \geq 1) \wedge (x' \leq 19) \wedge (y' = y - 1 \vee y' = y + 1) \wedge (y' \geq 1) \wedge (y' \leq 19)\}$.

Formally, during the game, at a given time step t, we can distinguish three kinds (pairwise disjoint) of intersections in the set I_t of all intersections, namely, $I_t^{\emptyset}$—the set of all vacant intersections at t, $I_t^{\bullet}$—the set of all intersections occupied by plasmodia of *Physarum polycephalum* at t (black stones), and I_t°—the set of all intersections occupied by plasmodia of *Badhamia utricularis*) at t (white stones). The plasmodium of *Physarum polycephalum* and the plasmodium of *Badhamia utricularis* cannot occupy the same attractants.

For the set I_t^{π} of all intersections occupied by given plasmodia π (either the plasmodia of *Physarum polycephalum* or the plasmodia of *Badhamia utricularis*) at a given time step t we define neighborhood approximations.

The lower neighborhood approximation of I_t^{π} is defined as

$$\underline{\mathrm{IN}}(I_t^{\pi}) = \{i \in I_t^{\pi} : \mathrm{IN}(i) \neq \emptyset \wedge \mathrm{IN}(i) \subseteq I_t^{\pi}\},$$

where π is either $\bullet$ or $\circ$. Each intersection $i \in I$ such that $i \in \underline{\mathrm{IN}}(I_t^{\pi})$ is called a full generator of the payoff of the player playing for the plasmodia π.

The upper neighborhood approximation of I_t^{π} is given by

$$\overline{\mathrm{IN}}(I_t^{\pi}) = \{i \in I_t^{\pi} : \mathrm{IN}(i) \cap I_t^{\pi} \neq \emptyset\},$$

where π is either $\bullet$ or $\circ$.

The set $\mathrm{BN}_{\mathrm{IN}}(I_t^{\pi}) = \overline{\mathrm{IN}}(I_t^{\pi}) - \underline{\mathrm{IN}}(I_t^{\pi})$ is the boundary region of neighborhood approximation of I_t^{π} at time step t. Each intersection $i \in I$ such that $i \in \mathrm{BN}_{\mathrm{IN}}(I_t^{\pi})$ is called a partial generator of the payoff of the player playing for the plasmodia π.

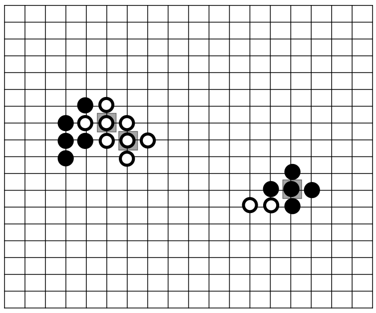

Fig. 6. Configuration of the Go game after several moves (payoffs defined on the basis of a standard definition of rough sets).

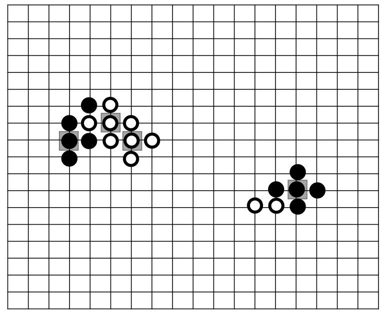

Fig. 7. Configuration of the Go game after several moves (payoffs defined based on the VPRSM approach for $\beta = 0.25$).

In the case of the variable precision rough set model, we obtain the β-lower neighborhood approximation:

$$\underline{\mathrm{IN}}^{\beta}(I_t^{\pi}) = \{i \in I_t^{\pi} : \mathrm{IN}(i) \neq \emptyset \wedge \mathrm{IN}(i) \overset{\beta}{\subseteq} I_t^{\pi}\},$$

Each intersection $i \in I$ such that $i \in \underline{\mathrm{IN}}^{\beta}(I_t^{\pi})$ is called a full quasi-generator of the payoff of the player playing for the plasmodia π.

On the basis of lower neighborhood approximations, we define a measure assessing payoffs of the players. For the first player playing for the *Physarum polycephalum* plasmodia we have $\Theta^{\bullet} = \mathrm{card}(\underline{\mathrm{IN}}(I_t^{\bullet}))$. For the second player playing for the *Badhamia utricularis* plasmodia we have $\Theta^{\circ} = \mathrm{card}(\underline{\mathrm{IN}}(I_t^{\circ}))$. In a more relaxed case, we have $\Theta^{\bullet} = \mathrm{card}(\underline{\mathrm{IN}}^{\beta}(I_t^{\bullet}))$ and $\Theta^{\circ} = \mathrm{card}(\underline{\mathrm{IN}}^{\beta}(I_t^{\circ}))$, respectively.

Consider an illustrative configuration of the Go game after several moves. Intersections belonging to lower neighborhood approximations $\underline{\mathrm{IN}}(I_t^{\bullet})$ and $\underline{\mathrm{IN}}(I_t^{\circ})$ of $(I_t^{\bullet})$ and (I_t°), respectively, are marked with grey rectangles in Fig. 6. It is worth noting that all of the intersections from $\underline{\mathrm{IN}}(I_t^{\bullet})$ and $\underline{\mathrm{IN}}(I_t^{\circ})$ are full generators of the payoffs of the players playing for the *Physarum polycephalum* plasmodia and *Badhamia utricularis* plasmodia, respectively. Hence, we obtain $\Theta^{\circ} = 1$ and $\Theta^{\bullet} = 2$. The second player, playing for the *Badhamia utricularis* plasmodia, wins.

In the case of the VPRSM approach, for $\beta = 0.25$, intersections belonging to lower neighborhood approximations $\underline{\mathrm{IN}}^{0.25}(I_t^{\bullet})$ and $\underline{\mathrm{IN}}^{0.25}(I_t^{\circ})$ of $(I_t^{\bullet})$ and (I_t°), respectively, are marked with grey rectangles in Fig. 7. It is worth noting that some of intersections from $\underline{\mathrm{IN}}^{0.25}(I_t^{\bullet})$ and $\underline{\mathrm{IN}}^{0.25}(I_t^{\circ})$ are full quasi-generators of the payoffs of the players playing for the *Physarum polycephalum* plasmodia and *Badhamia utricularis* plasmodia, respectively. Hence, we obtain $\Theta^{\circ} = 3$ and $\Theta^{\bullet} = 3$. No player wins.

We can define strategies in the Go game using the neighborhood approximation. A mapping from the intersections belonging to upper neighborhood approximations $\overline{\mathrm{IN}}(I_t^{\pi})$ at t to the intersections belonging to lower neighborhood approximations $\underline{\mathrm{IN}}(I_{t+k}^{\pi})$ at $t+k$ is said to be a set of rational strategies of π of radius k to win. A mapping from the intersections belonging to β-lower neighborhood approximations $\underline{\mathrm{IN}}^{\beta}(I_t^{\bullet})$ at t to the intersections belonging to the union $\underline{\mathrm{IN}}^{\gamma}(I_{t+k}^{\circ}) \cup \underline{\mathrm{IN}}^{\beta}(I_{t+k}^{\bullet})$ at $t+k$, where $0 < \beta \leq \gamma$ and

$$\mathrm{card}(\underline{\mathrm{IN}}^{\beta}(I_t^{\bullet})) \leq \mathrm{card}(\underline{\mathrm{IN}}^{\gamma}(I_{t+k}^{\circ}) \cup \underline{\mathrm{IN}}^{\beta}(I_{t+k}^{\bullet})) < \mathrm{card}(\underline{\mathrm{IN}}(I_{t+k}^{\bullet})),$$

is said to be a set of rational strategies of $\circ$ (the player playing for *Badhamia utricularis*) of radius k not to lose. A mapping from the intersections belonging to β-lower neighborhood approximations $\underline{\mathrm{IN}}^{\beta}(I_t^{\circ})$ at t to the intersections belonging to the union $\underline{\mathrm{IN}}^{\gamma}(I_{t+k}^{\bullet}) \cup \underline{\mathrm{IN}}^{\beta}(I_{t+k}^{\circ})$ at $t+k$, where $0 < \beta \leq \gamma$ and

$$\mathrm{card}(\underline{\mathrm{IN}}^{\beta}(I_t^{\circ})) \leq \mathrm{card}(\underline{\mathrm{IN}}^{\gamma}(I_{t+k}^{\bullet}) \cup \underline{\mathrm{IN}}^{\beta}(I_{t+k}^{\circ})) < \mathrm{card}(\underline{\mathrm{IN}}(I_{t+k}^{\circ})),$$

is said to be a set of rational strategies of $\bullet$ (the player playing for *Physarum polycephalum*) of radius k not to lose.

The agent is rational if (s)he follows one of the rational strategies to win or not to lose in moves. Also, we can define strategies in the Go game if we deal with the VPRSM neighborhood approximation. A mapping from the intersections belonging to γ-lower neighborhood approximations $\underline{\mathrm{IN}}^{\gamma}(I_t^{\pi})$ at t to the intersections belonging to β-lower neighborhood approximations $\underline{\mathrm{IN}}^{\beta}(I_{t+k}^{\pi})$ at $t+k$ is said to be a set of rational β-strategies of π of radius k to win if $\gamma < beta$. A mapping from the intersections belonging to γ-lower neighborhood approximations $\underline{\mathrm{IN}}^{\gamma}(I_t^{\bullet})$ at t to the intersections belonging to the union $\underline{\mathrm{IN}}^{\delta}(I_{t+k}^{\circ}) \cup \underline{\mathrm{IN}}^{\gamma}(I_{t+k}^{\bullet})$ at $t+k$, where $0 < \beta \leq \gamma$ and $0 < \beta \leq \delta$ and

$$\mathrm{card}(\underline{\mathrm{IN}}^{\gamma}(I_t^{\bullet})) \leq \mathrm{card}(\underline{\mathrm{IN}}^{\delta}(I_{t+k}^{\circ}) \cup \underline{\mathrm{IN}}^{\gamma}(I_{t+k}^{\bullet})) < \mathrm{card}(\underline{\mathrm{IN}}^{\beta}(I_{t+k}^{\bullet})),$$

is said to be a set of rational β-strategies of $\circ$ (the player playing for *Badhamia utricularis*) of radius k not to lose. A mapping from the intersections belonging to γ-lower neighborhood approximations $\underline{\mathrm{IN}}^{\gamma}(I_t^{\circ})$ at t to the intersections belonging to the union $\underline{\mathrm{IN}}^{\delta}(I_{t+k}^{\bullet}) \cup \underline{\mathrm{IN}}^{\gamma}(I_{t+k}^{\circ})$ at $t+k$, where $0 < \beta \leq \gamma$ and $0 < \beta \leq \delta$ and

$$\begin{aligned}\mathrm{card}(\underline{IN}^{\gamma}(I_t^{\circ})) &\leq \mathrm{card}(\underline{IN}^{\delta}(I_{t+k}^{\bullet}) \cup \underline{\mathrm{IN}}^{\gamma}(I_{t+k}^{\circ})) \\ &< \mathrm{card}(\underline{\mathrm{IN}}^{\beta}(I_{t+k}^{\circ})),\end{aligned}$$

is said to be a set of rational β-strategies of $\bullet$ (the player playing for *Physarum polycephalum*) of radius k not to lose.

The agent is β-rational if (s)he follows one of the rational β-strategies to win or not to lose in moves. In the Go games, we base on the following presuppositions: (i) the universe for all game moves is 5-adic valued, (ii) the game is antagonistic, (iii) the game is sequential, not concurrent. Hence, the Go games defined above are a simple version of slime mould games as such. The latter can process in the p-adic valued universes for different $p > 0$, they can be not only antagonistic, but also cooperative, they can be concurrent or even massively parallel. If an experiment involves several attractants, we deal with a concurrent game, and if an experiment is based on placing several hundred attractants, we talk about a massive-parallel game.

7. Conclusions

We have proposed a bio-inspired experimental game theory on the medium of *Physarum polycephalum* and *Badhamia utricularis*. In our approach, we share the following interpretations of basic entities: (i) attractants as payoffs; (ii) attractants occupied by the plasmodium as states of the game; (iii) active zones of plasmodium as players; (iv) logic gates for behaviors as moves (available actions) for the players; (v) propagation of the plasmodium as the transition table which associates, with a given set of states and a given move of the players, the set of states resulting from that move. As a result, in slime mould games, we base on the following assumptions: each slime mould game is concurrent or massively parallel; players can change or modify their strategies; the game can be infinite in the p-adic valued universe for any integer $p > 0$.

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